

5. 夹挤原理

$$(1) \lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \underline{\quad 1 \quad};$$

(2) $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = \underline{0}$;

注重 $x \rightarrow ?$

(3) $\lim_{x \rightarrow 0} x \cdot \left\lceil \frac{1}{x} \right\rceil = 1 : \left\lceil \frac{1}{x} \right\rceil \in \left(\frac{1}{x} - 1, \frac{1}{x} \right]$

夹挤原理

(1) $\lim_{x \rightarrow \infty} \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}$

$$\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} = \frac{x + \sqrt{x + \sqrt{x}} - x}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}} = \frac{\sqrt{x + \sqrt{x}}}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}}$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x + \sqrt{x}}}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}} = \frac{1}{2}$$

对比: ① $\sqrt{x} \left(\sqrt{1 + \frac{\sqrt{x}}{x}} - 1 \right)$

$$\begin{aligned} &= \sqrt{x} \cdot \left(1 + \frac{1}{2} \frac{\sqrt{x}}{x} + o\left(\frac{\sqrt{x}}{x}\right) - 1 \right) \\ &= \frac{1}{2} \cdot \frac{\sqrt{x} \cdot \sqrt{x}}{\sqrt{x}} + o\left(\frac{\sqrt{x} \cdot \sqrt{x}}{\sqrt{x}}\right) \\ &= \frac{1}{2} \sqrt{x} + o\left(\sqrt{x}\right) \\ &= \frac{1}{2} + o(1) = \frac{1}{2} \end{aligned}$$

② $\frac{\sqrt{x + \sqrt{x}}}{\sqrt{x} \cdot \sqrt{1 + \frac{\sqrt{x}}{x}}} + \sqrt{x} = \frac{\sqrt{1 + \frac{\sqrt{x}}{x}}}{\sqrt{1 + \frac{\sqrt{x}}{x}}} + 1$

$$\begin{aligned} &= 1 + \frac{1}{2} \frac{\sqrt{x}}{x} \\ &= 1 + \frac{1}{2} \frac{\sqrt{x}}{x} + o\left(\frac{\sqrt{x}}{x}\right) \\ &= \frac{1 + o(1)}{2 + o(1)} = \frac{1}{2} \end{aligned}$$

对比下这道题

分子有理化
次数比较低时写的更清楚
次数高了反而难顶

$$\begin{aligned} \lim_{x \rightarrow \infty} & \left(\sqrt{x^{100} - 3x^{99} + 3} - \sqrt{x^{100} + 3x^{98} - 4} \right) \\ &= \lim_{x \rightarrow \infty} x \cdot \left(\sqrt{1 - \frac{3}{x} + \frac{3}{x^{100}}} - \sqrt{1 + \frac{3}{x^2} - \frac{4}{x^{100}}} \right) \\ &= \lim_{x \rightarrow \infty} x \cdot \left(1 - \frac{3}{100x} + o\left(\frac{1}{x}\right) - 1 \right) \\ &= -\frac{3}{100} \end{aligned}$$

这个方法更普遍

提公因式
然后泰勒

在求渐近线时
也可以广泛应用

典例
先猜后证
记得充要

4. $\infty - \infty$ 型

方法:设法转化为 $\frac{0}{0}, \frac{\infty}{\infty} \dots$

$$(x^3 + x^2)^c \approx (x^3)^c = x^{3c}$$

(3) 求常数 c , s. t. $\lim_{x \rightarrow +\infty} (x^3 + x^2)^c - x$ 极限存在, 并求值.

直观猜测 $c = \frac{1}{3}$

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{x^3}{(x^3 + x^2)^{1/3}} - x &= \lim_{x \rightarrow +\infty} \frac{x^3 + x^2 - x^3}{(x^3 + x^2)^{1/3} + x(x^3 + x^2)^{1/3} + x^2} \\ &= \lim_{x \rightarrow +\infty} \frac{x^2}{(x^3 + x^2)^{2/3} + x(x^3 + x^2)^{1/3} + x^2} = \lim_{x \rightarrow +\infty} \frac{1}{(1 + \frac{1}{x})^{2/3} + (1 + \frac{1}{x})^{1/3} + 1} = 1/3 \end{aligned}$$

$$a-b = \frac{a^3 - b^3}{a^2 + ab + b^2}$$

$x < \frac{1}{3}$ 时, $(x^3 + x^2)^c - x < 2^c \cdot x^{3c} - x$ 而后者 $\rightarrow -\infty$

$x > \frac{1}{3}$ 时, $(x^3+x^2)^{\frac{1}{2}} - x > x^{\frac{3}{2}} - x$ 而后者 $\rightarrow +\infty$

套路性太强

是否是1(不是1不行)
是否X趋于0(不是有0误差太大)

1.1[∞]型

$$\lim_{x \rightarrow x_0} g(x) = 1, \lim_{x \rightarrow x_0} h(x) = \infty, \lim_{x \rightarrow x_0} \frac{g(x)}{h(x)} = 0$$

$$\lim_{x \rightarrow \infty} \left(\frac{x^2 + 1}{x^2 - 1} \right)^{x^2}$$

$$\lim_{x \rightarrow \infty} \left(\frac{x^2+1}{x^2-1}\right)^{x^2} = \lim_{x \rightarrow \infty} e^{\ln\left(\frac{x^2+1}{x^2-1}\right)x^2} = e$$

$$\lim_{x \rightarrow \infty} (x^2 - 1) = \infty$$

$$\lim_{x \rightarrow \infty} \ln\left(1 + \frac{2}{x^2 - 1}\right) x^2$$

路本明显，

套路太明显，

① 判定为 1^∞ ② 取为 e^+

$$\textcircled{3} \ln g(x) = \ln^{(+g(x)-1)} = g(x)-1 + O(g(x)-1)$$

1(∞)型

$$\begin{aligned} & \lim_{x \rightarrow 0} \left(\frac{\cos x}{\cos 2x} \right)^{\frac{1}{x^2}} \\ & \lim_{x \rightarrow 0} e^{\frac{1}{x^2} \ln \frac{\cos x}{\cos 2x}} \quad \text{不要写这一串} \\ & = \lim_{x \rightarrow 0} e^{\frac{1}{x^2} \cdot \left(\frac{\cos x - \cos 2x}{\cos 2x} + 0 \right) \frac{\cos x - \cos 2x}{\cos 2x}} \\ & \text{即 } \frac{\cos x - \cos 2x}{\cos 2x} = \frac{-\frac{1}{2}x^2 - (-2x^2) + 0(x^4)}{1 - 2x^2 + 0(x^2)} \\ & = \frac{\frac{3}{2}x^2 + 0(x^2)}{1 - 2x^2 + 0(x^2)} \\ & \text{原式} = \lim_{x \rightarrow 0} e^{\frac{\frac{3}{2}x^2 + 0(x^2)}{1 - 2x^2 + 0(x^2)}} \\ & = \lim_{x \rightarrow 0} e^{\frac{\frac{3}{2} + 0(1)}{1 + 0(1)}} = e^{\frac{3}{2}} \end{aligned}$$

具体的过程

注意到:在实际用等价无穷小的时候
不要写出复杂的0(一堆)
真正只有在用泰勒的时候
才写出 $O(X)$

常见的泰勒公式
(见补充资料)

0/0型

直接上泰勒

有对 $(x+1)^{\alpha}$ ($a^2e^{-1}-x, a^2-1-x\ln(a>0)$)
与 x^α 的展 $(1+x)^{-1} = -ax.$

$$\begin{aligned}\text{Ex } \lim_{n \rightarrow \infty} \frac{x^n}{\sqrt[n]{n}} &= \lim_{n \rightarrow \infty} \frac{x^{n+1}}{\sqrt[n+1]{n+1}} = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{(n+1)} = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{n+1} \\&= \lim_{n \rightarrow \infty} \frac{x^{n+1}}{n+1} = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{n+1} = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{n+1} = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{n+1} \\&= \lim_{n \rightarrow \infty} \frac{x^{n+1}}{n+1} = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{n+1} = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{n+1} = \lim_{n \rightarrow \infty} \frac{x^{n+1}}{n+1}\end{aligned}$$

错误来源:

①厘清 $(1+x)^\alpha$ 与 x^α 的区别:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n + o(x^n)$$
$$= \frac{x^\alpha}{0!} + \frac{x^\alpha}{1!} + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n + o(x^n)$$

②注意条件:

泰勒在阶数足够时一定可用,但阶数不够时应 $x \rightarrow 0$

$O(X)$ 的性质

一. $O(X + O(X)) = O(X) \quad (X \rightarrow 0)$
即实际上占主导地位的是最大的数

二. $O(\cdot) \in O(k)$
即 $x \rightarrow 0$ 时, $O(x^2) \in O(x)$
 $O(x^3 + \frac{1}{8}x^4) \subset O(1)$

三. $O(x)$ 可乘:

$$x \rightarrow 0: x^2 \cdot O(x) = O(x^3) \in O(x)$$
$$x \rightarrow +\infty: x^2 \cdot O(x) = O(x^3) \in O(x^*)$$