

第五周作业110.

$$2.127. \quad x_0=0: \quad \lim_{x \rightarrow x_0} \frac{\sin x}{|x|} = \begin{cases} \lim_{x \rightarrow 0} \frac{\sin x}{x} = 0 & x > 0 \\ -\lim_{x \rightarrow 0} \frac{\sin x}{x} = 0 & x < 0 \end{cases}$$

$\therefore f(x_0) \neq \lim_{x \rightarrow x_0} f(x)$ 不连续

$\therefore f(x)$ 在 $x=x_0$ 处不连续. 称 x_0 为 f 的可去间断点

$$2.151. \quad \lim_{x \rightarrow x_0} x^2(1 - e^{\frac{x}{x-2}}) = -\lim_{x \rightarrow x_0} \frac{e^{\frac{x}{x-2}} - 1}{\frac{x}{x-2}} \\ = -\lim_{x \rightarrow x_0} \frac{e^{\frac{x}{x-2}} - 1}{(\frac{x}{x-2})} \cdot \frac{1}{x-2} \\ = -1 \cdot \lim_{x \rightarrow x_0} \frac{1}{x-2}$$

$$x_0=0, \quad -1 \cdot \lim_{x \rightarrow 0} \frac{1}{x-2} = \frac{1}{2} = f(0) \quad \lim_{x \rightarrow 2^-} = \frac{1}{2} \quad \lim_{x \rightarrow 2^+} = -\infty$$

$$x_0=2, \quad -1 \cdot \lim_{x \rightarrow 2} \frac{1}{x-2} \rightarrow \infty \neq f(\frac{1}{2}). \quad x \rightarrow 2^- \quad x \rightarrow 2^+$$

$\therefore f(x)$ 在 $x_0=0$ 处连续, 在 $x=2$ 时不连续, 且 $x=2$ 是 $f(x)$ 的第一类间断点

3.110. 对 $a \in \mathbb{R}, x > 0, f(x) = 3e^x + a$ 为初等函数

$\therefore x > 0$ 时, $3e^x + a$ 连续.

$$\lim_{x \rightarrow 0^-} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x^2} = \lim_{x \rightarrow 0} \left(\frac{\sqrt{x^2+1} - 1}{x^2} + \frac{\sqrt{x^2+1} - 1}{-x^2} \right) = \frac{1}{2} + \frac{1}{2} = 1$$

$$\text{令 } f(0) = \lim_{x \rightarrow 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x^2} = 1 = 3 + a$$

$$\therefore a = -2.$$

4. $\sin x - 2, ax^2 + b, 2^x + x$ 均为初等函数. 分别在 $(-\infty, 0), (0, 1), (1, +\infty)$ 连续

$$\text{由 } \lim_{x \rightarrow 0} (\sin x - 2) = -2; \quad \lim_{x \rightarrow 1^+} (2^x + x) = 3.$$

$$\therefore \text{令 } \lim_{x \rightarrow 0^+} (ax^2 + b) = -2, \quad \lim_{x \rightarrow 1^-} (ax^2 + b) = 3$$

$$\text{即 } b = -2, \quad a + b = 3, \quad \text{解得 } a = 5, \quad b = -2.$$

2.6.3. 证: " $f \in C[a, b]$ $\therefore f$ 有界.

即 $\exists x_M \in [a, b]$, 使得 $f(x_M) = \sup f(x), x \in [a, b]$:

$\exists x_m \in [a, b]$, 使得 $f(x_m) = \inf f(x), x \in [a, b]$.

$\therefore f(x_m) \leq f(x_i) \leq f(x_M) \quad i=1, 2, \dots, n.$

$\therefore$ 必有: $\frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n} \in [f(x_m), f(x_M)]$

即 $f(\xi) \in f(I)$. 由连续函数的介值定理知.

$\exists \xi \in [a, b]$ 使得 $f(\xi) \in \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}$

2.6.2. 证: 记 $f(x) = x^{2m} + a_1 x^{2m-1} + \dots + a_{2m-1} x + a_{2m}$.

" $f(x)$ 是初等函数 $\therefore f(x)$ 符合连续函数的介值定理.

$f(0) = a_{2m} < 0$.

要证 $f(x)$ 至少有 2 个零点, 即证 $\exists x_1, x_2 < 0$, 使 $f(x_1) > 0, f(x_2) > 0$.

当 $m=1$ 时, $f(x) = x^2 + a_1 x + a_2$. 其中 $\Delta = a_1^2 - 4a_2 > 0$. $f(x)$ 有 2 个零点.

假设 $m=k$ 时, $f_k(x_1) = x_1^{2k} + a_1 x_1^{2k-1} + \dots + a_{2k-1} x_1 + a_{2k} > 0$

$f_k(x_2) > 0$, 且 $x_1, x_2 < 0$.

则当 $m=k+1$ 时, $f_{k+1}(x) = x^{2k+2} + a_1 x^{2k+1} + \dots + a_{2k+1} x^3 + a_{2k} x^2 + a_{2k+1} x + a_{2k+2}$
 $= x^2 (x^{2k} + a_1 x^{2k-1} + \dots + a_{2k-1} x + a_{2k}) + a_{2k+1} x + a_{2k+2}$
 $= f_k(x) x^2 + a_{2k+1} x + a_{2k+2}.$

其中 $x = x_1$ 或 x_2 时, $\Delta = a_{2k+1}^2 - 4 f_k(x) a_{2k+2} > 0$.

$\therefore f_{k+1}(x)$ 至少有 2 个零点.

$\therefore a_{2m} < 0$ 时, 实系数多项式 $x^{2m} + a_1 x^{2m-1} + \dots + a_{2m-1} x + a_{2m}$ 至少有 2 个零点.

8. 证明: 设 $A = \lim_{x \rightarrow +\infty} f(x)$, 则对 $\forall \varepsilon > 0, \exists N \in \mathbb{R}$, 当 $x > N$ 时 $|f(x) - A| < \varepsilon$

$\therefore x > N$ 时, $|f(x)| = |f(x) - A + A| \leq |f(x) - A| + |A| < |A| + \varepsilon$.

$x \leq N$ 时, $f(x) \in C[a, N]$ $\therefore f(x)$ 在 $[a, N]$ 上有界

即 $\exists M > 0$, 使 $|f(x)| \leq |M|$.

$\therefore x \in [a, +\infty)$ 时, $|f(x)| \leq \max\{|M|, |A| + \varepsilon\}$

$\therefore f(x)$ 在 $[a, +\infty)$ 上有界.

12. 证明: 先证 $f(a) = a, f(b) = b$.

反证, 若不然, 则 $\exists x_1, x_2 \in (a, b)$, 使得 $f(x_1) = a, f(x_2) = b$.

又 $f(x)$ 在 $[a, b]$ 上单调递增 $\therefore \exists x \in (a, x_1)$, 使得 $f(x) < a$, 与 $f \in [a, b]$ 矛盾.

$\therefore$ 假设不成立, 则 $x_1 = a, f(a) = a$, 同理有 $f(b) = b$.

① 若 f 为 $[a, b]$ 上的凹函数, 则有 $f(x) \leq x$

又 $x_{n+1} = f(x_n) \therefore x_{n+1} \leq x_n$. 即 $\{x_n\}_{n=1}^{+\infty}$ 为单调不增数列

又 $x \in [a, b] \therefore \{x_n\}$ 有下界 a . $\therefore \{x_n\}_{n=1}^{+\infty}$ 的极限存在. 记 $\lim_{n \rightarrow \infty} x_n = A$

又 $f \in C[a, b]$ 取 $x_1 \in [a, b]$, 当 $x_1 = a$ 时, $x_1 = x_2 = \dots = x_n = a$.

$\therefore \lim_{n \rightarrow \infty} x_n = a$ 即 $\lim_{n \rightarrow \infty} x_{n+1} = a$. $\therefore \lim_{n \rightarrow \infty} f(x_n) = a$. $\therefore f(a) = a \therefore f(\xi) = \xi$.

[又 $x_{n+1} = f(x_n) \therefore \lim_{n \rightarrow \infty} f(x_n) = a$. $\therefore \lim_{x \rightarrow a^+} f(x) = a$]

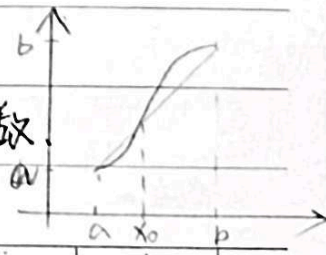
② 若 f 为 $[a, b]$ 上的凹增函数, 则有 $f(x) \geq x, x_{n+1} \geq x_n, \{x_n\}_{n=1}^{+\infty}$ 单调不减

又 $\sup \{x_n\} = b \therefore \lim_{n \rightarrow \infty} x_n = b$.

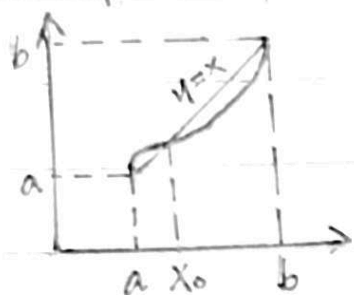
$\therefore$ 极限 $\lim_{n \rightarrow \infty} x_n = b$ 存在且 $f(b) = b$.

③ 若 f 为 $[a, x_0]$ 上的凹增函数, $[x_0, b]$ 上的凹增函数.

若 $x_1 \in [a, x_0]$, 同情况①; 若 $x_1 \in [x_0, b]$, 同情况②



④ 若 $f(x)$ 是 $[a, x_0]$ 上的凹增函数, $[x_0, b]$ 上的凸增函数. 此时 $f(x_0) = x_0$.



若 $x_1 \in [a, x_0]$, 则 $\{x_n\}_{n=1}^{\infty}$ 是单调不减数列.

且 $\sup \{x_n\} = x_0$.

$\therefore$ 极限 $\lim_{n \rightarrow \infty} x_n = x_0$ 存在且 $f(x_0) = x_0$.

若 $x_1 \in [x_0, b]$, 则 $\{x_n\}_{n=1}^{\infty}$ 是单调不增数列, 且 $\inf \{x_n\} = x_0$.

$\therefore$ 极限 $\lim_{n \rightarrow \infty} x_n = x_0$ 存在且 $f(x_0) = x_0$.

综上, 极限 $\lim_{n \rightarrow \infty} x_n = \xi$ 存在, 且 $f(\xi) = \xi$.

① $L > 0$

8. 证明: 对 $\forall \varepsilon > 0$, 取 $\delta = \frac{\varepsilon}{L}$. 则对 $\forall x, y \in I$, $|x - y| < \delta$ 时

$$|f(x) - f(y)| \leq L|x - y| < L\delta = \varepsilon.$$

② $L = 0$, $\forall x, y \in I$, $f(x) - f(y) = 0$, 对 $\forall \varepsilon > 0$, $|f(x) - f(y)| < \varepsilon$

综上, $f(x)$ 在 I 上一致连续.

9. (2). 证明: 对 $\forall x_1, x_2 \in (2, +\infty)$, 不妨设 $x_2 > x_1$,

对 $\forall \varepsilon > 0$, 取 $\delta = 2(e^\varepsilon - 1)$. $|x_2 - x_1| < \delta$ 时, 记 $x_1, x_2 \in (x_0, x_0 + \delta)$ ($x_0 \geq 2$).

$$\therefore |f(x_1) - f(x_2)| = \ln x_2 - \ln x_1 = \ln\left(1 + \frac{x_2 - x_1}{x_1}\right)$$

$$< \ln\left(1 + \frac{\delta}{2}\right) = \ln e^\varepsilon = \varepsilon.$$

$\therefore f(x) = \ln x$ 在 $(2, +\infty)$ 上一致连续.

15.12) 要证 $f(x) = \ln x, x \in (0, +\infty)$ 不一致连续.

即证 $\exists \varepsilon_0 > 0$, 对 $\forall \delta > 0$, $\exists x_1, x_2 \in I$, $|x_1 - x_2| < \delta$, 但 $|f(x_1) - f(x_2)| \geq \varepsilon_0$.

不妨取 $\varepsilon_0 = 1$, 对 $\forall \delta > 0$, 取 $\hat{\delta} < \delta$, 记 $x_2 = x_1 + \hat{\delta}$, 取 $x_1 = \frac{\hat{\delta}}{e-1}$ ($\hat{\delta} > 0$)

$$\text{则 } |x_1 - x_2| < \delta, \quad |\ln x_2 - \ln x_1| = \ln\left(1 + \frac{\hat{\delta}}{x_1}\right) = \ln e = 1 = \varepsilon_0$$

$\therefore$ 得证.

16. 证明: $(\Rightarrow)$ $\because f(x)$ 在 (a, b) 内一致连续.

$\therefore \forall \varepsilon > 0, \exists \delta$, 当 $x_1, x_2 \in (a, a+\delta)$ 时, $|x_1 - x_2| < \delta, |f(x_1) - f(x_2)| < \varepsilon$.

由 Cauchy 准则知 $\lim_{x \rightarrow a^+} f(x)$ 存在. 同理 $\lim_{x \rightarrow b^-} f(x)$ 存在.

$\Leftarrow$ $\because f \in C(a, b)$, $\lim_{x \rightarrow a^+} f(x)$, $\lim_{x \rightarrow b^-} f(x)$ 都存在.

$$\therefore \text{令 } g(x) = \begin{cases} \lim_{x \rightarrow a^+} f(x) & x = a \\ f(x) & x \in (a, b) \\ \lim_{x \rightarrow b^-} f(x) & x = b \end{cases}$$

$\therefore g \in [a, b] \quad \therefore g(x)$ 在 $[a, b]$ 上一致连续

$\therefore f(x)$ 在 (a, b) 内一致连续.

$\because y = \frac{\sin x}{x}$ 是初等函数 $\therefore f \in C(0, 1)$

$$\text{又 } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \quad \lim_{x \rightarrow 1} \frac{\sin x}{x} = \frac{\lim_{x \rightarrow 1} \sin x}{\lim_{x \rightarrow 1} x} = \frac{\sin 1}{1} = \sin 1$$

$\therefore y = \frac{\sin x}{x}$ 在 $(0, 1)$ 上一致连续.

HW: 设 I 是 $\mathbb{R}$ 区间, $f \in C(I)$, 若 f 是单射, 求证 $f^{-1}: f(I) \rightarrow I$ 是连续的.

证明: $\because f \in C(I)$, 且 f 是单射 $\therefore f$ 在 I 上单调递增(减)

$\therefore$ 不妨设 f 在 I 上单增.

记 $x_0 = f^{-1}(y_0)$, 则 $f(x_0) = y_0$.

不妨 $\exists \varepsilon_0 > 0$, 使得 $x_0 - \varepsilon_0, x_0 + \varepsilon_0 \in I$.

$\therefore \forall \varepsilon > 0$, 有 $x_0 - \varepsilon_0 \leq x_0 - \min\{\varepsilon_0, \varepsilon\} < x_0 < x_0 + \min\{\varepsilon_0, \varepsilon\} \leq x_0 + \varepsilon_0$.

由 f 的单调性知:

$$f(x_0 - \varepsilon_0) \leq f(x_0 - \min\{\varepsilon_0, \varepsilon\}) < f(x_0) < f(x_0 + \min\{\varepsilon_0, \varepsilon\}) \leq f(x_0 + \varepsilon_0)$$

$$\text{取 } \delta = \min\{f(x_0 + \min\{\varepsilon_0, \varepsilon\}) - f(x_0), f(x_0) - f(x_0 - \min\{\varepsilon_0, \varepsilon\})\}.$$

则 $\delta > 0$; 当 $|y - y_0| < \delta$ 时,

$$f(x_0 - \min\{\varepsilon_0, \varepsilon\}) \leq y_0 - \delta < y < y_0 + \delta \leq f(x_0 + \min\{\varepsilon_0, \varepsilon\})$$

由 f^{-1} 的单调性知,

$$x_0 - \min\{\varepsilon_0, \varepsilon\} < f^{-1}(y) < x_0 + \min\{\varepsilon_0, \varepsilon\}.$$

$$\therefore x_0 - \varepsilon < f^{-1}(y) < x_0 + \varepsilon.$$

$$\therefore f^{-1}(y_0) - \varepsilon < f^{-1}(y) < f^{-1}(y_0) + \varepsilon, |f^{-1}(y) - f^{-1}(y_0)| < \varepsilon$$

$\therefore f^{-1}$ 在 y_0 处连续

第五周作业(2)

Date.

$$1. (1) f'(x) = \left(\frac{1}{x}\right)' = \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = - \lim_{h \rightarrow 0} \frac{1}{x(x+h)} = -\frac{1}{x^2}$$

$$x_0 = -3, f'(x_0) = f'(-3) = -\frac{1}{9}$$

$$(2) f'(x) = (2^{-x})' = \lim_{h \rightarrow 0} \frac{2^{-(x+h)} - 2^{-x}}{h} = \lim_{h \rightarrow 0} \frac{2^{-x}(2^{-h} - 1)}{h} = 2^{-x} \ln(2^{-1})$$

$$x_0 = 0, f'(x_0) = f'(0) = \ln 2^{-1} = -\ln 2.$$

$$2. \text{解: } \lim_{x \rightarrow 0^-} (3x^2 + 4x) = 0 \neq f(0) = -1$$

$\therefore f(x)$ 在 $x_0 = 0$ 处不连续.

$$x_0 = 0, f(x) = x^2 - 1, f'(x) = (x^2 - 1)' = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} (h + 2x) = 2x$$

$$\therefore f'(x_0) = f'(0) = 2 \cdot 0 = 0. \quad \text{不连续则不可导甲鸟.}$$

... 最多也是左右导

3. 解: $\because f(x)$ 在 $x=1$ 处可导 $\Leftrightarrow f'_+(x)$ 都存在且相等. 叭?

$$x \leq 1 \text{ 时, } f'(x) = (x^2 + 1)' = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = 2x$$

$$\therefore \lim_{x \rightarrow 1^-} f'(x) = 2. \text{ 即 } f'_-(1) = 2x = 2$$

$$x > 1 \text{ 时, } f'(x) = (ax + b)' = \lim_{h \rightarrow 0} \frac{a(x+h) - ax}{h} = a$$

$$\therefore \lim_{x \rightarrow 1^+} f'(x) = a. \text{ 即 } f'_+(1) = a$$

$$\therefore \lim_{x \rightarrow 1} f'(x) = \lim_{x \rightarrow 1^+} f'(x) \quad a = 2.$$

$\therefore f(x)$ 在 $x=1$ 处可导 $\therefore f(x)$ 在 $x=1$ 处连续.

也即 $f(x)$ 在 $x=1$ 处左连续且右连续

$$\lim_{x \rightarrow 1} f(x) = f(x) = f(1) = 2$$

$$\text{又 } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (ax + b) = a + b = 2 + b$$

$$2 + b = 2, b = 0$$

综上, $a = 2, b = 0.$

$$5. (2). \lim_{n \rightarrow \infty} n [f(x_0 + \frac{2}{n}) - f(x_0)]$$

$$= 2 \lim_{n \rightarrow \infty} \frac{f(x_0 + \frac{2}{n}) - f(x_0)}{(x_0 + \frac{2}{n}) - x_0} = 2 f'(x_0)$$

$$\begin{aligned} (4) \lim_{h \rightarrow 0} (f(x_0 + h))^{\frac{1}{h}} &= \lim_{h \rightarrow 0} (f(x_0 + h) - f(x_0) + f(x_0))^{\frac{1}{h}} \cdot \frac{f(x_0 + h) - f(x_0)}{h} \\ &= \lim_{h \rightarrow 0} (f(x_0 + h) - f(x_0) + 1)^{\frac{1}{f(x_0 + h) - f(x_0)}} \cdot \frac{f(x_0 + h) - f(x_0)}{h} \\ &= \lim_{h \rightarrow 0} e^{\frac{f(x_0 + h) - f(x_0)}{h}} \\ &= e^{f'(x_0)} \end{aligned}$$

12. 证明: $\because f'(a) f'(b) > 0$, 不妨设 $f'(a) > 0, f'(b) > 0$.

则 $\exists \delta_1 > 0$, 使得 $\forall x_1 \in (a, a + \delta_1), \frac{f(x_1) - f(a)}{x_1 - a} > 0, f(x_1) > f(a) = 0$

$\exists \delta_2 < 0$, 使得 $\forall x_2 \in (b - \delta_2, b), \frac{f(x_2) - f(b)}{x_2 - b} > 0, f(x_2) < f(b) = 0$

$\because f \in C([a, b])$

$\therefore f \in C([x_1, x_2])$, 且 $f(x_1) \cdot f(x_2) < 0$

$\therefore$ 由连续函数零值定理知, $\exists \xi \in [x_1, x_2]$, 也即 $\exists \xi \in (a, b)$, 使 $f(\xi) = 0$

$$3.2.2. (1). y' = 3x^2 + \frac{2}{3}x^{-\frac{2}{3}} + 3 \cdot \frac{1}{x^4}$$

$$(3) y = (x^3 - x^2 + x - 1)(3 - x^3)$$

$$= 3x^3 - 3x^2 + 3x - 3 - x^6 + x^5 - x^4 + x^3 = -x^6 + x^5 - x^4 + 4x^3 - 3x^2 + 3x - 3$$

$$y' = -6x^5 + 5x^4 - 4x^3 + 12x^2 - 6x + 3$$

$$(5) y' = \frac{\frac{1}{\cos^2 x} \cdot x - \tan x}{x^2} = \frac{\frac{x}{\cos^2 x} - \tan x}{x^2}$$

$$4(11). y' = 2 \cos 3x \cdot 3 = 6 \cos 3x$$

$$(13) y' = \frac{3}{2} (1-x^3)^{\frac{1}{2}} (-3x^2) = -\frac{9}{2} x^2 (1-x^3)^{\frac{1}{2}}$$

$$5. (1) [f(-x)]' = f(-x) \cdot (-1) = -f(-x) \text{ 这个 ' 很 fatal 鸭...!}$$

$$(2) [f(\sin^2 x) f(\cos^2 x)]'$$

$$= [f(\sin^2 x)]' f(\cos^2 x) + f(\sin^2 x) [f(\cos^2 x)]'$$

$$= f'(\sin^2 x) (\sin^2 x)' f(\cos^2 x) + f(\sin^2 x) f'(\cos^2 x) (\cos^2 x)'$$

$$= 2 \sin x \cos x f'(\sin^2 x) f(\cos^2 x) - 2 \sin^2 x \cdot x f(\sin^2 x) f'(\cos^2 x)$$

$$6. (3) \sqrt{x} g(x) = \sqrt{x}, h(x) = x^{\ln(1-x)}$$

$$\ln|g(x)| = \frac{1}{x} \ln|x|, \ln|h(x)| = \ln(1-x) \cdot \ln|x|$$

$$g'(x) = g(x) \ln|g(x)| = (\sqrt{x}) \cdot \frac{1}{x} \ln|x|$$

$$h'(x) = h(x) \ln|h(x)| = x^{\ln(1-x)} \cdot \ln(1-x) \cdot \ln|x|$$

$$y' = g'(x) + h'(x) = \sqrt{x} \cdot \frac{1}{x} \ln|x| + x^{\ln(1-x)} \cdot \ln(1-x) \cdot \ln|x|$$

见下?

$$(5). \ln|y| = \sin x \ln(\ln x)$$

$$y' = y \cdot \ln|y| = (\ln x)^{\sin x} \cdot \sin x \ln(\ln x) (\ln x)^{\sin x} \cdot (\cos x \cdot \ln \ln x + \frac{\sin x}{x \ln x})$$

$$x^{\frac{1}{x}} = e^{\frac{1}{x} \ln x} \quad e^{\frac{1}{x} \ln x} \cdot (-\frac{1}{x^2} \ln x + \frac{1}{x^2}) = x^{\frac{1}{x}} \cdot \frac{1 - \ln x}{x^2}$$

$$e^{\ln(1-x) \cdot \ln x} = e^{\ln(1-x) \cdot \ln x} \left(\frac{-1}{1-x} \ln x + \frac{1}{x} \cdot \ln(1-x) \right)$$

$$= x^{\ln(1-x)} \cdot \frac{(1-x) \ln^{1-x} - x \ln x}{x(1-x)}$$