

习题 1.1

1.12 当 $n=1$ 时

$$1^2 = 1$$

$$\frac{1}{6} \cdot 1 \cdot (1+1) \cdot (2+1) = 1$$

成立

假设 $n=k$ 时成立. 即

$$1^2 + 2^2 + \dots + k^2 = \frac{1}{6} k \cdot (k+1) \cdot (2k+1)$$

当 $n=k+1$ 时

$$1^2 + 2^2 + \dots + (k+1)^2 = \frac{1}{6} k(k+1)(2k+1) + (k+1)^2$$

$$= \frac{1}{6} (k+1)(2k^2 + 7k + 6)$$

$$= \frac{1}{6} (k+1)(k+2)(2k+3)$$

 $\therefore$ 得证

2.13 上确界: 3

下确界: -1

3. $x^2 < 2$

$$-\sqrt{2} < x < \sqrt{2}$$

 $\therefore \sqrt{2}$ 为 A 的一个上界

$$\forall \varepsilon > 0, \text{ 取 } x_0 = \max\{0, \sqrt{2} - \frac{\varepsilon}{2}\}$$

$$x_0 \in A, x_0 > \sqrt{2} - \varepsilon$$

 $\therefore \sqrt{2}$ 为 A 的上确界6. 当 $a \geq b$ 时

$$\max\{a, b\} = a$$

$$\frac{a+b+|a-b|}{2} = \frac{a+b+a-b}{2} = a$$

$$\min\{a, b\} = b$$

$$\frac{a+b-|a-b|}{2} = \frac{a+b-a+b}{2} = b$$

当 $a < b$ 时

$$\max\{a, b\} = b$$

$$\frac{a+b+|a-b|}{2} = \frac{a+b+b-a}{2} = b$$

$$\min\{a, b\} = a$$

$$\frac{a+b-|a-b|}{2} = \frac{a+b+a-b}{2} = a$$

 $\therefore$ 得证8.111 不妨设 $\inf A \leq \inf B$

$$\min\{\inf A, \inf B\} = \inf A$$

$$\forall x \in B, x \geq \inf B \geq \inf A$$

$$\therefore \forall x \in A \cup B, x \geq \inf A$$

 $\therefore \inf A$ 为 $A \cup B$ 的一个下界 $\therefore \inf A$ 为 A 的下确界

$$\therefore \forall a > \inf A, \exists x \in A, \text{ 使得 } x < a$$

$$\therefore A \subset A \cup B$$

$$\therefore \forall a > \inf A, \exists x \in A \cup B, \text{ 使得 } x < a$$

$$\therefore \inf A = \inf(A \cup B)$$

 $\therefore$ 得证

13) 不妨设 $\inf A \leq \inf B$

$$\forall x \in B, x \geq \inf B$$

$$\therefore B \cap A \subset B$$

$$\therefore \forall x \in A \cap B, x \geq \inf B$$

$\therefore \inf B$ 为 $A \cap B$ 的一个下界

$$\therefore \inf(A \cap B) \geq \inf B = \max\{\inf A, \inf B\}$$

习题 1.2

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1. (3) 等价

$\Rightarrow$ $\forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \text{当 } n > N \text{ 时, 有 } |a_n - A| < \varepsilon$

$\therefore \forall \varepsilon \in (0, 1), \exists N \in \mathbb{N}^*, \text{当 } n > N \text{ 时, 有 } |a_n - A| < \varepsilon$

$\Leftarrow$ 取 $\varepsilon = \frac{1}{2}, \exists N_0 \in \mathbb{N}^*, \text{当 } n > N_0 \text{ 时, 有 } |a_n - A| < \frac{1}{2}$

$\therefore \forall \varepsilon \geq 1, \exists N = N_0 \in \mathbb{N}^*, \text{当 } n > N \text{ 时, 有 } |a_n - A| < \frac{1}{2} < \varepsilon$

$\therefore \forall \varepsilon > 0, \exists N \in \mathbb{N}^*, \text{当 } n > N \text{ 时, 有 } |a_n - A| < \varepsilon$

(7) 不等价

$$\text{令 } a_n = \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} a_n = 0, \text{ 即 } A = 0$$

$$|a_n - A| = \frac{1}{n}$$

即不存在满足条件的 N

2. $\exists \varepsilon_0 > 0, \forall N \in \mathbb{N}^*, \exists n > N, \text{使得 } |a_n - A| \geq \varepsilon_0$

(3) 不等价 无穷多项

$$\text{令 } a_n = \begin{cases} A, & n \text{ 为奇数} \\ A+1, & n \text{ 为偶数} \end{cases}$$

$\{a_n\}$ 不收敛于 A

当 n 为奇数时, $|a_n - A| = 0$

即 $\forall \varepsilon > 0, |a_n - A| < \varepsilon$

即 $\forall \varepsilon > 0, \{a_n\}$ 中有无限项不满足 $|a_n - A| \geq \varepsilon$

(4) 等价

$\Rightarrow$ 反证法. 假设 $\forall \varepsilon > 0, \{a_n\}$ 中有有限多项满足 $|a_n - A| \geq \varepsilon$

设有限项中项数最大的为 a_{n_ε}

取 $N = n_\varepsilon$

$\therefore \exists \varepsilon_0 > 0, \exists n_1 > n_\varepsilon, \text{使得 } |a_{n_1} - A| \geq \varepsilon_0$

即 a_{n_1} 也满足 $|a_n - A| \geq \varepsilon_0$

$\therefore n_1 > n_\varepsilon$

$\therefore$ 与项数最大的为 a_{n_ε} 矛盾

(=) 反证法. 假设 $\{a_n\}$ 收敛于 A

$$\forall \varepsilon > 0. \exists N \in \mathbb{N}^*. \text{当 } n > N \text{ 时, } |a_n - A| < \varepsilon$$

即至多有 N 项满足 $|a_n - A| \geq \varepsilon$

$\therefore$ 矛盾

3.13) $\forall \varepsilon > 0$. 取 $N = [\frac{1}{\varepsilon}] + 1 (n \in \mathbb{N}^*)$

当 $n > N$ 时, $n > \frac{1}{\varepsilon}$

$$\therefore \left| \frac{\cos n}{n} \right| < \frac{1}{n} < \varepsilon$$

$$\therefore \lim_{n \rightarrow +\infty} \frac{\cos n}{n} = 0$$

6.14) 不妨设 $A > 0$

取 $\varepsilon = A$. $\exists N_1 \in \mathbb{N}^*$. 使得当 $n > N_1$ 时, $|a_n - A| < A$. 即 $a_n > 0$

$$\forall \varepsilon > 0. \text{取 } \delta = \frac{\varepsilon A}{2 + \varepsilon}. \exists N_2 \in \mathbb{N}^*. \text{使得 } |a_n - A| < \delta, \text{即 } \frac{2}{2 + \varepsilon} A < a_n < \frac{2 + 2\varepsilon}{2 + \varepsilon} A$$

$$\therefore \text{当 } n > N_1 + N_2 \text{ 时, } 1 + \varepsilon > \frac{a_{n+1}}{a_n} > \frac{1}{1 + \varepsilon} > 1 - \varepsilon$$

$$\text{即 } \left| \frac{a_{n+1}}{a_n} - 1 \right| < \varepsilon$$

7.12) $\lim_{n \rightarrow +\infty} \frac{|a_1 + a_2 + \dots + a_{n+1}| - |a_1 + a_2 + \dots + a_n|}{n+1 - n}$

用定义.

$$= \lim_{n \rightarrow +\infty} a_{n+1} = A$$

由 Stolz 命题, 可得 $\lim_{n \rightarrow +\infty} \frac{a_1 + a_2 + \dots + a_n}{n} = A$

反之不成立

$$\text{令 } a_n = \begin{cases} A-1, & n \text{ 为奇数} \\ A+1, & n \text{ 为偶数} \end{cases}$$

$$\text{当 } n \text{ 为奇数时, } \left| \frac{a_1 + a_2 + \dots + a_n}{n} - A \right| = \frac{1}{n}$$

$$\text{当 } n \text{ 为偶数时, } \left| \frac{a_1 + a_2 + \dots + a_n}{n} - A \right| = 0$$

$$\forall \varepsilon > 0. \exists N = [\frac{1}{\varepsilon}] + 1 (n \in \mathbb{N}^*). \text{当 } n > N \text{ 时, } \left| \frac{a_1 + a_2 + \dots + a_n}{n} - A \right| < \varepsilon$$

$$\text{即 } \lim_{n \rightarrow +\infty} \frac{a_1 + a_2 + \dots + a_n}{n} = A. \text{但 } a_n \text{ 不收敛于 } A$$

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习题 1.3

1.11) 正确

$$\text{设 } \lim_{n \rightarrow +\infty} (2x_n - y_n) = A, \lim_{n \rightarrow +\infty} (3x_n + 4y_n) = B$$

由极限的四则运算

$$\begin{aligned} \lim_{n \rightarrow +\infty} x_n &= \lim_{n \rightarrow +\infty} \left(\frac{4}{11} (2x_n - y_n) + \frac{1}{11} (3x_n + 4y_n) \right) \\ &= \lim_{n \rightarrow +\infty} \left(\frac{4}{11} (2x_n - y_n) \right) + \lim_{n \rightarrow +\infty} \left(\frac{1}{11} (3x_n + 4y_n) \right) \\ &= \frac{4}{11} A + \frac{1}{11} B \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow +\infty} y_n &= \lim_{n \rightarrow +\infty} (2x_n - (2x_n - y_n)) \\ &= \lim_{n \rightarrow +\infty} 2x_n - \lim_{n \rightarrow +\infty} (2x_n - y_n) \\ &= 2 \left(\frac{4}{11} A + \frac{1}{11} B \right) - A \\ &= -\frac{3}{11} A + \frac{2}{11} B \end{aligned}$$

故 $\{x_n\}$ 与 $\{y_n\}$ 都收敛

3. 取 $\varepsilon = \min\{A-a, b-A\} > 0$

由 $\lim_{n \rightarrow +\infty} a_n = A, \exists N \in \mathbb{N}^*, \text{当 } n > N \text{ 时, 有 } |a_n - A| < \varepsilon$

$$\text{即 } A - \varepsilon < a_n < A + \varepsilon$$

$$\begin{aligned} A - \varepsilon &= A - \min\{A-a, b-A\} \\ &\geq A - (A-a) = a \end{aligned}$$

$$\begin{aligned} A + \varepsilon &= A + \min\{A-a, b-A\} \\ &\leq A + (b-A) = b \end{aligned}$$

故 $a < a_n < b$

$$4.18) \lim_{n \rightarrow +\infty} \left(\frac{1+2+\dots+n}{n+2} - \frac{n}{2} \right)$$

$$= \lim_{n \rightarrow +\infty} \left(\frac{n(n+1)}{2(n+2)} - \frac{n}{2} \right)$$

$$= \lim_{n \rightarrow +\infty} \left(-\frac{n}{2n+4} \right)$$

$$\text{设 } a_n = -\frac{n}{2n+4}$$

$$\forall \varepsilon > 0, \text{取 } N = \left[\frac{1}{\varepsilon} \right] \in \mathbb{N}^*$$

$$\text{当 } n > N \text{ 时, 有 } |a_n + \frac{1}{2}| = \frac{1}{n+2} < \varepsilon$$

$$\text{故 } \lim_{n \rightarrow +\infty} a_n = -\frac{1}{2} \quad \therefore \lim_{n \rightarrow +\infty} \left(\frac{1+2+\dots+n}{n+2} - \frac{n}{2} \right) = -\frac{1}{2}$$

$$(12) \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$$

$$< \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}$$

$$= 1$$

$$\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$$

$$> \frac{1}{n+1} + \frac{1}{n+1} + \dots + \frac{1}{n+1}$$

$$= \frac{n}{n+1}$$

$\forall \varepsilon > 0, \exists N = [\frac{1}{\varepsilon}] \in \mathbb{N}$, 当 $n > N$ 时, 有

$$|\frac{n}{n+1} - 1| = \frac{1}{n+1} < \varepsilon$$

$$\text{故 } \lim_{n \rightarrow +\infty} \frac{n}{n+1} = 1$$

由夹挤定理

$$\lim_{n \rightarrow +\infty} (\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}) = 1$$

6. $\forall \varepsilon > 0$. 由 $\lim_{n \rightarrow +\infty} (x_n - y_n) = 0$

$\exists N \in \mathbb{N}^*$. 当 $n > N$ 时, 有 $|x_n - y_n| < \varepsilon$

$$\text{即 } y_n - \varepsilon < x_n < y_n + \varepsilon$$

$$\therefore x_n \leq A \leq y_n$$

$$\therefore A - \varepsilon < x_n \leq A$$

故当 $n > N$ 时, $|x_n - A| < \varepsilon$

$$\text{即 } \lim_{n \rightarrow +\infty} x_n = A$$

同理可得 $\lim_{n \rightarrow +\infty} y_n = A$

习题1.4

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4.11) 当 $k \geq 2$ 时, $\frac{1}{k^k} < \frac{1}{k^2 - k}$

$$a_n < 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \dots + \frac{1}{(n-1)n}$$

$$= 1 + 1 - \frac{1}{n} < 2$$

$$a_{n+1} - a_n = \frac{1}{(n+1)(n+1)} > 0$$

即 $\{a_n\}_{n=1}^{+\infty}$ 单调不减且以 2 为上界

由单调有界定理可知, $\lim_{n \rightarrow +\infty} a_n$ 存在

5.11) $a_1 = \sqrt{2} < 2$

假设 $a_k < 2$, $a_{k+1} = \sqrt{2a_k} < 2$

$$a_{k+1} - a_k = \sqrt{2a_k} - a_k$$

$$= \sqrt{a_k}(\sqrt{2} - \sqrt{a_k}) > 0$$

即 $a_n < 2$ 且 $\{a_n\}_{n=1}^{+\infty}$ 单调不减

由单调有界定理, $\{a_n\}_{n=1}^{+\infty}$ 收敛

设 $\lim_{n \rightarrow +\infty} a_n = A$

$$a_{n+1}^2 = 2a_n$$

$$\text{取极限, } \lim_{n \rightarrow +\infty} a_{n+1}^2 = \lim_{n \rightarrow +\infty} (2a_n)$$

$$\text{即 } A^2 = 2A$$

$$A = 2$$

$$(4) 0 < a_1 < \frac{1+\sqrt{5}}{2}$$

假设 $0 < a_k < \frac{1+\sqrt{5}}{2}$

$$a_{k+1} = 2 - \frac{1}{1+a_k} > 0$$

$$< 2 - \frac{1}{1+\frac{1+\sqrt{5}}{2}}$$

$$= \frac{1+\sqrt{5}}{2}$$

故 $0 < a_n < \frac{1+\sqrt{5}}{2}$ $n=1, 2, \dots$

$$a_{n+1} - a_n = \frac{1+a_n - a_n^2}{1+a_n} > 0$$

即 a_n 单调不减且以 $\frac{1+\sqrt{5}}{2}$ 为上界

由单调有界定理, $\lim_{n \rightarrow +\infty} a_n$ 存在

设 $\lim_{n \rightarrow +\infty} a_n = A$

$$a_{n+1} = 2 - \frac{1}{1+a_n}$$

$$\text{取极限, } \lim_{n \rightarrow +\infty} a_{n+1} = \lim_{n \rightarrow +\infty} (2 - \frac{1}{1+a_n})$$

$$\text{即 } A = 2 - \frac{1}{1+A}$$

$$A = \frac{1+\sqrt{5}}{2}$$

$$12.13) \lim_{n \rightarrow +\infty} \frac{1}{n\sqrt{n}} (1 + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n})$$

$$= \lim_{n \rightarrow +\infty} (\sqrt{1} \cdot \frac{1}{n} + \sqrt{2} \cdot \frac{1}{n} + \dots + \sqrt{n} \cdot \frac{1}{n})$$

$$= \int_0^1 \sqrt{x} \cdot dx$$

$$= \frac{2}{3}$$

$$\lim_{n \rightarrow +\infty} \frac{\sqrt{n+1}}{(n+1)\sqrt{n+1} - n\sqrt{n}}$$

$$= \lim_{n \rightarrow +\infty} \frac{1}{n+1 - n\sqrt{\frac{n}{n+1}}}$$

$$= \lim_{n \rightarrow +\infty} \frac{1}{1 + \frac{n}{1+\sqrt{n+1}}}$$

$$= \frac{2}{3}$$

$$\because \lim_{n \rightarrow +\infty} n\sqrt{n} = +\infty$$

由 Stolz 定理

$$\lim_{n \rightarrow +\infty} \frac{1 + \sqrt{2} + \dots + \sqrt{n}}{n\sqrt{n}} = \lim_{n \rightarrow +\infty} \frac{(1 + \sqrt{2} + \dots + \sqrt{n+1}) - (1 + \sqrt{2} + \dots + \sqrt{n})}{(n+1)\sqrt{n+1} - n\sqrt{n}}$$

$$= \lim_{n \rightarrow +\infty} \frac{\sqrt{n+1}}{(n+1)\sqrt{n+1} - n\sqrt{n}}$$

$$= \frac{2}{3}$$

$$13. \lim_{n \rightarrow +\infty} \frac{(a_1 + 2a_2 + \dots + (n+1)a_{n+1}) - (a_1 + 2a_2 + \dots + na_n)}{(n+1)^2 - n^2}$$

$$= \lim_{n \rightarrow +\infty} \frac{(n+1)a_{n+1}}{2n+1}$$

$$= \lim_{n \rightarrow +\infty} \frac{a_{n+1}}{2 - \frac{1}{n+1}}$$

$$= \frac{a}{2}$$

$$\because \lim_{n \rightarrow +\infty} n^2 = +\infty$$

由Stolz定理

$$\lim_{n \rightarrow +\infty} \frac{a_1 + 2a_2 + \dots + na_n}{n^2} = \frac{a}{2}$$

习题 1.5

1. $\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n, m > N$ (n, m 为整数), 使得 $|x_n - x_m| \geq \varepsilon$

2. (3) $\forall \varepsilon > 0$, 设 $\{c_k\}$ 的界为 M , 取 $N = \lceil \log_{|q|} \frac{(1-|q|)\varepsilon}{M} \rceil$, 当 $m, n > N$ 时

不妨设 $m \geq n$

$$\begin{aligned} |a_m - a_n| &= |c_{n+1}q^{n+1} + c_{n+2}q^{n+2} + \dots + c_m q^m| \\ &\leq |c_{n+1}|q^{n+1} + |c_{n+2}|q^{n+2} + \dots + |c_m|q^m \\ &\leq M(|q|^{n+1} + |q|^{n+2} + \dots + |q|^m) \\ &= M \cdot \frac{|q|^{n+1}(1 - |q|^{m-n+1})}{1 - |q|} \\ &< M \cdot \frac{|q|^{n+1}}{1 - |q|} < \varepsilon \end{aligned}$$

故 $\{a_n\}$ 为柯西列

由柯西收敛定理, $\lim_{n \rightarrow \infty} a_n$ 存在

(b) $\forall \varepsilon > 0$, 取 $N = \lceil \varepsilon^{-\frac{1}{\alpha}} \rceil + 1$, 当 $m, n > N$ 时

不妨设 $m \geq n$

① $m - n$ 为偶数

$$\begin{aligned} |a_m - a_n| &= \left| \frac{1}{(n+1)^\alpha} - \frac{1}{(n+2)^\alpha} + \dots - \frac{1}{m^\alpha} \right| \\ &= \left| \frac{1}{(n+1)^\alpha} - \left(\frac{1}{(n+2)^\alpha} - \frac{1}{(n+3)^\alpha} \right) - \dots - \left(\frac{1}{(m-2)^\alpha} - \frac{1}{(m-1)^\alpha} \right) - \frac{1}{m^\alpha} \right| \\ &< \frac{1}{(n+1)^\alpha} < \varepsilon \end{aligned}$$

② $m - n$ 为奇数

$$\begin{aligned} |a_m - a_n| &= \left| \frac{1}{(n+1)^\alpha} - \frac{1}{(n+2)^\alpha} + \dots + \frac{1}{m^\alpha} \right| \\ &= \left| \frac{1}{(n+1)^\alpha} - \left(\frac{1}{(n+2)^\alpha} - \frac{1}{(n+3)^\alpha} \right) - \dots - \left(\frac{1}{(m-1)^\alpha} - \frac{1}{m^\alpha} \right) \right| \\ &< \frac{1}{(n+1)^\alpha} < \varepsilon \end{aligned}$$

故 $\{a_n\}$ 为柯西列

由柯西收敛定理, $\lim_{n \rightarrow \infty} a_n$ 存在

3. (1) 取 $\varepsilon = 1, \forall N \in \mathbb{N}$, 取 $m = 3N + 1, n = 3N + 3$ ($m, n > N$)

$$\begin{aligned} |a_m - a_n| &= \left| \cos\left(2N\pi + \frac{\pi}{3}\right) - \cos\left(2N\pi + 2\pi\right) \right| \\ &= \left| -\frac{1}{2} - 1 \right| \\ &= \frac{3}{2} > \varepsilon \end{aligned}$$

即 a_n 不是柯西列

由柯西收敛定理, a_n 不收敛, 即 $\lim_{n \rightarrow \infty} a_n$ 不存在

b. $\because \{a_n\}$ 发散

由柯西收敛原理, $\{a_n\}$ 不是柯西列

即 $\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists m, n > N$, 使得 $|x_m - x_n| \geq \varepsilon$

设 $\{a_n\}$ 以 M 为界, 将区间 $[-M, M]$ 等分为 $[\frac{2M}{\varepsilon}] + 1$ 个区间

则每个区间的“长度”小于 ε

假设其中只有 1 个区间含有 $\{a_n\}$ 的无限多项

则其他所有区间所含的项为有限个

取其中项数最大的一项, 记为 a_{N_0}

故当 $n > N_0$ 时, 所有 a_n 均在一个区间内

即 $\forall m, n > N_0, |x_m - x_n| < \varepsilon$

矛盾

故至少有 2 个区间含有 $\{a_n\}$ 的无限多项, 记为 $[A, B], [C, D]$ ($B \leq C$)

将其中 $\{a_n\}$ 的项分别构造为数列 $\{b_n\}, \{c_n\}$

由 Bolzano 定理, $\{b_n\}, \{c_n\}$ 有界, 故存在子列 $\{b_{n_k}\}, \{c_{n_k}\}$ 收敛

且 $\{b_{n_k}\}, \{c_{n_k}\}$ 均为 $\{a_n\}$ 的子列

$\therefore b_{n_k} \leq B \leq C \leq c_{n_k}$

$\therefore \lim_{k \rightarrow \infty} b_{n_k} \leq \lim_{k \rightarrow \infty} c_{n_k}$

① $\lim_{k \rightarrow \infty} b_{n_k} < \lim_{k \rightarrow \infty} c_{n_k}$ ② $\lim_{k \rightarrow \infty} b_{n_k} = \lim_{k \rightarrow \infty} c_{n_k}$

得证

则 $[\frac{A+B}{2}, \frac{C+D}{2}]$ 中有 $\{a_n\}$ 的无限多项

$[A, \frac{A+B}{2}], [\frac{C+D}{2}, D]$ 及其他区间中至少有 1 个含有 $\{a_n\}$ 的无限多项
理由同上

则可在该区间内找到另一个收敛子列, 且其极限不等于 B
得证

$$\begin{aligned} 8. |a_{n+1} - a_n| &\leq q |a_n - a_{n-1}| \\ &\leq q^2 |a_{n-1} - a_{n-2}| \\ &\vdots \\ &\leq q^{n-1} |a_2 - a_1| \end{aligned}$$

$\forall \varepsilon > 0$, 取 $N = [\log_q \frac{\varepsilon(1-q)}{|a_2 - a_1|}] + 2$, 当 $m, n > N$ 时

不妨设 $m \geq n$

$$\begin{aligned} |a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} - a_{m-2} + \cdots - a_n| \\ &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\ &\leq (q^{m-2} + q^{m-3} + \cdots + q^{n-1}) |a_2 - a_1| \\ &= \frac{q^{n-1}(1 - q^{m-n})}{1 - q} \cdot |a_2 - a_1| \\ &< \frac{q^{n-1}}{1 - q} \cdot |a_2 - a_1| < \varepsilon \end{aligned}$$

故 $\{a_n\}$ 为柯西列, 由柯西收敛定理, $\{a_n\}$ 收敛

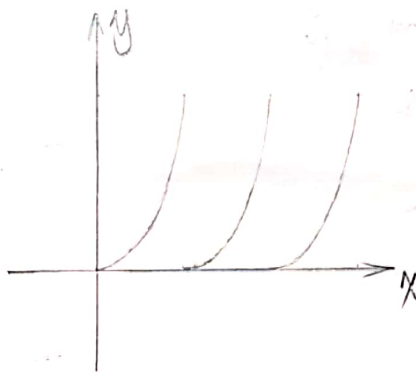
习题2.1

14. 当 $x \in [4, 6)$ 时

$$f(x) = f(x-4) = (x-4)^2$$

当 $x \in [2, 4)$ 时

$$f(x) = f(x-2) = (x-2)^2$$



18. (1) $\ln(x-1) = y-2$

$$x-1 = e^{y-2}$$

$$x = e^{y-2} + 1$$

$$\therefore y = e^{x-2} + 1$$

$$(-\infty, +\infty)$$

(2) $\cos^3 x = y-1$

$$\cos x = \sqrt[3]{y-1}$$

$$x = \arccos \sqrt[3]{y-1}$$

$$\therefore y = \arccos \sqrt[3]{x-1}$$

$$[0, 2]$$

(3) ① $x < -1$ 即 $y < -1$

$$2x^2 = 1-y$$

$$x = -\sqrt{\frac{1-y}{2}}$$

$$\therefore y = -\sqrt{\frac{1-y}{2}} \quad (x < -1)$$

② $-1 \leq x \leq 2$ 即 $-1 \leq y \leq 8$

$$x = \sqrt[3]{y}$$

$$\therefore y = \sqrt[3]{x} \quad (-1 \leq x \leq 8)$$

③ $x > 2$ 即 $y > 8$

$$12x = y+16$$

$$x = \frac{y+16}{12}$$

$$\therefore y = \frac{x+16}{12} \quad (x > 8)$$

第2章总复习题

1. $f(x) = \tan(\pi x - \frac{\pi}{2})$

$$f(x) = (-1)^{x+1} [\frac{x+1}{2}]$$

~~$$x = 0 \quad f(x) =$$~~

第1章总复习题

14. $\forall \varepsilon > 0$

由 $A = \inf_{n \in \mathbb{N}} \frac{x_n}{n}$, $\exists q \in \mathbb{N}^*$ 使得 $\frac{x_q}{q} < A + \frac{\varepsilon}{2}$ $\forall n \in \mathbb{N}^*$, $\exists p, r \in \mathbb{N}$ 且 $r < q$ 使得 $n = pq + r$

$$\frac{x_n}{n} = \frac{px_q + x_r}{pq + r} = \frac{px_q}{pq + r} + \frac{x_r}{n} < \frac{x_q}{q} + \frac{x_r}{n} < A + \frac{\varepsilon}{2} + \frac{x_r}{n}$$

令 $x_0 = \max\{x_1, x_2, \dots, x_{n-1}\} > x_r$, 取 $N_\varepsilon = [\frac{2x_0}{\varepsilon}] + 1$ 当 $n > N_\varepsilon$ 时, 有 $\frac{x_n}{n} < \frac{\varepsilon}{2}$, 即 $A \leq \frac{x_n}{n} < A + \varepsilon$ 故 $|\frac{x_n}{n} - A| < \varepsilon$ 由定义知, $\lim_{n \rightarrow \infty} \frac{x_n}{n} = A$

$\forall \varepsilon > 0$. 取 $\delta = x_0 - \frac{x_0}{e^\varepsilon}$. 当 $0 < |x - x_0| < \delta$ 时

即 $\frac{x_0}{e^\varepsilon} < x < 2x_0 - \frac{x_0}{e^\varepsilon} < x_0 \cdot e^\varepsilon$ 且 $x \neq x_0$.

取对数 $\ln x_0 - \varepsilon < \ln x < \ln x_0 + \varepsilon$

即 $|\ln x - \ln x_0| < \varepsilon$

故 $\lim_{x \rightarrow x_0} \ln x = \ln x_0$.

习题2.2

2.13) 等价

$\Rightarrow) \forall k \in \mathbb{N}^*$. 取 $\varepsilon = 2^{-k}$

由 $\lim_{x \rightarrow x_0} f(x) = A$, $\exists \delta_k > 0$, 当 $x \in U(x_0, \delta_k)$ 时, 有 $|f(x) - A| < \varepsilon = 2^{-k}$

$\therefore$ 得证

$\Leftarrow) \forall \varepsilon > 0$. 取 $k = [\log_2 \frac{1}{\varepsilon}] \in \mathbb{N}^*$

故 $\exists \delta_k > 0$, 当 $x \in U(x_0, \delta_k)$ 时, 有 $|f(x) - A| < 2^{-k} \leq \varepsilon$

由定义知, $\lim_{x \rightarrow x_0} f(x) = A$

14) 不等价

令 $f(x) = x$, $x_0 = 0$, $\lim_{x \rightarrow x_0} f(x) = 0$, 即 $A = 0$

取 $n = 1$. 当 $x = \frac{1}{2}$ 时, $0 < |x - x_0| = \frac{1}{2} < \frac{1}{n}$, 而 $|f(x) - A| = \frac{1}{2} > \frac{1}{n}$

$$4. |f(x)| = |f(x) - A + A| \leq |f(x) - A| + |A|$$

$$\text{故 } |f(x) - A| \geq |f(x)| - |A| \geq |f(x)| - |A|$$

由 $\lim_{x \rightarrow x_0} f(x) = A$, $\forall \varepsilon > 0$, $\exists \delta > 0$, 当 $x \in U(x_0, \delta)$ 时, 有 $|f(x) - A| < \varepsilon$

$$\text{故 } |f(x)| - |A| \leq |f(x) - A| < \varepsilon$$

由定义知, $\lim_{x \rightarrow x_0} |f(x)| = |A|$

6.11) 若 f 在 (a, b) 上有上界, 设其上确界为 A

故 $\forall \varepsilon > 0$, $\exists x_\varepsilon \in (a, b)$, 使得 $f(x_\varepsilon) > A - \varepsilon$

取 $\delta = b - x_\varepsilon$

由 f 在 (a, b) 上单调

当 $x \in (b - \delta, b)$ 时, $f(x) > f(x_\varepsilon) > A - \varepsilon$

由 $f(x) \leq A$, 故 $|f(x) - A| < \varepsilon$

由定义知, $\lim_{x \rightarrow b} f(x) = A$

习题2.3

$$6.14) \text{ 当 } x < 1 \text{ 时, } \frac{\sqrt{(x-1)^2}}{x-1} = \frac{1-x}{x-1} = -1$$

$$\text{故 } \lim_{x \rightarrow 1} \frac{\sqrt{(x-1)^2}}{x-1} = -1$$

$$(11) \lim_{x \rightarrow 1} \frac{x^m - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x^{m-1} + x^{m-2} + \dots + 1)}{x-1}$$

$$= \lim_{x \rightarrow 1} (1 + x + \dots + x^{m-1})$$

$$= m$$

$$(17) x[\frac{1}{x}] = x(\frac{1}{x} - \{\frac{1}{x}\})$$

$$= 1 - x\{\frac{1}{x}\}$$

$$\because 0 \leq \{\frac{1}{x}\} < 1$$

$$\therefore 1 - x < x[\frac{1}{x}] \leq 1$$

$$\therefore \lim_{x \rightarrow 0} (1 - x) = 1$$

由夹挤定理, $\lim_{x \rightarrow 0} x[\frac{1}{x}] = 1$

7.14) 令 $\theta = \arctan x$ 即 $x = \tan \theta, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\lim_{x \rightarrow 0} \arctan x = 0$ 且 $x \neq 0$ 时, $\theta \neq 0$

$$\lim_{x \rightarrow 0} \frac{\arctan x}{2x} = \frac{\theta}{2 \tan \theta} = \frac{\theta}{2 \sin \theta} \cdot \cos \theta = \frac{1}{2}$$

$$\begin{aligned} (g) \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3} &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1 - \cos x}{x^2} \cdot \cos x \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 \cdot \frac{\cos x}{2} \\ &= \frac{1}{2} \end{aligned}$$

8.15) 令 $u = x - 1$ 即 $x = u + 1$

$\lim_{x \rightarrow 1} (x - 1) = 0$ 且 $x \neq 1$ 时, $u \neq 0$

$$\begin{aligned} \lim_{x \rightarrow 1} (2x - 1)^{\frac{1}{x-1}} &= \lim_{u \rightarrow 0} (1 + 2u)^{\frac{1}{u}} \\ &= \lim_{u \rightarrow 0} (1 + 2u)^{\frac{1}{2u} \cdot 2} \\ &= e^2 \end{aligned}$$

$$\begin{aligned} (b) \lim_{x \rightarrow 0} (2 \sin x + \cos x)^{\frac{1}{x}} &= \lim_{x \rightarrow 0} (1 + 2 \sin x - 2 \sin^2 \frac{x}{2})^{\frac{1}{x}} \\ &= \lim_{x \rightarrow 0} (1 + 2 \sin x - 2 \sin^2 \frac{x}{2})^{\frac{1}{2 \sin x - 2 \sin^2 \frac{x}{2}}} \cdot \frac{2 \sin x - 2 \sin^2 \frac{x}{2}}{x} \\ &= \lim_{x \rightarrow 0} e^{\frac{2 \sin x}{x} - \frac{\sin \frac{x}{2}}{\frac{x}{2}} \cdot \sin \frac{x}{2}} \\ &= e^{2-1-0} = e^2 \end{aligned}$$

9.11) $a = -1, b = \frac{1}{2}$

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x^2 - x + 1} - ax - b) &= \lim_{x \rightarrow \infty} (\sqrt{x^2 - x + 1} + x - \frac{1}{2}) \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - x + 1} - \sqrt{x^2 - x + \frac{1}{4}}}{\frac{3}{4}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{3}{4}}{\sqrt{x^2 - x + 1} + \sqrt{x^2 - x + \frac{1}{4}}} \\ &= 0 \end{aligned}$$

(为什么直接就能知道

$a = -1, b = \frac{1}{2}$, 然后验证

极限为零呢!

难道就不存在其他的

a, b s.t. $\lim$ 为零,

(要证明或者直接计算)

习题 2.3

(2) $\Rightarrow$ (1)

由 $\lim_{x \rightarrow +\infty} f(x) = A, \forall \varepsilon > 0, \exists M > 0$, 当 $x > M$ 时, 有 $|f(x) - A| < \frac{\varepsilon}{2}$

故当 $x_2 > x_1 > M$ 时, 有 $|f(x_2) - f(x_1)| \leq |f(x_2) - A| + |f(x_1) - A| < \varepsilon$

(2) $\Rightarrow$ (1)

取 $x_n = n, \forall \varepsilon > 0, \exists M > 0$, 当 $m > n > M$ 时, 有 $|f(x_m) - f(x_n)| < \varepsilon$

故 $\{f(n)\}_{n=[M]+1}^{+\infty}$ 为 Cauchy 列

$$\text{设 } A = \lim_{n \rightarrow +\infty} f(n)$$

$\forall \varepsilon > 0, \exists N_\varepsilon > 0$, 当 $n > N_\varepsilon$ 时, 有 $|f(n) - A| < \frac{\varepsilon}{2}$

$\exists M' > 0$, 当 $x_2 > x_1 > M'$ 时, 有 $|f(x_2) - f(x_1)| < \frac{\varepsilon}{2}$

故当 $x > N_\varepsilon + M' + a$ 时, $|f(x) - A| \leq |f(x) - f(n)| + |f(n) - A| < \varepsilon$

(1) $\Rightarrow$ (3)

由 $\lim_{x \rightarrow +\infty} f(x) = A, \forall \varepsilon > 0, \exists M_1 > a$, 当 $x > M_1$ 时, 有 $|f(x) - A| < \varepsilon$

由 $x_n \rightarrow +\infty (n \rightarrow +\infty), \exists n_1 \in \mathbb{N}^*$, 当 $n > n_1$ 时, 有 $x_n > M_1$, 故 $|f(x_n) - A| < \varepsilon$

$$\therefore \lim_{n \rightarrow +\infty} f(x_n) = A$$

(3) $\Rightarrow$ (1)

反证法. 若当 $x \rightarrow +\infty$ 时, $f(x)$ 不收敛于 A

$\exists \varepsilon_0 > 0, \forall M > 0, \exists x > M$, 使得 $|f(x) - A| \geq \varepsilon_0$.

取 $M = [a] + n (n = 1, 2, \dots), \exists x_n > [a] + n$, 使得 $|f(x_n) - A| \geq \varepsilon_0$.

由 $x_n > [a] + n, \forall N > 0, \exists n_0 = [N] + 1$, 当 $n > n_0$ 时, $x_n > [a] + n > [a] + n_0 > N$

故 $x_n \rightarrow +\infty (n \rightarrow +\infty)$, 但 $f(x_n)$ 不收敛于 A

$$\therefore \lim_{x \rightarrow +\infty} f(x) = A$$

$$\therefore (2) \Leftrightarrow (1) \Leftrightarrow (3)$$

9. (1) 由 $x \rightarrow 0$, 不妨 $|x| < \frac{\pi}{2}$

$$\text{设 } u = \sin x$$

$$\lim_{x \rightarrow 0} \tan(\sin x) = \lim_{u \rightarrow 0} \tan u$$

$$= \lim_{u \rightarrow 0} u$$

$$= \lim_{x \rightarrow 0} \sin x$$

$$\text{设 } v = \tan x$$

$$\lim_{x \rightarrow 0} \sin(\tan x) = \lim_{v \rightarrow 0} \sin v$$

$$= \lim_{v \rightarrow 0} v$$

$$= \lim_{x \rightarrow 0} \tan x$$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{\sin(\tan x)}{\tan(\sin x)} \\ &= \lim_{x \rightarrow 0} \frac{\tan x}{\sin x} \\ &= \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1 \end{aligned}$$

(3) 由 $x \rightarrow 0$, 不妨 $|x| < \frac{\pi}{2}$

$$\text{设 } u = \sin x$$

$$\lim_{x \rightarrow 0} \frac{a^{\sin x} - 1}{\sin x} = \lim_{u \rightarrow 0} \frac{a^u - 1}{u} = \ln a$$

$$\therefore \lim_{x \rightarrow 0} \frac{a^{\sin x} - 1}{x} = \lim_{x \rightarrow 0} \frac{a^{\sin x} - 1}{\sin x} \cdot \frac{\sin x}{x} = \ln a$$

$$11. \lim_{x \rightarrow 0} \frac{(1+x^2)^{\frac{1}{2}} - 1}{1 - \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2}{1 - \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2(1 + \cos x)}{(1 - \cos x)(1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2(1 + \cos x)(\frac{x}{\sin x})^2}{x^2}$$

$$= \frac{2x}{x} = 1$$

$$\therefore x = \frac{2}{3}$$

$$12. \lim_{x \rightarrow +\infty} x^p (a^{\frac{1}{x}} - a^{\frac{1}{x+1}})$$

$$= \lim_{x \rightarrow +\infty} x^{p-1} \left(\frac{a^{\frac{1}{x}} - 1}{\frac{1}{x}} - \frac{a^{\frac{1}{x+1}} - 1}{\frac{1}{x+1}} \right) \cdot \frac{x}{x+1}$$

$$= \lim_{x \rightarrow +\infty} x^{p-1} (\ln a - \ln a \cdot \frac{x}{x+1})$$

$$= \lim_{x \rightarrow +\infty} \frac{x^{p-1}}{x+1} \cdot \ln a$$

$$\therefore p = 2$$

$$p \leq 2$$

第2章总复习题

$$11. (1) \lim_{x \rightarrow 1} \ln(2-x^2) = 0$$

$$\text{若 } \lim_{x \rightarrow 1} (x^2+ax+b) \neq 0, \lim_{x \rightarrow 1} \frac{\ln(2-x^2)}{x^2+ax+b} = 0$$

$$\text{故 } \lim_{x \rightarrow 1} (x^2+ax+b) = 0$$

$$\therefore x^2+ax+b \text{ 在 } 1 \text{ 处连续}$$

$$\therefore 1+a+b=0$$

$$\lim_{x \rightarrow 1} \frac{\ln(2-x^2)}{x^2+ax+b} = \lim_{x \rightarrow 1} \frac{1-x^2}{x^2-1+bx+a}$$

$$= \lim_{x \rightarrow 1} \frac{(1-x)(1+x)}{(1-x)(1+b-x)}$$

$$= \lim_{x \rightarrow 1} \frac{1+x}{1+b-x}$$

$$= \frac{2}{b-1} = -\frac{1}{2}$$

$$\therefore b = -3, a = 2$$

$$(2) \lim_{x \rightarrow +\infty} (x^3+x^2)^C - x$$

$$= \lim_{x \rightarrow +\infty} (x^{3C} (1 + \frac{1}{x})^C - x)$$

$$= \lim_{x \rightarrow +\infty} (x^{3C} (1 + \frac{C}{x} + o(\frac{1}{x})) - x)$$

$$= \lim_{x \rightarrow +\infty} (x^{3C} + Cx^{3C-1} - x + o(x^{3C-1}))$$

$$\textcircled{1} C = \frac{1}{3}$$

$$\text{原式} = \lim_{x \rightarrow +\infty} (\frac{1}{3} + o(1)) = \frac{1}{3}$$

$$\textcircled{2} C > \frac{1}{3}$$

$$\text{原式} = \lim_{x \rightarrow +\infty} (x^{3C} + o(x^{3C})) \text{ 极限不存在}$$

$$\textcircled{3} C < \frac{1}{3}$$

$$\text{原式} = \lim_{x \rightarrow +\infty} (-x + o(x)) \text{ 极限不存在}$$

$$\therefore C = \frac{1}{3} \text{ 极限值为 } \frac{1}{3}$$

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习题 2.5

$$2.12) \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} -\frac{\sin x}{x} = -1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$

故 $\lim_{x \rightarrow 0} f(x)$ 不存在, $f(x)$ 在 0 处不连续

$$15) \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} -x^{-1} \cdot \frac{x}{x-2} = \lim_{x \rightarrow 0} \frac{1}{2-x} = \frac{1}{2} = f(0)$$

故 $f(x)$ 在 $x=0$ 处连续

$$\lim_{x \rightarrow 2} e^{\frac{2}{x-2}} \text{ 不存在}$$

故 $\lim_{x \rightarrow 2} f(x)$ 不存在, $f(x)$ 在 $x=2$ 处不连续

$$3.11) \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{2x^2}{x^2(\sqrt{1+x^2} - \sqrt{1-x^2})} = 1$$

$$\lim_{x \rightarrow 0^+} f(x) = 3+a$$

$$f(0) = 3+a$$

$$3+a = 3+a = 1 \quad \therefore a = -2$$

$$4.12) \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (\sin x - 2) = -2$$

$$\lim_{x \rightarrow 0^+} f(x) = b$$

$$f(0) = b$$

$$b = b = -2 \quad \therefore b = -2$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (2^x + x) = 3$$

$$\lim_{x \rightarrow 1^-} f(x) = a+b$$

$$f(1) = a+b$$

$$a+b = a+b = 3 \quad \therefore a = 1$$

习题 2.6

$$2. \text{ 设 } f(x) = x^{2m} + a_1 x^{2m-1} + \dots + a_{2m}$$

$$\text{设 } M = \max\{|a_1|, |a_2|, \dots, |a_{2m}|\}$$

$$f(x) = x^{2m} \left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_{2m}}{x^{2m}} \right)$$

$$\left| \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_{2m}}{x^{2m}} \right|$$

$$\leq \left| \frac{a_1}{x} \right| + \left| \frac{a_2}{x^2} \right| + \dots + \left| \frac{a_{2m}}{x^{2m}} \right|$$

$$\leq M \left(\frac{1}{|x|} + \frac{1}{|x|^2} + \dots + \frac{1}{|x|^{2m}} \right)$$

$$= M \cdot \frac{1 - \frac{1}{|x|^{2m+1}}}{|x| - 1}$$

$$< \frac{M}{|x| - 1}$$

$$\text{当 } |x| \geq M+1 \text{ 时, } \left| \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_{2m}}{x^{2m}} \right| < 1, f(x) > 0$$

$$f(M+1) > 0$$

$$f(0) = a_{2m} < 0$$

$$f(-M-1) > 0$$

$$\therefore f(x) \in C[-M-1, 0], f(x) \in C[0, M+1]$$

$$\text{由零值定理, } \exists x_1 \in (-M-1, 0), x_2 \in (0, M+1)$$

$$\text{使 } f(x_1) = 0, f(x_2) = 0$$

$$3. \because f \in C[x_1, x_2]$$

$\frac{f(x_1) + f(x_2)}{2}$ 介于 $f(x_1)$ 与 $f(x_2)$ 之间

由介值定理, $\exists \eta_1 \in [a, b], f(\eta_1) = \frac{f(x_1) + f(x_2)}{2}$

$$\therefore f \in C[\eta_1, x_3]$$

$\frac{2f(\eta_1) + f(x_3)}{3}$ 介于 $f(\eta_1)$ 与 $f(x_3)$ 之间

由介值定理, $\exists \eta_2 \in [a, b], f(\eta_2) = \frac{2f(\eta_1) + f(x_3)}{3}$

同理可得 $\eta_i \in [a, b] (i=1, 2, \dots, n-1), f(\eta_i) = \frac{i f(\eta_{i-1}) + f(x_{i+1})}{i+1}$

$$\therefore \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}$$

$$= \frac{2f(\eta_1) + f(x_3) + \dots + f(x_n)}{n}$$

$$= \frac{(n-1)f(\eta_{n-2}) + f(x_n)}{n}$$

$$= f(\eta_{n-1})$$

$$\therefore \exists \xi = \eta_{n-1} \in [a, b], \text{使得 } f(\xi) = \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}$$

$$8. \text{ 设 } A = \lim_{x \rightarrow \infty} f(x)$$

取 $\varepsilon = 1, \exists b > a$, 当 $x > b$ 时, $|f(x) - A| < 1$

$$\text{即 } A - 1 < f(x) < A + 1$$

故 $x > b$ 时, $f(x)$ 有界, 设为 M

$$\therefore f \in C[a, b]$$

由定理知, $f(x)$ 在 $[a, b]$ 有界, 设为 N

故 $\forall x \in [a, +\infty), |f(x)| < M + N$, 即 $f(x)$ 有界

$$10. \text{ 取 } N = 1. \text{ 由 } \lim_{x \rightarrow \infty} f(x) = +\infty, \exists M > 0, \text{ 当 } |x| > M \text{ 时, } f(x) > 1$$

$$\therefore f \in C[-M, M]$$

由最值定理, f 在 $[-M, M]$ 有最小值, 设为 A

$$\text{故 } f(x) \geq \min\{A, 1\}$$

$$12. \text{ 若 } x_{k+1} - x_k > 0, x_{k+2} - x_{k+1} = f(x_{k+1}) - f(x_k) > 0$$

反之亦然, 故 $\{x_n\}$ 单调

由单调有界定理, $\lim_{n \rightarrow \infty} x_n$ 存在

$$\text{因为 } \xi \in [a, b], f \text{ 在 } \xi \text{ 处连续, } f(\xi) = \lim_{x \rightarrow \xi} f(x) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = \xi$$

只说明了有界
未必有最小值!

习题5.1

8. $L < 0$ 时, 矛盾 $L = 0$ 时, $f(x)$ 为常值函数, 得证下设 $L > 0$ $\forall \varepsilon > 0, \exists \delta = \frac{\varepsilon}{L}$, 当 $x, y \in I, |x - y| < \delta$ 时, 有 $|f(x) - f(y)| \leq L|x - y| < L\delta = \varepsilon$ 故 $f(x)$ 在 I 上一致连续9. (2) $\exists \varepsilon > 0$, 取 $\delta = 2e^\varepsilon - 2 > 0$ 当 $x_1, x_2 \in (2, +\infty), |x_1 - x_2| < \delta$ 时, 不妨设 $x_1 > x_2$

$$\begin{aligned}
 |f(x_1) - f(x_2)| &= \ln x_1 - \ln x_2 \\
 &= \ln \left(\frac{x_1 - x_2}{x_2} + 1 \right) \\
 &< \ln \left(\frac{2e^\varepsilon - 2}{2} + 1 \right) = \varepsilon
 \end{aligned}$$

故 $f(x)$ 在 $(2, +\infty)$ 上一致连续13. $\forall \varepsilon > 0, \exists \delta > 0$, 当 $x_1, x_2 \in (a, b), |x_1 - x_2| < \delta$ 时, 有 $|f(x_1) - f(x_2)| < \varepsilon$ 若 $b - \delta < x_1, x_2 < b, |x_1 - x_2| < \delta$, 故 $|f(x_1) - f(x_2)| < \varepsilon$ 由 Cauchy 定理, $\lim_{x \rightarrow b} f(x)$ 存在, 设为 B 同理 $\lim_{x \rightarrow a} f(x)$ 存在, 设为 A 取 $\varepsilon = 1, \exists \delta_1 > 0$, 当 $b - \delta_1 < x < b$ 时, $|f(x) - B| < 1$, 即 $B - 1 < f(x) < B + 1$ $\exists \delta_2 > 0$, 当 $a < x < a + \delta_2$ 时, $|f(x) - A| < 1$, 即 $A - 1 < f(x) < A + 1$ $f \in C[a + \delta_2, b - \delta_1]$, 故 $f(x)$ 在 $[a + \delta_2, b - \delta_1]$ 上有界, 设为 M , 即 $-M < f(x) < M$ 故 $f(x)$ 以 $|A| + |B| + |M| + 1$ 为界15. (2) 取 $\varepsilon_0 = \frac{1}{2}, \forall \delta > 0$, 取 $x_1 = \frac{\delta}{2}, x_2 = \delta$ $x_1, x_2 \in (0, +\infty), |x_1 - x_2| < \delta$

$$|f(x_1) - f(x_2)| = \left| \ln \frac{\delta}{2} - \ln \delta \right| = \ln 2 > \varepsilon_0$$

故 $f(x)$ 在 $(0, +\infty)$ 上不一致连续16. $\Rightarrow) \forall \varepsilon > 0, \exists \delta > 0$, 当 $x_1, x_2 \in (a, b), |x_1 - x_2| < \delta$ 时, 有 $|f(x_1) - f(x_2)| < \varepsilon$ 当 $a < x_1, x_2 < a + \delta$ 时, $|x_1 - x_2| < \delta$, 故 $|f(x_1) - f(x_2)| < \varepsilon$ 由 Cauchy 定理, $\lim_{x \rightarrow a} f(x)$ 存在同理 $\lim_{x \rightarrow b} f(x)$ 存在

$\Leftarrow$) 由 $\lim_{x \rightarrow a} f(x)$ 存在, $\forall \varepsilon > 0, \exists \delta_1 > 0$, 当 $a < x_1, x_2 < a + \delta_1$ 时, $|f(x_1) - f(x_2)| < \varepsilon$

同理, $\exists \delta_2 > 0$, 当 $b - \delta_2 < x_1, x_2 < b$ 时, $|f(x_1) - f(x_2)| < \varepsilon$

$f \in C[a + \delta_1, b - \delta_2]$, 由 Cantor 定理, $f(x)$ 在 $[a + \delta_1, b - \delta_2]$ 上一致连续

即 $\exists \delta_3 > 0$, 当 $x_1, x_2 \in [a + \delta_1, b - \delta_2], |x_1 - x_2| < \delta_3$ 时, $|f(x_1) - f(x_2)| < \varepsilon$

由 $f \in C(a, b)$, $\lim_{x \rightarrow a+0} f(x)$ 存在, $\exists \delta_4 > 0$, 当 $|x_1 - (a + \delta_1)| < \delta_4, |x_2 - (a + \delta_1)| < \delta_4, x_1, x_2 \in (a, b)$ 时有 $|f(x_1) - f(x_2)| < \varepsilon$

同理, $\exists \delta_5 > 0$, 当 $|x_1 - (b - \delta_2)| < \delta_5, |x_2 - (b - \delta_2)| < \delta_5, x_1, x_2 \in (a, b)$ 时, 有 $|f(x_1) - f(x_2)| < \varepsilon$

取 $\delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$

当 $x_1, x_2 \in (a, b), |x_1 - x_2| < \delta$ 时, $|f(x_1) - f(x_2)| < \varepsilon$

即 $f(x)$ 在 (a, b) 内一致连续

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1 \quad \lim_{x \rightarrow 1^-} \frac{\sin x}{x} = \sin 1 \quad y \in C(0, 1)$$

故 $y = \frac{\sin x}{x}$ 在 $(0, 1)$ 内一致连续

习题 3.1

$$1. (1) f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{1}{x x_0} = -\frac{1}{x_1}$$

$$(2) f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \rightarrow 0} \frac{1}{2^{x_0 + h}} \cdot \frac{1 - 2^h}{h} = -\ln 2$$

$$2. (4) \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (3x^2 + 4x) = 0$$

$$f(0) = -1$$

$$\lim_{x \rightarrow 0} f(x) \neq f(0)$$

故 $f(x)$ 在 $x=0$ 处不连续, 不可导

$$3. \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (ax + b) = a + b \quad f'(1) = a$$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x^2 + 1) = 2 \quad f'(1) = 2$$

$$f(1) = 2$$

$$\therefore a = 2$$

$$\therefore a + b = 2 = 2$$

$$\text{故 } a = 2, b = 0$$

$$5. (2) \text{ 设 } h = \frac{2}{n} \neq 0$$

$$\text{原式} = \lim_{n \rightarrow \infty} 2 \cdot \frac{f(x_0 + \frac{2}{n}) - f(x_0)}{\frac{2}{n}}$$

$$\lim_{n \rightarrow \infty} \frac{2}{n} = 0$$

$$= 2f'(x_0)$$

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$$

$$\therefore \lim_{n \rightarrow \infty} \frac{f(x_0 + \frac{2}{n}) - f(x_0)}{\frac{2}{n}} = f'(x_0)$$

$$(4) \lim_{h \rightarrow 0} (f(x_0+h))^{\frac{1}{h}}$$

$$= \lim_{h \rightarrow 0} (1 + f(x_0+h) - f(x_0))^{\frac{1}{f(x_0+h) - f(x_0)}} \cdot \frac{f(x_0+h) - f(x_0)}{h}$$

$$= e^{f'(x_0)}$$

12. 不妨设 $f'(a) = A > 0$, $f'(b) = B > 0$

取 $\varepsilon_1 = A$, $\exists \delta_1 > 0$, 当 $a < x < a + \delta_1$ 时, $|\frac{f(x) - f(a)}{x - a} - A| > A$, 即 $f(x) > f(a) = 0$

故 $f(a + \frac{\delta_1}{2}) > 0$. 同理 $\exists \delta_2 > 0$, 使得 $f(b - \frac{\delta_2}{2}) < 0$

$f \in C[a + \frac{\delta_1}{2}, b - \frac{\delta_2}{2}]$, 由零值定理, $f(x)$ 在 $[a + \frac{\delta_1}{2}, b - \frac{\delta_2}{2}]$ 存在零点

习题3.2

$$2. (1) y' = (x^3)' + 2(\sqrt[3]{x})' - (\frac{1}{x^3})' + (12)'$$

$$= 3x^2 + \frac{2}{3\sqrt[3]{x^3}} + \frac{2}{x^4}$$

$$(3) y' = (-x^6 + x^5 - x^4 + 4x^3 - 3x^2 + 3x - 3)'$$

$$= -6x^5 + 5x^4 - 4x^3 + 12x^2 - 6x + 3$$

$$(5) y' = \frac{(\tan x)' \cdot x - x' \cdot \tan x}{x^2}$$

$$= \frac{\frac{1}{\cos^2 x} \cdot x - 1 \cdot \tan x}{x^2}$$

$$= \frac{x - \sin x}{x^2 \cdot \cos^2 x}$$

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$$4. (1) y' = 2 \cos 3x \cdot 3$$

$$= 6 \cos 3x$$

$$(3) y' = \frac{3}{2} \sqrt{1-x^3} \cdot (-3x^2)$$

$$= -\frac{9}{2} x^2 \sqrt{1-x^3}$$

$$5. (1) (f(1-x))' = f'(1-x) \cdot (1-x)'$$

$$= -f'(1-x)$$

$$(2) (f(\sin^2 x) f(\cos^2 x))' = (f(\sin^2 x))' f(\cos^2 x) + f(\sin^2 x) (f(\cos^2 x))'$$

$$= f'(\sin^2 x) \cdot 2 \sin x \cdot \cos x \cdot f(\cos^2 x) + f(\sin^2 x) \cdot f'(\cos^2 x) \cdot 2 \cos x \cdot (-\sin x)$$

$$= 2 \sin x \cos x (f'(\sin^2 x) f(\cos^2 x) - f(\sin^2 x) f'(\cos^2 x))$$

$$6. (3) y' = (e^{\frac{1}{x} \ln x} + e^{\ln x \ln(1-x)})'$$

$$= e^{\frac{1}{x} \ln x} \cdot (\frac{1}{x} \ln x)' + e^{\ln x \ln(1-x)} \cdot (\ln x \ln(1-x))'$$

$$= \sqrt{x} \cdot (-\frac{1}{x^2} \ln x + \frac{1}{x} \cdot \frac{1}{x}) + x^{\ln(1-x)} \cdot (\frac{1}{x} \ln(1-x) + \ln x \cdot \frac{1}{1-x} \cdot (-1))$$

$$= \frac{1 - \ln x}{x^2} \sqrt{x} + \frac{x \ln x + (1-x) \ln(1-x)}{x(1-x)} \cdot x^{\ln(1-x)}$$

$$(5) y' = (e^{\sin x \ln \ln x})'$$

$$= e^{\sin x \ln \ln x} \cdot (\sin x \cdot \ln \ln x + \sin x \cdot (\ln \ln x)')$$

$$= (\ln x)^{\sin x} \cdot (\cos x \cdot \ln \ln x + \sin x \cdot \frac{1}{\ln x} \cdot \frac{1}{x})$$

$$7. (1) y' = 1 + \frac{1}{x} = \frac{x+1}{x}$$

$$x'(y) = \frac{1}{y'(x)} = \frac{x}{x+1}$$

$$(2) y' = 1 + e^x$$

$$x'(y) = \frac{1}{y'(x)} = \frac{1}{e^x + 1}$$

$$10. (2) y'(x) = \frac{y'(t)}{x'(t)}$$

$$= \frac{2+2t}{e^t + te^t}$$

$$= \frac{2}{e^t}$$

$$11. (1) d(|x|) = \frac{1}{x} \cdot d(-x) \\ = \frac{1}{x} \cdot dx$$

$$(2) d\left(\frac{x}{\sqrt{1-x^3}}\right) = \left(\frac{x}{\sqrt{1-x^3}}\right)' \cdot dx \\ = \frac{1 \cdot \sqrt{1-x^3} - x \cdot \frac{1}{2} \cdot (-3x^2)}{1-x^3} \cdot dx \\ = \frac{\sqrt{1-x^3} - x \cdot \frac{3x^2}{2\sqrt{1-x^3}}}{1-x^3} \cdot dx \\ = \frac{2+x^3}{2(1-x^3)\sqrt{1-x^3}} \cdot dx$$

$$(3) d\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right) = \frac{1}{2\sqrt{\frac{1-\sin x}{1+\sin x}}} \cdot d\left(\frac{1-\sin x}{1+\sin x}\right) \\ = \frac{1}{2} \sqrt{\frac{1+\sin x}{1-\sin x}} \cdot \frac{-\cos x \cdot (1+\sin x) - \cos x \cdot (1-\sin x)}{(1+\sin x)^2} \cdot dx \\ = \frac{-2\cos x}{2\sqrt{(1-\sin x)(1+\sin x)} \cdot (1+\sin x)} \cdot dx \\ = -\frac{1}{1+\sin x} \cdot dx$$

$$(4) d\left(\arctan \frac{u(x)}{v(x)}\right) = \frac{1}{1+\left(\frac{u(x)}{v(x)}\right)^2} \cdot d\left(\frac{u(x)}{v(x)}\right) \\ = \frac{v^2(x)}{u^2(x)+v^2(x)} \cdot \frac{u'(x) \cdot v(x) - v'(x) \cdot u(x)}{v^2(x)} \cdot dx \\ = \frac{u'(x) \cdot v(x) - v'(x) \cdot u(x)}{u^2(x)+v^2(x)} \cdot dx$$

$$12. \text{ 当 } |x| \ll a \text{ 时 } \frac{x}{na^n} \rightarrow 0 \quad ! 1) \sqrt[3]{27} = \sqrt[3]{27+2}$$

$$\text{故 } \left(1 \pm \frac{x}{na^n}\right)^n \approx 1 \pm \frac{x}{a^n} \quad \approx 3 + \frac{2}{3 \cdot 3^2}$$

$$\Rightarrow 1 + \frac{x}{na^n} \approx \sqrt[n]{1 \pm \frac{x}{a^n}} \quad = \frac{83}{27}$$

$$\Rightarrow a + \frac{x}{na^{n-1}} \approx \sqrt[n]{a^n + x}$$

习题3.3

1.11) $y' = e^x \cdot 2x$

$$\begin{aligned} y'' &= (e^x)' \cdot 2x + e^x \cdot (2x)' \\ &= e^x \cdot 2x + e^x \cdot 2 \\ &= (4x^2 + 2) \cdot e^x \end{aligned}$$

3.16) 设 $f(x) = \frac{1}{1-x}$, $g(x) = \frac{1}{2-x}$

下面证 $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$

$n=1$ 时, $f'(x) = -\frac{1}{(1-x)^2} \cdot (-1) = \frac{1}{(1-x)^2}$

假设 $n=k$ 时, $f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$

$n=k+1$ 时, $f^{(k+1)}(x) = -(k+1) \cdot \frac{k!}{(1-x)^{k+2}} \cdot (-1)$
 $= \frac{(k+1)!}{(1-x)^{k+2}}$

同理, $g^{(n)}(x) = \frac{n!}{(2-x)^{n+1}}$

$y = f(x) \cdot g(x)$

$y^{(20)} = (f \cdot g)^{(20)}(x)$

$= \sum_{k=0}^{20} C_{20}^k f^{(k)}(x) g^{(20-k)}(x)$

$= \sum_{k=0}^{20} \frac{20!}{k!(20-k)!} \cdot \frac{k!}{(1-x)^{k+1}} \cdot \frac{(20-k)!}{(2-x)^{20-k+1}}$

$= 20! \sum_{k=0}^{20} \frac{1}{(1-x)^{k+1} (2-x)^{20-k+1}}$

19) 设 $f(x) = x^3$, $g(x) = e^x$

$f^{(n)}(x) = 3x^2$ $g^{(n)}(x) = e^x$

$f^{(1)}(x) = 3x$

$f^{(2)}(x) = 3$

$n \geq 4$ 时, $f^{(n)}(x) = 0$

$y = f(x) \cdot g(x)$

$y^{(n)} = (f \cdot g)^{(n)}(x)$

$= \sum_{k=0}^n C_n^k f^{(k)}(x) g^{(n-k)}(x)$

$= e^x \cdot \sum_{k=0}^n C_n^k f^{(k)}(x)$

$y^{(4)} = (x^3 + 3x^2)e^x$ $y^{(3)} = (x^3 + 6x^2 + 6x)e^x$

$n \geq 3$ 时, $y^{(n)} = e^x (C_n^0 f^{(0)}(x) + C_n^1 f^{(1)}(x) + C_n^2 f^{(2)}(x) + C_n^3 f^{(3)}(x))$
 $= (x^3 + 3nx^2 + 3n(n-1)x + 2n(n-1)(n-2))e^x$

综上, $y^{(n)} = (x^3 + 3nx^2 + 3n(n-1)x + 2n(n-1)(n-2))e^x$

4.11) $y'(x) = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \sin t}{a - a \cos t}$

$y''(x) = \frac{d \frac{dy}{dx}}{dx} = \frac{d \frac{dy}{dt} / \frac{dx}{dt}}{\frac{dx}{dt}} = \frac{\frac{\cos t - 1}{a - a \cos t}}{-1/a} = -\frac{1}{a}$

$y'''(x) = \frac{d \frac{dy''}{dx}}{dx} = 0$

第3章总复习题

1. (1) ① $k > 0$

$\forall \varepsilon > 0, \exists \delta = \varepsilon^{\frac{1}{k}} > 0$. 当 $|x| < \delta$ 时, $|x|^k < \varepsilon$

故 $\lim_{x \rightarrow 0} |x|^k = 0$

$0 \leq \lim_{x \rightarrow 0} f(x) \leq \lim_{x \rightarrow 0} |x|^k = 0$

由夹挤定理, $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$

故 $f(x)$ 在 $x=0$ 处连续

(2) ① $k > 1$

$\forall \varepsilon > 0, \exists \delta = \varepsilon^{\frac{1}{k-1}} > 0$. 当 $|x| < \delta$ 时, $|x|^{k-1} < \varepsilon$

故 $\lim_{x \rightarrow 0} |x|^{k-1} = 0$

$|f(0)| = \left| \lim_{x \rightarrow 0} \frac{|x|^k \sin \frac{1}{x}}{x} \right| \leq \lim_{x \rightarrow 0} |x|^{k-1} = 0$

由夹挤定理, $f(0) = 0$

② $k \leq 0$

$\exists \varepsilon = 1, \forall \delta > 0, \exists x = \frac{1}{\left[\frac{1}{\delta\pi}\right]\pi + \frac{3}{2}\pi}, |x| < \delta$

$\therefore |x| < 1, k \leq 0$

$\therefore |x|^k \sin \frac{1}{x} = |x|^k \geq \varepsilon$

故 $f(x)$ 在 $x=0$ 处不收敛于 0

即 $f(x)$ 在 $x=0$ 处不连续

② $k \leq 1$

$\exists \varepsilon = 1, \forall \delta > 0, \exists x_m = \frac{1}{\left[\frac{1}{\delta\pi}\right]\pi + \frac{3}{2}\pi}$
 $|x_m|, |x_n| < \delta$

由 $|x_m| < 1, k-1 \leq 0$ $x_n = \frac{1}{\left[\frac{1}{\delta\pi}\right]\pi + \pi}$

$\left| \frac{|x_m|^k \sin \frac{1}{x_m}}{x_m} - \frac{|x_n|^k \sin \frac{1}{x_n}}{x_n} \right| = |x_m|^{k-1} \geq \varepsilon$

故当 $x \rightarrow 0$ 时, $\frac{|x|^k \sin \frac{1}{x}}{x}$ 不收敛

即 $f(x)$ 在 $x=0$ 处不可导

(3) 当 $x > 0$ 时, $f'(x) = kx^{k-1} \sin \frac{1}{x} + x^k \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2}\right)$

$= (kx \sin \frac{1}{x} - \cos \frac{1}{x}) \cdot x^{k-2}$

同理可知, 当且仅当 $k > 2$ 时, $\lim_{x \rightarrow 0} f'(x) = 0 = f'(0)$

因为 $f(x)$ 为奇函数, $f'(-x) = f'(x)$

故 $\lim_{x \rightarrow 0} f'(x) = 0$, 即导函数连续

4. $f'(a) = \lim_{n \rightarrow \infty} \frac{f(a+\frac{1}{n}) - f(a)}{\frac{1}{n}}$

$\lim_{n \rightarrow \infty} \frac{f(a+\frac{1}{n}) - f(a)}{f(a)} = \lim_{n \rightarrow \infty} \frac{1}{nf(a)} \cdot \lim_{n \rightarrow \infty} \frac{f(a+\frac{1}{n}) - f(a)}{\frac{1}{n}} = 0 \cdot f'(a) = 0$

$\lim_{n \rightarrow \infty} \left(\frac{f(a+\frac{1}{n})}{f(a)} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{f(a+\frac{1}{n}) - f(a)}{f(a)} \right)^n$

$= e^{\lim_{n \rightarrow \infty} \frac{f(a+\frac{1}{n}) - f(a)}{f(a)} \cdot n}$

$= e^{\frac{f'(a)}{f(a)}}$

$$5. y'(0) = \cos 0 = 1$$

$$f(0) = 0 \quad f'(0) = 1$$

$$\lim_{n \rightarrow \infty} \frac{f(\frac{2}{n}) - f(0)}{\frac{2}{n} - 0} = f'(0) = 1$$

$$\text{故 } \lim_{n \rightarrow \infty} n f(\frac{2}{n}) = 2$$

$$\text{设 } A = \lim_{n \rightarrow \infty} \sqrt{n} \cdot \sqrt{f(\frac{2}{n})} > 0$$

$$A^2 = \lim_{n \rightarrow \infty} n f(\frac{2}{n}) = 2$$

$$\therefore A = \sqrt{2}$$

习题4.4

6. (1) $y \in C[-1, 2]$ 且 y 在 $[-1, 2]$ 上可导 (2) $y \in C[-10, 10]$

$$y'(x) = 5x^4 - 20x^3 + 15x^2$$

$$= 5x^2(x-3)(x-1)$$

$$y(0) = 1 \quad y(1) = 2$$

$$y(-1) = -10 \quad y(2) = -7$$

最大值为2, 最小值为-10

$$y = \begin{cases} x^2 - 3x + 2, & x \in [-10, 1] \cup [2, 10] \\ -x^2 + 3x - 2, & x \in (1, 2) \end{cases}$$

$$y' = \begin{cases} 2x - 3, & x \in [-10, 1] \cup [2, 10] \\ -2x + 3, & x \in (1, 2) \end{cases}$$

y 在 $x=1, x=2$ 处不可导

$$y(1) = y(2) = 0 \quad y(\frac{3}{2}) = \frac{1}{4}$$

$$y(10) = 72 \quad y(-10) = 132$$

最大值为132, 最小值为0

11. 设漏斗高为 h , 口的半径为 r

$$h^2 + r^2 = R^2$$

$$V \in (0, +\infty)$$

$$V = \frac{1}{3} \pi r^2 h$$

$$V'(h) = -\pi h^2 + \frac{1}{3} \pi R^2$$

$$= \frac{1}{3} \pi (R^2 - h^2) \cdot h$$

$$V'(\frac{2}{3}R) = 0$$

$$= -\frac{1}{3} \pi h^3 + \frac{1}{3} \pi R^3 h$$

V 在 $(0, \frac{2}{3}R)$ 单增, 在 $(\frac{2}{3}R, +\infty)$ 单减

故 $h = \frac{2}{3}R$ 时, 容积最大

$$\theta = \frac{2\pi r}{R} = \frac{2\sqrt{3}}{3} \pi$$

习题4.1

1. $f \in C[a, b]$, f 在 $[a, b]$ 可导

由Rolle定理, $\exists \xi_1 \in (a, c), \xi_2 \in (c, b)$, 使 $f'(\xi_1) = 0, f'(\xi_2) = 0$

f' 在 (a, b) 内可导, 故 $f' \in C[\xi_1, \xi_2]$ 且 f' 在 $[\xi_1, \xi_2]$ 可导

由Rolle定理, $\exists \xi \in (\xi_1, \xi_2) \subset (a, b)$, $f''(\xi) = 0$

b. $f(x)$ 不是常值函数

故 $\exists x_0 \in (a, b)$, 使 $f(x_0) \neq f(a)$

不妨 $f(x_0) > f(a)$

$$\text{令 } g(x) = f(x) - \frac{f(x_0) - f(a)}{x_0 - a} (x - a)$$

$$g(a) = f(a) = g(x_0)$$

11. (2) 设 $A = \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow +\infty} f(x)$

若 $\forall x \in (a, +\infty)$, $f(x) = A$, 即 $f(x)$ 为常值函数, 多可取 $(a, +\infty)$ 上任意点

若 $\exists x_0 \in (a, +\infty)$, $f(x_0) \neq A$, 不妨 $f(x_0) > A$

取 $\varepsilon = f(x_0) - A$. 由 $\lim_{x \rightarrow a} f(x) = A$, $\exists \delta > 0$, 当 $x \in (a, a + \delta)$ 时, $|f(x) - A| < \varepsilon$

由 $\lim_{x \rightarrow +\infty} f(x) = A$, $\exists M > 0$, 当 $x > M$ 时, $|f(x) - A| < \varepsilon$

$$\text{取 } x_1 = a + \frac{\delta}{2}, x_2 = M + |a + \frac{\delta}{2}|$$

$$f(x_1) < f(x_0), f(x_2) < f(x_0)$$

由 $f \in C[x_1, x_2]$, f 在 $[x_1, x_2]$ 上有最大值

故 f 在 (x_1, x_2) 上的某点多取到最大值, 由 $f(x)$ 在 (x_1, x_2) 可导, $f'(\xi) = 0$

同理, 将 $(a, +\infty)$ 改为 $(-\infty, a)$ 或 $(-\infty, +\infty)$ 后, 相应的结论也成立

14. (2) 设 $g(x) = \frac{f(x)}{e^x}$

$$g'(x) = \frac{f'(x) - f(x)}{e^x}$$

$$\begin{aligned} g'(a)g'(b) &= \frac{f'(a) - f(a)}{e^a} \cdot \frac{f'(b) - f(b)}{e^b} \\ &= \frac{f'(a)f'(b) - f'(a)f(b) - f(a)f'(b) + f(a)f(b)}{e^a e^b} > 0 \end{aligned}$$

不妨 $g'(a) > 0, g'(b) > 0$

$$g'(a) = \lim_{x \rightarrow a^+} \frac{g(x) - g(a)}{x - a} > 0 \quad \text{故当 } x \rightarrow a^+ \text{ 时, } g(x) > g(a) = 0$$

$$g'(b) = \lim_{x \rightarrow b^-} \frac{g(x) - g(b)}{x - b} > 0 \quad \text{当 } x \rightarrow b^- \text{ 时, } g(x) < g(b) = 0$$

由零值定理, $\exists x_0 \in (a, b)$, $g(x_0) = 0 = g(a) = g(b)$

由Roll定理, $\exists \xi_1 \in (a, x_0)$, $g'(\xi_1) = 0$

$$\exists \xi_2 \in (x_0, b), g'(\xi_2) = 0$$

$$g'(\xi_1) = g'(\xi_2), \text{ 由Roll定理, } \exists \theta \in (\xi_1, \xi_2), g''(\theta) = \frac{f''(\theta) - 2f'(\theta) + f(\theta)}{e^\theta} = 0$$

$$\text{故 } \exists \theta \in (a, b), f''(\theta) - 2f'(\theta) + f(\theta) = 0$$

15.11) 由Lagrange定理, $\exists \xi$ 在 x 与 x_0 之间, 使 $f'(\xi) = \frac{f(x) - f(x_0)}{x - x_0}$

$$\text{故 } f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$= \lim_{x \rightarrow x_0} f'(\xi)$$

$$= \lim_{\xi \rightarrow x_0} f'(\xi) = A$$

12) 反证法. 若不然, $f(x)$ 在 I 内存在第一类间断点 x_0 .

$$\text{即 } \lim_{x \rightarrow x_0^-} f(x) \neq \lim_{x \rightarrow x_0^+} f(x)$$

$$\text{由(1), 同理可知 } f_+'(x_0) = \lim_{x \rightarrow x_0^+} f'(x), f_-'(x_0) = \lim_{x \rightarrow x_0^-} f'(x)$$

故 $f_+'(x_0) \neq f_-'(x_0)$. 与 $f(x)$ 可导矛盾

光线折射

$$T(x) = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(l-x)^2 + h_2^2}}{v_2}$$

$T(x) \in C[0, l]$, $T(x)$ 在 $[0, l]$ 可导

下面证 $[0, l]$ 中至多有一个点满足 $T'(x) = 0$

若不然, 设 $x_1, x_2 \in [0, l]$, $T'(x_1) = T'(x_2) = 0$

由Rolle定理, $\exists \xi \in (x_1, x_2)$, $T''(\xi) = 0$

$$T''(\xi) = \frac{h_1^2}{v_1(\xi^2 + h_1^2)^{\frac{3}{2}}} + \frac{h_2^2}{v_2((l-\xi)^2 + h_2^2)^{\frac{3}{2}}} > 0$$

与 $T''(\xi) = 0$ 矛盾

$$T'(x) = \frac{x}{v_1 \sqrt{x^2 + h_1^2}} + \frac{x-l}{v_2 \sqrt{(l-x)^2 + h_2^2}}$$

$$T'(0) = \frac{-l}{v_2 \sqrt{l^2 + h_2^2}} < 0, \quad T'(l) = \frac{l}{v_1 \sqrt{l^2 + h_1^2}} > 0$$

故 $T(x)$ 不在 $x=0$ 或 l 时取最小值

由于 $T(x) \in C[0, l]$, 由零值定理, $\exists x_0 \in (0, l)$, $T'(x_0) = 0$

故 $T(x)$ 在 $[0, l]$ 上有唯一最小值点

习题4.4

4.12) $y = e^{x \ln(1+\frac{1}{x})}$

设 $g(x) = x \ln(1+\frac{1}{x})$

$g'(x) = \ln(1+\frac{1}{x}) - \frac{1}{x+1} > 0$

故 $g(x)$ 在 $(0, +\infty)$ 单增由于 e^x 单增, y 在 $(0, +\infty)$ 单增

无极值

14) $y' = 6x^2 - 6x - 12$

$= 6(x-2)(x+1)$

故 y 在 $(-\infty, -1)$ 和 $(2, +\infty)$ 单增在 $(-1, 2)$ 单减 $x = -1$ 时取极大值 8 $x = 2$ 时取极小值 -19

5.16) 设 $f(x) = x - \sin x$

$f'(x) = 1 - \cos x \geq 0$

故 $f(x)$ 在 $\mathbb{R}$ 上单增

$x > 0$ 时, $f(x) > f(0) = 0$

即 $x > \sin x$

设 $g(x) = \sin x - x + \frac{x^3}{6}$

$g'(x) = \cos x - 1 + \frac{x^2}{2}$

$= -2\sin^2 \frac{x}{2} + \frac{x^2}{2}$

$= 2(\frac{x}{2} - \sin \frac{x}{2})(\frac{x}{2} + \sin \frac{x}{2})$

当 $x > 2$ 或 $0 < x \leq 2$ 时, 易知 $\frac{x}{2} + \sin \frac{x}{2} > 0$ 故 $g'(x) > 0$, $g(x)$ 在 $(0, +\infty)$ 单增由 $g(x) \in C[0, +\infty)$, $g(x) > g(0) = 0$

即 $x - \frac{x^3}{6} < \sin x$

17) 设 $f(x) = \ln(x+1) - x + \frac{1}{2}x^2$

$f'(x) = \frac{1}{x+1} - 1 + x$

$= \frac{x^2}{x+1} > 0$

故 $f(x)$ 在 $[0, +\infty)$ 上单增

$x > 0$ 时, $f(x) > f(0) = 0$

即 $\ln(1+x) > x - \frac{1}{2}x^2$

由 $\ln(1+\frac{1}{x}) < \frac{1}{x}$, $\ln(1+x) < x$

习题4.5

1.11) $y' = 6x - 3x^2$

当 $x \in (-\infty, 1)$ 时, $y'' > 0$, 即 y 为下凸

$y'' = 6 - 6x$

当 $x \in (1, +\infty)$ 时, $y'' < 0$, 即 y 为上凸

$y''(1) = 0$

(1, 2) 为拐点

14) $y' = \frac{1-x}{e^x}$ 当 $x \in (-\infty, 0)$ 时, $y' < 0$, 即 y 为上凸
 $y'' = \frac{x}{e^x}$ 当 $x \in (0, +\infty)$ 时, $y' > 0$, 即 y 为下凸
 $y'(0) = 0$, 即 $(0, 0)$ 为拐点

5.12) $n=1$ 时, 成立

假设 $n=k$ 时成立, 即 $(\frac{x_1+x_2+\dots+x_k}{k})^p \leq \frac{x_1^p+x_2^p+\dots+x_k^p}{k}$

$n=k+1$ 时, 设 $f(x) = x^p$ ($x \geq 0$)

$f''(x) = p(p-1)x^{p-2} \geq 0$

即 $f(x)$ 在 $[0, +\infty)$ 下凸

故 $\forall x_a, x_b \in [0, +\infty), \forall \lambda \in (0, 1), (\lambda x_a + (1-\lambda)x_b)^p \leq \lambda x_a^p + (1-\lambda)x_b^p$

取 $x_a = x_{k+1}, x_b = \frac{x_1+x_2+\dots+x_k}{k}, \lambda = \frac{1}{k+1}$

$$(\frac{x_{k+1}}{k+1} + \frac{k}{k+1} \cdot \frac{x_1+x_2+\dots+x_k}{k})^p \leq \frac{x_{k+1}^p}{k+1} + \frac{k}{k+1} \cdot (\frac{x_1+x_2+\dots+x_k}{k})^p$$

$$\leq \frac{x_{k+1}^p}{k+1} + \frac{k}{k+1} \cdot \frac{x_1^p+x_2^p+\dots+x_k^p}{k}$$

故 $(\frac{x_1+x_2+\dots+x_{k+1}}{k+1})^p \leq \frac{x_1^p+x_2^p+\dots+x_{k+1}^p}{k+1}$

$\therefore$ 得证

7. 由下凸函数定义

$\forall x_1, x_2 \in [a, b], \forall \lambda \in (0, 1), f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$

$\forall x \in [a, b],$ 取 $x_1 = a, x_2 = b, \lambda = \frac{b-x}{b-a}$

$x = \lambda x_1 + (1-\lambda)x_2$

故 $f(x) \leq (1-\lambda)f(a) + \lambda f(b) \leq (1-\lambda)\max\{f(a), f(b)\} + \lambda\max\{f(a), f(b)\}$
 $= \max\{f(a), f(b)\}$

故 $\max_{a \leq x \leq b} f(x) \leq \max\{f(a), f(b)\}$

8. 由 f 在 $[a, b]$ 下凸, $f'(x)$ 单增

故 $x > x_0$ 时, $f'(x) > f'(x_0) = 0$, 即 f 在 $[x_0, b]$ 单增

$x < x_0$ 时, $f'(x) < f'(x_0) = 0$, 即 f 在 $[a, x_0]$ 单减

故 x_0 是 f 的最小值点

习题4.6

1. (1) $\lim_{x \rightarrow 1^+} \sqrt{\frac{x^3}{x-1}} = +\infty$

故垂直渐近线为 $x=1$

$$\lim_{x \rightarrow +\infty} \frac{y}{x} = \lim_{x \rightarrow +\infty} \sqrt{\frac{x}{x-1}} = 1$$

$$\lim_{x \rightarrow +\infty} (y-x) = \lim_{x \rightarrow +\infty} \frac{\frac{x^3}{x-1} - x^2}{\sqrt{\frac{x^3}{x-1}} + x} = \lim_{x \rightarrow +\infty} \frac{\frac{x}{x-1}}{\sqrt{\frac{x}{x-1}} + 1} = \frac{1}{2}$$

故 $y = x + \frac{1}{2}$ 为 $x \rightarrow +\infty$ 时的斜渐近线

$$\lim_{x \rightarrow -\infty} \frac{y}{x} = \lim_{x \rightarrow -\infty} -\sqrt{\frac{x}{x-1}} = -1$$

$$\lim_{x \rightarrow -\infty} (y+x) = \lim_{x \rightarrow -\infty} \frac{\frac{x^3}{x-1} - x^2}{-\sqrt{\frac{x^3}{x-1}} - x} = \lim_{x \rightarrow -\infty} \frac{\frac{x}{x-1}}{-\sqrt{\frac{x}{x-1}} - 1} = -\frac{1}{2}$$

故 $y = -x - \frac{1}{2}$ 为 $x \rightarrow -\infty$ 时的斜渐近线

无水平渐近线

2. (4) 定义域 $\mathbb{R}$

$$y' = 1 + \frac{1}{1+x^2} > 0$$

y 在 $\mathbb{R}$ 上单增

$$\lim_{x \rightarrow +\infty} \frac{y}{x} = \lim_{x \rightarrow +\infty} \left(1 + \frac{\arctan x}{x}\right) = 1$$

$$\lim_{x \rightarrow +\infty} (y-x) = \lim_{x \rightarrow +\infty} \arctan x = \frac{\pi}{2}$$

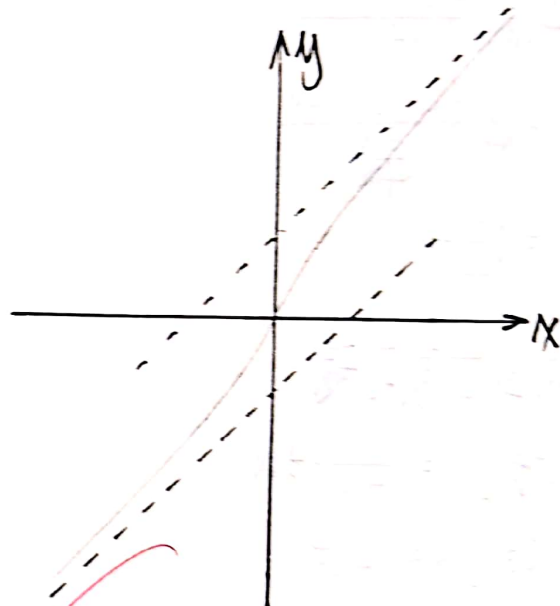
$$\lim_{x \rightarrow -\infty} (y-x) = \lim_{x \rightarrow -\infty} \arctan x = -\frac{\pi}{2}$$

故 $y = x + \frac{\pi}{2}$ 为 $x \rightarrow +\infty$ 时的斜渐近线

$y = x - \frac{\pi}{2}$ 为 $x \rightarrow -\infty$ 时的斜渐近线

$$y(-x) = -x - \arctan x = -y(x)$$

y 为奇函数



习题4.2

2. (2) $\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

$$= \lim_{x \rightarrow 0} \frac{e^x - e^x \cos x}{1 - \cos x}$$

$$= \lim_{x \rightarrow 0} e^x = 1$$

(10) $\lim_{x \rightarrow +\infty} \frac{\ln(1+x)}{x^2}$

$$= \lim_{x \rightarrow +\infty} \frac{1}{2x(1+x)}$$

$$= 0$$

$$\begin{aligned}
 113) \lim_{x \rightarrow 1} (x-1) \tan \frac{\pi x}{2} \\
 &= \lim_{x \rightarrow 1} \frac{x-1}{\cot \frac{\pi x}{2}} \\
 &= \lim_{x \rightarrow 1} \frac{1}{\frac{1}{\sin^2(\frac{\pi x}{2})} \cdot \frac{\pi}{2}} \\
 &= -\frac{2}{\pi}
 \end{aligned}$$

$$\begin{aligned}
 117) \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\pi}{x} - 1 \right)^{\tan x} \\
 &= \lim_{x \rightarrow \frac{\pi}{2}} e^{\tan x \ln \left(\frac{\pi}{x} - 1 \right)} \\
 &= e^{\lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln \left(\frac{\pi}{x} - 1 \right)}{\cot x}} \\
 &= e^{\lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{\pi}{\pi - x} - 1}{-\frac{1}{\sin^2 x}}} \\
 &= e^{\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 3. \lim_{h \rightarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \\
 &= \lim_{h \rightarrow 0} \frac{f'(a+h) - f'(a) + f'(a) - f'(a-h)}{2h} \\
 &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{f'(a+h) - f'(a)}{h} + \frac{1}{2} \lim_{h \rightarrow 0} \frac{f'(a) - f'(a-h)}{h} \\
 &= f''(a)
 \end{aligned}$$

$$\begin{aligned}
 4. \lim_{x \rightarrow 0} \frac{f(\sin x) - 1}{\ln f(x)} \\
 &= \lim_{x \rightarrow 0} \frac{f'(\sin x) \cdot \cos x}{\frac{f'(x)}{f(x)}} \\
 &= \frac{1 \cdot 1}{1} = 1
 \end{aligned}$$

第4章总复习题

$$4. \text{ 设 } g(x) = \frac{x}{1+x^2} - f(x)$$

$$0 \leq g(x) \leq \frac{x}{1+x^2}$$

$$\textcircled{1} g(x) \equiv 0$$

$$\text{任取 } x, g'(x) = \frac{1-x^2}{(1+x^2)^2} - f'(x) = 0$$

$$\text{即 } f'(x) = \frac{1-x^2}{(1+x^2)^2}$$

$$\textcircled{2} \exists x_0 > 0, g(x_0) > 0$$

$$\text{设 } g(x_0) = m$$

$$\text{由 } \frac{x}{1+x^2} \leq \frac{1}{2}, m \in (0, \frac{1}{2}]$$

$$\text{取 } x_1 = \frac{1-\sqrt{1-m^2}}{m}, x_2 = \frac{1+\sqrt{1-m^2}}{m}$$

$$g(x_1) \leq \frac{x_1}{1+x_1^2} = \frac{m}{2} < g(x_0)$$

$$g(x_2) \leq \frac{x_2}{1+x_2^2} = \frac{m}{2} < g(x_0)$$

易知 $x_1 < x_0 < x_2$

不妨 $g(x_1) \leq g(x_2)$

因为 $f(x) \in C[x_1, x_2]$, 由介值定理, $\exists x_3 \in [x_1, x_2], g(x_3) = g(x_2)$

由Rolle定理, $\exists \xi \in (x_3, x_2), g'(\xi) = 0$

$$\text{即 } f'(\xi) = \frac{1-\xi^2}{(1+\xi^2)^2}$$

$$5. \text{ 令 } h(x) = f(x) - \frac{f(+\infty) - f(-\infty)}{g(+\infty) - g(-\infty)} (g(x) - g(-\infty))$$

$$h(+\infty) = f(+\infty) = h(-\infty)$$

$$\text{设 } h(+\infty) = h(-\infty) = A$$

$$\textcircled{1} h(x) \equiv A$$

$$\textcircled{2} \exists x_0, h(x_0) \neq A$$

$$\text{任取 } \varepsilon, h'(\xi) = 0 \quad \text{不此 } h(x_0) > A$$

取 $\varepsilon = h(x_0) - A, \exists M > 0$, 当 $|x| < M$ 时, $|h(x) - A| < \varepsilon$. 即 $h(x) < h(x_0)$

由 $h(x) \in C(\mathbb{R})$, $h(x)$ 在 $[-M, M]$ 上有最大值

故 $h(x)$ 在 $\mathbb{R}$ 上有最大值

由 $h(x)$ 在 $\mathbb{R}$ 上可导, $\exists \xi, h'(\xi) = 0$

$$h'(\xi) = f'(\xi) - \frac{f(+\infty) - f(-\infty)}{g(+\infty) - g(-\infty)} g'(\xi) = 0$$

$$\text{故 } \frac{f(+\infty) - f(-\infty)}{g(+\infty) - g(-\infty)} = \frac{f'(\xi)}{g'(\xi)}$$

$$7. \text{ 设 } g(x) = \frac{f(x)}{x}, h(x) = -\frac{1}{x}$$

$$g'(x) = \frac{x f'(x) - f(x)}{x^2}$$

$$h'(x) = \frac{1}{x^2}$$

$$\text{由 Cauchy 定理, } \exists \xi \in (a, b), \frac{g(b) - g(a)}{h(b) - h(a)} = \frac{g'(\xi)}{h'(\xi)}$$

$$\text{即 } \frac{\frac{f(b)}{b} - \frac{f(a)}{a}}{-\frac{1}{b} + \frac{1}{a}} = \frac{\frac{x f'(x) - f(x)}{x^2}}{\frac{1}{x^2}}$$

$$\text{即 } \frac{a f(b) - b f(a)}{b - a} = \xi f'(\xi) - f(\xi)$$

$$8. \text{ 设 } g(x) = x^2$$

$$g'(x) = 2x$$

$$\text{由 Cauchy 定理, } \exists \eta \in (a, b), \frac{g(b) - g(a)}{g'(\eta)} = \frac{f(b) - f(a)}{f'(\eta)}$$

$$\text{即 } \frac{f(b) - f(a)}{b^2 - a^2} = \frac{f'(\eta)}{2\eta}$$

$$\text{由 Lagrange 定理, } \exists \xi \in (a, b), f'(\xi) = \frac{f(b) - f(a)}{b - a} = \frac{1a + b f'(\eta)}{2\eta}$$

$$\text{或取 } \eta = \xi = \frac{a+b}{2}, \text{ 满足条件}$$

习题 4.3

$$3. (2) y = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + o((x-x_0)^3)$$

$$\sin \frac{\pi}{4} = \sin \frac{\pi}{4} + \cos \frac{\pi}{4}(x-\frac{\pi}{4}) + \frac{1}{2} \sin \frac{\pi}{4}(x-\frac{\pi}{4})^2 + \frac{1}{6} \cos \frac{\pi}{4}(x-\frac{\pi}{4})^3 + o((x-\frac{\pi}{4})^3)$$

$$= 1 + (x-\frac{\pi}{4}) - \frac{1}{2}(x-\frac{\pi}{4})^2 - \frac{1}{6}(x-\frac{\pi}{4})^3 + o((x-\frac{\pi}{4})^3)$$

$$4. (1) y = 1 + \frac{2x}{1-x+x^2}$$

$$= 1 + 2x + \frac{2x^2-2x^3}{1-x+x^2}$$

$$= 1 + 2x + 2x^2 - \frac{2x^4}{1-x+x^2}$$

$$= 1 + 2x + 2x^2 - 2x^4 - \frac{2x^5-2x^6}{1-x+x^2}$$

$$= 1 + 2x + 2x^2 - 2x^4 + o(x^4)$$

$p(x)$

$$(\sin x)^{(n)} = \sin(x_0 + \frac{n\pi}{2})$$

$x=x_0$

$$y = \frac{1+x+x^2}{1-x+x^2} = (1+x+x^2) \frac{1}{1-x+x^2}$$

$$= (1+x+x^2) (1 + (x-x^2) + (x-x^2)^2 + (x-x^2)^3 + \dots)$$

$$= 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + o(x^4)$$

$$a_1 = 1+1=2$$

$$a_2 = 1+1-1+1=2$$

$$a_3 = \cancel{1+1-1-1} + 1 \times (-2) + \cancel{1 \times (-2) + 1 \times 1} = 0$$

$$+ 1 \times (-1+1) + 1 \times 1 = 0$$

$$a_4 = 1 \times (C_2^2 + C_3^2 + C_4^2 + 1) + C_4^4 \cdot 1^4 = -2$$

$$y = \frac{x-2}{(x-2)^2-4} = \frac{-(x-2)}{4-(x-2)^2}$$

$$= -(x-2) \cdot \frac{1}{4} \cdot \frac{1}{1 - (\frac{x-2}{2})^2}$$

$$(b) y = \frac{x-2}{(x-2)^2-4}$$

$$= -\frac{1}{4}(x-2) + \frac{\frac{(x-2)^3}{4}}{(x-2)^2-4}$$

$$= -\frac{1}{4}(x-2) - \frac{1}{4}(x-2)^3 + \frac{\frac{(x-2)^5}{4^2}}{(x-2)^2-4}$$

$$= -\frac{1}{4}(x-2) - \frac{1}{4}(x-2)^3 - \dots - \frac{1}{4^n}(x-2)^{2n-1} + o((x-2)^{2n})$$

$$(8) y = \frac{1}{2} \ln(1+x) - \frac{1}{2} \ln(1-x)$$

$$= \frac{1}{2} (x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n}) - \frac{1}{2} (-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n}) + o(x^n)$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{1+(-1)^{n-1}}{2} \cdot \frac{x^n}{n} + o(x^n)$$

$$(9) \lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = \lim_{x \rightarrow 0} \frac{\frac{1}{x^n}}{e^{\frac{1}{x^2}}}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{x^{n+1}}}{e^{\frac{1}{x^2} - \frac{2}{x^3}}}$$

$$= \frac{n}{2} \lim_{x \rightarrow 0} \frac{\frac{1}{x^{n+2}}}{e^{\frac{1}{x^2}}}$$

$$\Rightarrow 0 \quad \text{why?}$$

$$\text{故 } y = o(x^n) \quad (x \rightarrow 0)$$

$$\text{即 } y \text{ 在 } x_0=0 \text{ 处的 } n \text{ 次泰勒多项式为 } y = o(x^n) + o((x-0)^{2n})$$

$$= -\frac{1}{4}(x-2) - \frac{(x-2)^3}{4^2} - \dots - \frac{1}{4^n}(x-2)^{2n-1}$$

$$5. (1) \lim_{x \rightarrow \infty} (x - x^2 \ln(1 + \frac{1}{x}))$$

$$= \lim_{x \rightarrow \infty} (x - x^2 (\frac{1}{x} - \frac{1}{2x^2} + o(\frac{1}{x^2})))$$

$$= \lim_{x \rightarrow \infty} (x - x + \frac{1}{2}) = \frac{1}{2}$$

先代换再展开?

$$\begin{aligned}
 (2) \lim_{x \rightarrow 0} \frac{\frac{x^2}{2} + 1 - \sqrt{1+x^2}}{(\cos x - e^{x^2}) \sin x^2} \\
 = \lim_{x \rightarrow 0} \frac{\frac{x^2}{2} + 1 - \sqrt{1 + \frac{x^2}{2} + \frac{x^4}{8} + O(x^4)}}{(1 - \frac{x^2}{2} - 1 - x^2 + O(x^2)) x^2} \\
 = \lim_{x \rightarrow 0} \frac{\frac{x^4}{8} + O(x^4)}{-\frac{3x^4}{2} + O(x^4)} = -\frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 \sqrt{1+x^2} &= (1+x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^4 + o(x^4) \\
 &= 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4)
 \end{aligned}$$

$$9. f(x) = f(\frac{1}{2}) + f'(\frac{1}{2})(x - \frac{1}{2}) + \frac{f''(\frac{1}{2})}{2}(x - \frac{1}{2})^2 + \frac{f'''(\xi_1)}{6}(x - \frac{1}{2})^3 \quad \exists \xi_1 \text{ 介于 } x, \frac{1}{2}$$

$$x=0 \text{ 时, } 0 = f(\frac{1}{2}) + \frac{1}{6}f''(\frac{1}{2}) - \frac{1}{48}f'''(\xi_1) \quad \exists \xi_1 \in (0, \frac{1}{2})$$

$$x=1 \text{ 时, } \frac{1}{2} = f(\frac{1}{2}) + \frac{1}{6}f''(\frac{1}{2}) + \frac{1}{48}f'''(\xi_2) \quad \exists \xi_2 \in (\frac{1}{2}, 1)$$

$$\text{故 } \frac{1}{2} = \frac{1}{48}f'''(\xi_1) + \frac{1}{48}f'''(\xi_2)$$

$$\text{由介值定理, } \exists \xi \in (\xi_1, \xi_2), f'''(\xi) = \frac{f'''(\xi_1) + f'''(\xi_2)}{2} = 12$$

$$10. f(x) = f(\frac{a+b}{2}) + f'(\frac{a+b}{2})(x - \frac{a+b}{2}) + \frac{f''(\xi_1)}{2}(x - \frac{a+b}{2})^2 \quad \exists \xi_1 \text{ 介于 } x, \frac{a+b}{2}$$

$$f(a) = f(\frac{a+b}{2}) + f'(\frac{a+b}{2}) \cdot \frac{b-a}{2} + \frac{f''(\xi_1)}{2} \cdot \frac{(b-a)^2}{8} \quad \exists \xi_1 \in (a, \frac{a+b}{2})$$

$$f(b) = f(\frac{a+b}{2}) + f'(\frac{a+b}{2}) \cdot \frac{a-b}{2} + \frac{f''(\xi_2)}{2} \cdot \frac{(b-a)^2}{8} \quad \exists \xi_2 \in (\frac{a+b}{2}, b)$$

$$\text{故 } f(b) - f(a) = \frac{(b-a)^2}{4} \cdot \frac{f''(\xi_1) + f''(\xi_2)}{2}$$

$$\text{由介值定理, } \exists \xi \in (\xi_1, \xi_2), f''(\xi) = \frac{f''(\xi_1) + f''(\xi_2)}{2} \quad \text{达布定理.}$$

$$12. f(x) = f(a) + f'(a)(x-a) + \frac{f''(\xi_1)}{2}(x-a)^2 \quad \exists \xi_1 \in (a, x)$$

$$f(\frac{a+b}{2}) = f(a) + \frac{(b-a)}{2} f'(a) + \frac{f''(\xi_1)}{2} \cdot \frac{(b-a)^2}{8} \quad \exists \xi_1 \in (a, \frac{a+b}{2})$$

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(\xi_2)}{2}(x-b)^2 \quad \exists \xi_2 \in (x, b)$$

$$f(\frac{a+b}{2}) = f(b) + \frac{(b-a)}{2} f'(b) + \frac{f''(\xi_2)}{2} \cdot \frac{(b-a)^2}{8} \quad \exists \xi_2 \in (\frac{a+b}{2}, b)$$

$$\frac{4}{(b-a)^2} |f(b) - f(a)| = \left| \frac{f''(\xi_1) - f''(\xi_2)}{2} \right|$$

$$\leq \frac{|f''(\xi_1)| + |f''(\xi_2)|}{2}$$

$$\leq \max\{|f''(\xi_1)|, |f''(\xi_2)|\}$$

$$\therefore \exists \xi \in (a, b) \text{ s.t. } |f''(\xi)| \geq \frac{4}{(b-a)^2} |f(b) - f(a)|$$

~~使用两次 Lagrange 定理!~~

第4章总复习题

9. $\forall x_0 \in [0, 1], f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi_1)}{2}(x - x_0)^2 \quad \exists \xi_1 \text{ 介于 } x, x_0$

取 $x=0, f(0) = f(x_0) - f'(x_0) \cdot x_0 + \frac{f''(\xi_1)}{2} \cdot x_0^2$

取 $x=1, f(1) = f(x_0) + f'(x_0) \cdot (1 - x_0) + \frac{f''(\xi_2)}{2} \cdot \frac{x_0^2 - 2x_0 + 1}{2}$

$f'(x_0) = \frac{f''(\xi_1)}{2} \cdot \frac{x_0^2}{2} - \frac{f''(\xi_2)}{2} \cdot \frac{x_0^2 - 2x_0 + 1}{2}$

$|f'(x_0)| \leq \left| \frac{f''(\xi_1)}{2} \cdot \frac{x_0^2}{2} \right| + \left| \frac{f''(\xi_2)}{2} \cdot \frac{x_0^2 - 2x_0 + 1}{2} \right|$

$\leq M \cdot \frac{x_0^2}{2} + M \cdot \frac{x_0^2 - 2x_0 + 1}{2}$

$= M \cdot \frac{1 + 12x_0 - 11^2}{4}$

$\leq \frac{M}{2}$

11. 设 $f(x)$ 在 x_0 处取最小值

x_0 不为区间端点, 故 $f'(x_0) = 0$

$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi_1)}{2}(x - x_0)^2 \quad \exists \xi_1 \text{ 介于 } x, x_0$

即 $f(x) = -1 + \frac{f''(\xi_1)}{2}(x - x_0)^2$

令 $x=0, 0 = -1 + \frac{f''(\xi_1)}{2} \cdot x_0^2 \quad \xi_1 \in (0, x_0)$

令 $x=1, 0 = -1 + \frac{f''(\xi_2)}{2} \cdot (1 - x_0)^2 \quad \xi_2 \in (x_0, 1)$

$f''(\xi_1) = \frac{2}{x_0^2} \quad f''(\xi_2) = \frac{2}{(1 - x_0)^2}$

当 $x_0 \leq \frac{1}{2}$ 时, $f''(\xi_1) \geq 8$

当 $x_0 > \frac{1}{2}$ 时, $f''(\xi_2) \geq 8$

故 $\exists \xi \in (0, 1), f''(\xi) \geq 8$

$\triangleq g(x_0) = \frac{(1-x_0)(b-x_0)}{2} + \frac{(a-x_0)^2}{2(b-x_0)}$
 $g_{\min} x = \frac{(b-a)^2}{8}$

12. (1) $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi_1)}{2}(x - x_0)^2 \quad \exists \xi_1 \text{ 介于 } x, x_0$

令 $x=a, 0 = f(x_0) + \frac{f''(\xi_1)}{2}(a - x_0)^2 \quad \xi_1 \in (a, x_0)$

令 $x=b, 0 = f(x_0) + \frac{f''(\xi_2)}{2}(b - x_0)^2 \quad \xi_2 \in (x_0, b)$

$(b-a)f(x_0) = \frac{f''(\xi_1)}{2}(a - x_0)^2(b - x_0) - \frac{f''(\xi_2)}{2}(a - x_0)(b - x_0)^2$

$(b-a)|f(x_0)| \leq \frac{|x_0 - a|^2 |b - x_0|}{2} |f''(\xi_1)| + \frac{|x_0 - a| |b - x_0|^2}{2} |f''(\xi_2)|$

$\leq \frac{|x_0 - a| |b - x_0| (b-a)}{2} \max_{a \leq x \leq b} |f''(x)|$

$\leq \max_{0 \leq x \leq b} |f''(x)| \left(\frac{(a-x_0)(b-x_0)}{2} + \frac{(a-x_0)^2}{2(b-x_0)} \right)$

$$\leq \frac{|b-a|^3}{6} \max_{a \leq x \leq b} |f''(x)|$$

$$\text{故 } \max_{a \leq x \leq b} |f(x)| \leq \frac{1}{6} |b-a|^3 \max_{a \leq x \leq b} |f''(x)|$$

$$12) |b-a| f'(x_0) = \frac{|a-x_0|^2}{2} f''(\xi_1) - \frac{|b-x_0|^2}{2} f''(\xi_2) \quad \text{如何得到作证明!}$$

$$\begin{aligned} |b-a| f'(x_0) &\leq \frac{|a-x_0|^2}{2} |f''(\xi_1)| + \frac{|b-x_0|^2}{2} |f''(\xi_2)| \\ &\leq \frac{a^2 + b^2 - 2ax_0 - 2bx_0 + 2x_0^2}{2} \max_{a \leq x \leq b} |f''(x)| \\ &= \frac{|b-a|^2 + 2|a-x_0||b-x_0|}{2} \max_{a \leq x \leq b} |f''(x)| \\ &\leq \frac{|b-a|^2}{2} \max_{a \leq x \leq b} |f''(x)| \end{aligned}$$

$$\text{故 } \max_{a \leq x \leq b} |f(x)| \leq \frac{1}{2} |b-a| \max_{a \leq x \leq b} |f''(x)|$$

习题4.3

$$7.12) m(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} \quad \exists \xi \in (1, x)$$

$$\begin{aligned} m(1.02) &= 0.02 - \frac{0.02^2}{2} + \frac{0.02^3}{3} - \frac{0.02^4}{4} \\ &= 0.019797333\ldots - \frac{4}{3} \times 10^{-8} \\ &> 0.01979729333\ldots \end{aligned}$$

$$\text{故 } m(1.02) \approx 0.01979$$

12 舍去

习题5.1

$$1.13) \int_{-1}^1 (3+4x) dx$$

$$\begin{aligned} &= \frac{1}{2} (1 - (-\frac{3}{4})) \cdot 7 - \frac{1}{2} (-\frac{3}{4} - (-1)) \cdot 1 \\ &= \frac{49}{8} - \frac{1}{8} \\ &= 6 \end{aligned}$$

$$14) \int_0^1 (\sqrt{2x-x^2} - x) dx$$

$$\begin{aligned} &= \frac{\pi}{4} \cdot 1^2 - \frac{1}{2} \cdot 1 \cdot 1 \\ &= \frac{\pi-2}{4} \end{aligned}$$

画个图?

↑
用几何意义求.

例. 设 $T: a = x_0 < x_1 < \dots < x_n = b$

由 f, g 在 $[a, b]$ 可积. $\forall \varepsilon > 0, \exists \delta > 0$, 当 $|T| < \delta$ 时

$$U(f; T) - L(f; T) < \frac{\varepsilon}{\sqrt{2}} \quad U(g; T) - L(g; T) < \frac{\varepsilon}{\sqrt{2}}$$

$$\text{则 } U(\sqrt{f^2 + g^2}; T) - L(\sqrt{f^2 + g^2}; T)$$

$$= \sum_{i=1}^n \sup_{\xi_i, \eta_i \in [x_{i-1}, x_i]} |\sqrt{f(\xi_i)^2 + g(\xi_i)^2} - \sqrt{f(\eta_i)^2 + g(\eta_i)^2}| (x_i - x_{i-1})$$

$$\leq \sum_{i=1}^n \sup_{\xi_i, \eta_i \in [x_{i-1}, x_i]} \sqrt{(f(\xi_i) - f(\eta_i))^2 + (g(\xi_i) - g(\eta_i))^2} (x_i - x_{i-1})$$

$$\leq \frac{1}{\sqrt{2}} \sum_{i=1}^n \sup_{\xi_i, \eta_i \in [x_{i-1}, x_i]} (|f(\xi_i) - f(\eta_i)| + |g(\xi_i) - g(\eta_i)|) (x_i - x_{i-1})$$

$$\leq \frac{1}{\sqrt{2}} \sum_{i=1}^n \left(\sup_{\xi_i, \eta_i \in [x_{i-1}, x_i]} |f(\xi_i) - f(\eta_i)| (x_i - x_{i-1}) + \sup_{\xi_i, \eta_i \in [x_{i-1}, x_i]} |g(\xi_i) - g(\eta_i)| (x_i - x_{i-1}) \right)$$

$$= \frac{1}{\sqrt{2}} (U(f; T) - L(f; T) + U(g; T) - L(g; T))$$

$$< \frac{1}{\sqrt{2}} \left(\frac{\varepsilon}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} \right) = \varepsilon$$

故 $\sqrt{f^2 + g^2}$ 在 $[a, b]$ 可积

习题 5.1

5. $\forall \varepsilon > 0$, 由 $f(x) \in R[a, b]$, $\exists \delta > 0$, 当 $[a, b]$ 分割 T 满足 $|T| < \delta$ 时

$$U(f; T) - L(f; T) < \varepsilon$$

$$\text{则 } U(\cos f; T) - L(\cos f; T)$$

$$= \sum_{i=1}^n \sup_{\xi_i, \eta_i \in [x_{i-1}, x_i]} |\cos f(\xi_i) - \cos f(\eta_i)| (x_i - x_{i-1})$$

$$= \sum_{i=1}^n \sup_{\xi_i, \eta_i \in [x_{i-1}, x_i]} \left| \sin \left(\frac{f(\xi_i) + f(\eta_i)}{2} \right) \sin \left(\frac{f(\xi_i) - f(\eta_i)}{2} \right) \right| (x_i - x_{i-1})$$

$$\leq \sum_{i=1}^n \sup_{\xi_i, \eta_i \in [x_{i-1}, x_i]} \left| \frac{f(\xi_i) - f(\eta_i)}{2} \right| (x_i - x_{i-1})$$

$$= \frac{1}{2} (U(f; T) - L(f; T)) < \frac{\varepsilon}{2}$$

故 $\cos(f(x)) \in R[a, b]$

习题 5.2

5.11) 当 $x \in [0, 1]$ 时, $e^{-x} \leq e^{-x^2}$

由单调性知 $\int_0^1 e^{-x} dx < \int_0^1 e^{-x^2} dx$

b. 11) $\frac{\sin x}{x}$ 在 $[\frac{\pi}{4}, \frac{\pi}{2}]$ 单调减, 故 $\frac{\sin x}{x} \geq \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi}$

$$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1$$

由单调性, $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2}{\pi} dx \leq \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x} dx \leq \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 dx$

$$\text{故 } \frac{1}{2} \leq \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{x} dx \leq \frac{\pi}{4}$$

$$13) |a \cos x + b \sin x| = \sqrt{a^2 + b^2} |\sin(x + \varphi)| \leq \sqrt{a^2 + b^2}$$

$$\text{由单调性, } \int_0^{2\pi} |a \cos x + b \sin x| dx \leq \int_0^{2\pi} \sqrt{a^2 + b^2} dx = 2\pi \sqrt{a^2 + b^2}$$

$$10. \text{ 设 } g(x) = f(a) + \frac{f(b) - f(a)}{b - a} (x - a)$$

由 $f''(x) < 0$, $f(x)$ 下凸, 故 $f(x) \leq g(x)$

由 $f(x)$ 在 $[a, b]$ 单增, $f(x) \geq f(a)$

故由单调性, $\int_a^b f(a) dx < \int_a^b f(x) dx < \int_a^b g(x) dx$

$$\begin{aligned} \text{即 } (b-a)f(a) &< \int_a^b f(x) dx < f(a) \cdot x + \frac{f(b) - f(a)}{b - a} \left(\frac{x^2}{2} - ax \right) \Big|_a^b \\ &= (b-a) \frac{f(a) + f(b)}{2} \end{aligned}$$

习题 5.3

$$12. 11) \int_0^2 |1-x| dx$$

$$= \int_0^1 (1-x) dx + \int_1^2 (x-1) dx$$

$$= \left(x - \frac{x^2}{2} \right) \Big|_0^1 + \left(\frac{x^2}{2} - x \right) \Big|_1^2$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$12) \int_{-2}^3 |x^2 - 2x - 3| dx$$

$$= \int_{-2}^{-1} (x^2 - 2x - 3) dx + \int_{-1}^3 (3 + 2x - x^2) dx$$

$$= \left(\frac{x^3}{3} - x^2 - 3x \right) \Big|_{-2}^{-1} + \left(3x + x^2 - \frac{x^3}{3} \right) \Big|_{-1}^3$$

$$= \left(\frac{1}{3} - 1 - 3 \right) + \left(9 - 1 - \frac{27}{3} \right) = 13$$

习题 5.2

8. $\forall \varepsilon > 0$, 由 $f, g \in R[a, b]$, $\exists \delta > 0$, 当 $[a, b]$ 的分割 T 满足 $|T| < \delta$ 时

$$U(f; T) - L(f; T) < \frac{\varepsilon}{2} \quad U(g; T) - L(g; T) < \frac{\varepsilon}{2}$$

$$U(\min\{f, g\}; T) - L(\min\{f, g\}; T) = \sum_{i=1}^n \sup_{\eta \in [x_{i-1}, x_i]} |\min\{f(\eta), g(\eta)\} - \min\{f(\xi_i), g(\xi_i)\}| (x_i - x_{i-1})$$

不妨 $\min\{f(\xi_i), g(\xi_i)\} \geq \min\{f(\eta_i), g(\eta_i)\}$

$$\begin{aligned}
& |\min\{f(\xi_i), g(\xi_i)\} - \min\{f(\eta_i), g(\eta_i)\}| \\
&= \min\{f(\xi_i), g(\xi_i)\} - \min\{f(\eta_i), g(\eta_i)\} \\
&= \begin{cases} \min\{f(\xi_i), g(\xi_i)\} - f(\eta_i) \\ \min\{f(\xi_i), g(\xi_i)\} - g(\eta_i) \end{cases} \\
&\leq \begin{cases} f(\xi_i) - f(\eta_i) \\ g(\xi_i) - g(\eta_i) \end{cases} \\
&\leq |f(\xi_i) - f(\eta_i)| + |g(\xi_i) - g(\eta_i)|
\end{aligned}$$

$$\begin{aligned}
& \text{故 } U(\min\{f, g\}; T) - L(\min\{f, g\}; T) \\
&\leq \sum_{i=1}^n \sup_{\xi_i \in [x_{i-1}, x_i]} |f(\xi_i) - f(\eta_i)| + \sup_{\xi_i \in [x_{i-1}, x_i]} |g(\xi_i) - g(\eta_i)| (x_i - x_{i-1}) \\
&= U(f; T) - L(f; T) + U(g; T) - L(g; T) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

即 $\min\{f(x_i), g(x_i)\} \in R[a, b]$

同理 $\max\{f(x_i), g(x_i)\} \in R[a, b]$

9. 由柯西不等式

$$\begin{aligned}
\left| \int_a^b f(x) dx \right| \left| \int_a^b \frac{1}{f(x)} dx \right| &\geq \left| \int_a^b \sqrt{f(x)} \cdot \frac{1}{\sqrt{f(x)}} dx \right|^2 \\
&= \left| \int_a^b 1 \cdot dx \right|^2 \\
&= (b-a)^2
\end{aligned}$$

习题 5.3

$$\begin{aligned}
1. (3) F'(x) &= \ln(2+x^2) \cdot (x^2)' \\
&= 2x \ln(2+x^2)
\end{aligned}$$

$$\begin{aligned}
(4) F(x) &= \int_0^{x^2} e^{-t} dt - \int_0^{\sqrt{x}} e^{-t} dt \\
F'(x) &= \left(\int_0^{x^2} e^{-t} dt \right)' - \left(\int_0^{\sqrt{x}} e^{-t} dt \right)' \\
&= e^{-x^2} \cdot (x^2)' - e^{-\sqrt{x}} \cdot (\sqrt{x})' \\
&= 2xe^{-x^2} - \frac{1}{2\sqrt{x}} e^{-\sqrt{x}}
\end{aligned}$$

4. 由 $f \in C[0, +\infty)$, $F(x) = \int_0^x f(t) dt$ 在 $[0, +\infty)$ 可导且 $F'(x) = f(x)$

$$F(x) = \int_0^x f(t) dt = x^2 + \sin x^2$$

$$F'(x) = 2x + 2x \cos x^2$$

$$\text{故 } f(x) = 2x + 2x \cos x^2$$

$$5. F'(x) = x e^{-x^2}$$

$$F'(0) = 0$$

$$x > 0 \text{ 时, } F'(x) > 0$$

$$x < 0 \text{ 时, } F'(x) < 0$$

故 $x=0$ 为 $F(x)$ 的极值点

$$F''(x) = (1 - 2x^2)e^{-x^2}$$

$$F''(\frac{1}{\sqrt{2}}) = 0$$

$$x \in [0, \frac{1}{\sqrt{2}}) \text{ 时, } F''(x) > 0$$

$$x \in (\frac{1}{\sqrt{2}}, +\infty) \text{ 时, } F''(x) < 0$$

故 $x = \frac{1}{\sqrt{2}}$ 为 $F(x)$ 的拐点

~~1.2~~

$$7. \text{ 当 } x \in [-1, 0) \text{ 时, } F(x) = \int_{-1}^x (t+1) dt \\ = (\frac{t^2}{2} + t) \Big|_{-1}^x \\ = \frac{x^2}{2} + x + \frac{1}{2}$$

$$\text{当 } x \in [0, 1] \text{ 时, } F(x) = \int_{-1}^0 f(t) dt + \int_0^x f(t) dt \\ = \int_{-1}^0 (t+1) dt + \int_0^x t dt \\ = (\frac{t^2}{2} + t) \Big|_{-1}^0 + \frac{t^2}{2} \Big|_0^x \\ = \frac{x^2}{2} + \frac{1}{2}$$

$$\lim_{x \rightarrow 0^-} F(x) = \frac{1}{2} = F(0)$$

故 $F(x)$ 连续

$$F'_{-}(0) = \lim_{x \rightarrow 0^-} \frac{F(x) - F(0)}{x} = \lim_{x \rightarrow 0^-} (\frac{x}{2} + 1) = 1$$

$$F'_{+}(0) = \lim_{x \rightarrow 0^+} \frac{F(x) - F(0)}{x} = \lim_{x \rightarrow 0^+} \frac{x}{2} = 0$$

故 $F(x)$ 在 $x=0$ 处不可导, 在其他点可导

$$9. (1) \int_0^2 f(x) dx = \int_0^2 (x + \int_0^2 f(x) dx) dx \\ = 2 + 2 \int_0^2 f(x) dx$$

$$\int_0^2 f(x) dx = -2$$

$$f(x) = x + \int_0^2 f(x) dx = x - 2$$

$$13. (3) \lim_{n \rightarrow \infty} n^{-\frac{2}{3}} (\sqrt{n+1} + \sqrt{n+2} + \dots + \sqrt{n+n})$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} (\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{2}{n}} + \dots + \sqrt{1+\frac{n}{n}})$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{1+\frac{k}{n}} \cdot \frac{1}{n}$$

$$= \int_0^1 \sqrt{1+x} dx$$

$$= \frac{2}{3} (1+x)^{\frac{3}{2}} \Big|_0^1 = \frac{4\sqrt{2}-2}{3}$$

习题 5.4

$$2. (11) F(x) = \begin{cases} \frac{x^3}{3} + x + C, & x \leq 0 \\ \sin x + C, & x > 0 \end{cases}$$

$$(13) F(x) = \begin{cases} \cos x + C, & x \leq 0 \\ 2\sqrt{x} + C, & x > 0 \end{cases}$$

$$3. (11) \int (x - x^{-2}) \sqrt{x} dx \\ = \int (x^{\frac{7}{2}} - x^{-\frac{5}{2}}) dx \\ = \frac{4}{11} x^{\frac{11}{2}} + 4x^{-\frac{1}{2}} + C$$

$$(14) \int (x-1)(3x-2) dx \\ = \int (3x^2 - 5x + 2) dx \\ = x^3 - \frac{5}{2}x^2 + 2x + C$$

$$4. (12) \int \frac{x}{\sqrt{4-x^2}} dx = -\sqrt{4-x^2}$$

$$(14) \int \frac{2^x}{\sqrt{4-4^{x+1}}} dx = \frac{1}{2\ln 2} \int \frac{(2^x)'}{\sqrt{1-(2^x)^2}} dx \\ \underline{\underline{\text{令 } u = 2^x}} \quad \frac{1}{2\ln 2} \int \frac{1}{\sqrt{1-u^2}} du \\ = \frac{\arcsin u}{2\ln 2} + C \\ = \frac{\arcsin 2^x}{2\ln 2} + C$$

$$5. (14) \int \frac{x-1}{x^2-4x+9} dx = \int \frac{(x-2)+1}{(x-2)^2+4} dx$$

$$\underline{\underline{\text{令 } u = x-2}} \quad \int \frac{u+1}{u^2+4} du \\ = \int \frac{u}{u^2+4} du + \int \frac{1}{u^2+4} du$$

$$\underline{\underline{\text{令 } u^2 = v}} \quad \frac{1}{2} \int \frac{1}{v+4} dv + \frac{1}{2} \int \frac{1}{w^2+1} dw$$

$$= \frac{1}{2} \ln|v+4| + \frac{1}{2} \arctan w + C$$

$$= \frac{1}{2} \ln|u^2+4| + \frac{1}{2} \arctan \frac{u}{2} + C$$

$$= \frac{1}{2} \ln|x^2-4x+9| + \frac{1}{2} \arctan \frac{x-2}{2} + C$$

$$6. (5) \int \frac{2x-1}{\sqrt{4x^2+4x+5}} dx = \int \frac{(x+\frac{1}{2})-1}{\sqrt{(x+\frac{1}{2})^2+1}} dx$$

$$\underline{\underline{\text{令 } x+\frac{1}{2} = u}} \quad \int \frac{u-1}{\sqrt{u^2+1}} du$$

$$\begin{aligned}
 &= \frac{1}{2} \int \frac{2u}{\sqrt{u^2+1}} du - \int \frac{1}{\sqrt{u^2+1}} du \\
 &\stackrel{u^2=N}{=} \frac{1}{2} \int \frac{1}{\sqrt{N+1}} dN - \ln|\sqrt{1+u^2}+u| + C \\
 &= \sqrt{N+1} - \ln|\sqrt{1+u^2}+u| + C \\
 &= \frac{1}{2} \sqrt{4x^2+4x+5} - \ln|\sqrt{4x^2+4x+5}+2x+1| + C
 \end{aligned}$$

$$\begin{aligned}
 16) \int \frac{x^2}{\sqrt{3+2x-x^2}} dx &\stackrel{u=1+2\sin t}{\substack{t \in [-\frac{\pi}{2}, \frac{\pi}{2}]}} \int \frac{4\sin^2 t + 4\sin t + 1}{2\cos t} d(1+2\sin t) \\
 &= \int (4\sin^2 t + 4\sin t + 1) dt \\
 &= \int (4\sin t - 2\cos 2t + 3) dt \\
 &= -4\cos t - \sin 2t + 3t + C \\
 &= 3\arcsin \frac{x-1}{2} - \frac{x+3}{2} \sqrt{3+2x-x^2} + C
 \end{aligned}$$

$$\begin{aligned}
 7.11) \int x \cos 2x dx &= \frac{1}{2} \int x (\sin 2x)' dx \\
 &= \frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x dx \\
 &= \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C
 \end{aligned}$$

$$\begin{aligned}
 12) \int x e^{-x} dx &= -\frac{1}{2} \int x (e^{-x})' dx \\
 &= -\frac{1}{2} x e^{-x} + \frac{1}{2} \int e^{-x} dx \\
 &= -\frac{1}{2} x e^{-x} - \frac{1}{2} e^{-x} + C
 \end{aligned}$$

$$\begin{aligned}
 14) \int x \arctan x dx &= \frac{1}{2} \int \arctan x (x^2)' dx \\
 &= \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int x^2 d \arctan x \\
 &= \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\
 &= \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int (1 - \frac{1}{1+x^2}) dx \\
 &= \frac{1}{2} x^2 \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x + C
 \end{aligned}$$

$$\begin{aligned}
 15) \int x \ln(x-1) dx &= \frac{1}{2} \int (x^2)' \ln(x-1) dx \\
 &= \frac{1}{2} x^2 \ln(x-1) - \frac{1}{2} \int \frac{x^2}{x-1} dx \\
 &= \frac{1}{2} x^2 \ln(x-1) - \frac{1}{2} \int (x+1 + \frac{1}{x-1}) dx
 \end{aligned}$$

$$= \frac{1}{2}x^2 \ln|x-1| - \frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{2} \ln|x-1| + C$$

$$\begin{aligned} (10) \int e^x \sin^2 x dx &= \int (e^x)' \sin^2 x dx \\ &= e^x \sin^2 x - \int e^x 2 \sin x \cos x dx \\ &= e^x \sin^2 x - \int (e^x)' \sin 2x dx \\ &= e^x \sin^2 x - e^x \sin 2x + \int e^x 2 \cos 2x dx \\ &= e^x \sin^2 x - e^x \sin 2x + \int e^x (1 - \sin^2 x) dx \\ &= e^x (\sin^2 x - \sin 2x + 1) - \int e^x \sin^2 x dx + C \end{aligned}$$

$$\text{故 } \int e^x \sin^2 x dx = \frac{e^x}{2} (\sin^2 x - \sin 2x + 1) + C$$

习题 5.5

$$\begin{aligned} 1.11) \int \frac{1}{(x+1)(x+2)^2} dx &= \int \frac{(x+2) - (x-1)}{(x+1)(x+2)^2} dx \\ &= \int \frac{1}{(x+1)(x+2)} dx - \int \frac{1}{(x+2)^2} dx \\ &= \int \frac{1}{x+1} dx - \int \frac{1}{x+2} dx + \frac{1}{x+2} + C \\ &= \ln \left| \frac{x+1}{x+2} \right| + \frac{1}{x+2} + C \end{aligned}$$

$$\begin{aligned} 13) \int \frac{x^3+1}{x^3-5x^2+6x} dx &= \int \left(1 + \frac{5x^2-6x+1}{x(x-2)(x-3)} \right) dx \\ &= \int \left(1 + \frac{1}{6x} - \frac{9}{2x-4} + \frac{29}{2x-6} \right) dx \\ &= x + \frac{1}{6} \ln|x| - \frac{9}{2} \ln|x-2| + \frac{29}{2} \ln|x-3| + C \end{aligned}$$

$$\begin{aligned} 17) \int \frac{x^7}{(1-x^2)^5} dx &= \int \frac{\frac{1}{x^3}}{(\frac{1}{x^2}-1)^5} dx \\ &\stackrel{\text{令 } \frac{1}{x^2}=t}{=} -\frac{1}{2} \int \frac{1}{(t-1)^5} dt \\ &= \frac{1}{8(t-1)^4} + C \\ &= \frac{x^2}{8(1-x^2)^4} + C \end{aligned}$$

$$\begin{aligned} 2.11) \int \frac{\sin^4 x}{\cos^3 x} dx &= \int \frac{\sin^4 x (\sin x)'}{\cos^4 x} dx \\ &\stackrel{\text{令 } \sin x=t}{=} \int \frac{t^4}{(1-t^2)^2} dt \\ &= \int \left(1 + \frac{2t^2-1}{(1-t)^2(1+t)^2} \right) dt \end{aligned}$$

$$\begin{aligned}
 &= \int \left(1 - \frac{3}{4} \cdot \frac{1}{1-t} - \frac{3}{4} \cdot \frac{1}{1+t} + \frac{1}{4} \cdot \frac{1}{(1-t)^2} + \frac{1}{4} \cdot \frac{1}{(1+t)^2} \right) dt \\
 &= t + \frac{3}{4} \ln \left| \frac{1-t}{1+t} \right| + \frac{1}{4(1-t)} - \frac{1}{4(1+t)} + C \\
 &= \sin x + \frac{3}{4} \ln \left| \frac{1-\sin x}{1+\sin x} \right| + \frac{1}{4(1-\sin x)} - \frac{1}{4(1+\sin x)} + C
 \end{aligned}$$

$$\begin{aligned}
 12) \int \frac{1}{\sin^3 x \cos^4 x} dx &= - \int \frac{(\cos x)'}{\sin^3 x \cos^4 x} dx \\
 \underline{\underline{\cos x = t}} \quad &= - \int \frac{1}{(1-t^2)t^4} dt \\
 &= - \int \left(\frac{1}{t^4} + \frac{1}{t^2} + \frac{1}{2(1-t)} + \frac{1}{2(1+t)} \right) dt \\
 &= \frac{1}{3t^3} + \frac{1}{t} + \frac{1}{2} \ln \left| \frac{1-t}{1+t} \right| + C \\
 &= \frac{1}{3\cos^3 x} + \frac{1}{\cos x} + \frac{1}{2} \ln \left| \frac{1+\cos x}{1-\cos x} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 17) \int \frac{\sin x}{\sin^2 x + \cos^2 x} dx &= \int \frac{\sin x \cos x - \sin^3 x}{\cos^2 x - \sin^2 x} dx \\
 &= \frac{1}{2} \int \frac{\sin 2x + \cos 2x - 1}{\cos 2x} dx \\
 \underline{\underline{\cos 2x = t}} \quad &= \frac{1}{4} \int \left(\tan t + 1 - \frac{1}{\cos t} \right) dt \\
 &= \frac{1}{4} \left(\ln \left| \frac{1}{\cos t} \right| + t - \ln \left| \frac{1+\sin t}{\cos t} \right| \right) + C \\
 &= \frac{x}{2} - \frac{1}{4} \ln |1 + \sin 2x| + C
 \end{aligned}$$

$$\begin{aligned}
 19) \int \frac{\cos x}{\sin^2 x + \cos^2 x} dx &= \int \left(1 - \frac{\sin x}{\sin^2 x + \cos^2 x} \right) dx \\
 &= \int 1 \cdot dx - \int \frac{\sin x}{\sin^2 x + \cos^2 x} dx \\
 &= x - \left(\frac{x}{2} - \frac{1}{4} \ln |1 + \sin 2x| \right) + C \\
 &= \frac{x}{2} + \frac{1}{4} \ln |1 + \sin 2x| + C
 \end{aligned}$$

$$\begin{aligned}
 3.1b) \int x \sqrt{\frac{1+x}{1-x}} dx \quad &\underline{\underline{\frac{1}{2}x = \cos t}} \quad \int \cos t \cdot \frac{1+\cos t}{1-\cos^2 t} d\cos t \\
 &= - \int \cos t (1 + \cos t) dt \\
 &= - \int \cos t dt - \int \cos^2 t dt \\
 &= -\sin t - \frac{\sin t \cos t + t}{2} + C \\
 &= -\sqrt{1-x^2} - \frac{x\sqrt{1-x^2} + \arccos x}{2} + C
 \end{aligned}$$

$$\begin{aligned}
 18) \int \frac{1-x+x^2}{1+x-x^2} dx & \xrightarrow{\text{令 } x = \frac{\sqrt{5}\cos t + 1}{2}} \int \frac{5\cos^2 t + 3}{2\sqrt{5}\sin t} d\frac{\sqrt{5}\cos t + 1}{2} \\
 & = -\int \left(\frac{5}{4}\cos^2 t + \frac{3}{4} \right) dt \\
 & = -\int \left(\frac{5}{8}\cos 2t + \frac{11}{8} \right) dt \\
 & = -\frac{5}{16}\sin 2t + \frac{11}{8}t + C \\
 & = \frac{1}{4}(2x-1)\sqrt{1+x-x^2} + \frac{11}{8}\arccos \frac{2x-1}{\sqrt{5}} + C
 \end{aligned}$$

习题5.6

$$\begin{aligned}
 1.11) \int_0^{2\pi} |\sin x| dx &= \int_0^{\pi} \sin x dx - \int_{\pi}^{2\pi} \sin x dx \\
 &= (-\cos x)|_0^{\pi} - (-\cos x)|_{\pi}^{2\pi} \\
 &= 2 - (-2) = 4
 \end{aligned}$$

$$\begin{aligned}
 110) \int_{\frac{\pi}{2}}^1 \frac{\sqrt{1-x^2}}{x^2} dx & \xrightarrow{\text{令 } x = \cos t} \int_{\frac{\pi}{2}}^0 \frac{\sin t}{\cos^2 t} d\cos t \\
 &= -\int_{\frac{\pi}{2}}^0 \frac{\sin^2 t}{\cos^2 t} dt \\
 &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{\cos^2 t} - 1 \right) dt \\
 &= (t \tan t - t) \Big|_0^{\frac{\pi}{2}} = 1 - \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 112) \int_{-2}^{-\sqrt{2}} \frac{1}{x^2 \sqrt{x^2-1}} dx & \xrightarrow{\text{令 } x = \frac{\cos t}{t}} \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \cos t dx \\
 &= -\sin x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \frac{\sqrt{3}-\sqrt{2}}{2}
 \end{aligned}$$

$$\begin{aligned}
 15) \int_1^2 \frac{1}{x^2-2x-3} dx &= \int_1^2 \left(\frac{1}{4(x-3)} - \frac{1}{4(x+1)} \right) dx \\
 &= \frac{1}{4} (\ln|x-3| - \ln|x+1|) \Big|_1^2 \\
 &= \frac{1}{4} (-\ln 3 - 0) = -\frac{1}{4} \ln 3
 \end{aligned}$$

$$\begin{aligned}
 111) \int_0^1 \frac{1}{(x+1)\sqrt{1+x^2}} dx & \xrightarrow{\text{令 } x = \tan t} \int_0^{\frac{\pi}{4}} \frac{\cos^2 t}{\sin t + \cos t} dt \tan t \\
 &= \int_0^{\frac{\pi}{4}} \frac{1}{\sin t + \cos t} dt \\
 &= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{2}\cos(t-\frac{\pi}{4})} dt \\
 &= \int_0^{\frac{\pi}{4}} \frac{(\sin(t-\frac{\pi}{4}))'}{\sqrt{2}\cos^2(t-\frac{\pi}{4})} dt \\
 & \xrightarrow{\text{令 } \sin(t-\frac{\pi}{4}) = u} \int_{-\frac{\sqrt{2}}{2}}^0 \frac{du}{\sqrt{2}(1-u)(1+u)} \\
 &= \frac{\sqrt{2}}{4} \int_{-\frac{\sqrt{2}}{2}}^0 \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du \\
 &= \frac{\sqrt{2}}{4} \ln \left| \frac{1+u}{1-u} \right| \Big|_{-\frac{\sqrt{2}}{2}}^0 = -\frac{\sqrt{2}}{4} \ln(3-2\sqrt{2})
 \end{aligned}$$

$$\begin{aligned}
 2.11) \int_0^{\pi} (1-2x)\sin x dx &= \int_0^{\pi} (2x-1)d\cos x \\
 &= (2x-1)\cos x \Big|_0^{\pi} - \int_0^{\pi} 2\cos x dx \\
 &= 2 - 2\pi - 2\sin x \Big|_0^{\pi} = 2 - 2\pi
 \end{aligned}$$

$$\begin{aligned}
 15) \int_0^{\frac{\pi}{2}} e^{-x} \sin 2x dx &= -\frac{1}{2} \int_0^{\frac{\pi}{2}} e^{-x} d\cos 2x \\
 &= -\frac{1}{2} e^{-x} \cos 2x \Big|_0^{\frac{\pi}{2}} - \frac{1}{2} \int_0^{\frac{\pi}{2}} e^{-x} \cos 2x dx \\
 &= \frac{1}{2} - \frac{1}{4} \int_0^{\frac{\pi}{2}} e^{-x} d\sin 2x \\
 &= \frac{1}{2} - \frac{1}{4} e^{-x} \sin 2x \Big|_0^{\frac{\pi}{2}} - \frac{1}{4} \int_0^{\frac{\pi}{2}} e^{-x} \sin 2x dx \\
 \text{故 } \int_0^{\frac{\pi}{2}} e^{-x} \sin 2x dx &= \frac{4}{9} \left(\frac{1}{2} - \frac{1}{4} e^{-\frac{\pi}{2}} \right) = \frac{2}{9} - \frac{1}{9} e^{-\frac{\pi}{2}}
 \end{aligned}$$

$$3. (1) \text{ 设 } I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$$

$$I_0 = \frac{\pi}{2}$$

$$\begin{aligned} \text{当 } n \geq 2 \text{ 时, } I_n &= -\int_0^{\frac{\pi}{2}} \sin^{n-1} x d \cos x \\ &= -\sin^{n-1} x \cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1) \sin^{n-2} x \cos^2 x dx \\ &= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x dx \\ &= (n-1) I_{n-2} - (n-1) I_n \end{aligned}$$

$$\text{故 } I_n = \frac{n-1}{n} I_{n-2}$$

$$\int_0^{\frac{\pi}{2}} \sin^4 x dx = I_4 = \frac{3}{4} \cdot \frac{1}{2} \cdot I_0 = \frac{3\pi}{16}$$

$$\begin{aligned} 6. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-x) \sqrt{a^2-x^2} dx &\stackrel{\text{令 } x=a \sin t}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-a \sin t) a^2 \cos^2 t dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (a^2 - a^3 \sin t - a^2 \sin^2 t + a^3 \sin^3 t) dt \\ &= 2 \int_0^{\frac{\pi}{2}} (a^2 - a^2 \sin^2 t) dt \\ &= \pi a^2 - \frac{\pi}{2} a^2 = \frac{\pi}{2} a^2 \end{aligned}$$

$$5. \int_0^{\pi} (x - \frac{\pi}{2}) f(\sin x) dx \stackrel{\text{令 } x=t+\frac{\pi}{2}}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t f(\cos t) dt$$

$$\text{由于 } t f(\cos t) \text{ 为奇函数, } \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t f(\cos t) dt = 0$$

$$\text{故 } \int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx$$

$$7. (\int_0^x f(s) ds)' = f(x)$$

$$\begin{aligned} \text{故 } \int_0^x (x-t) f(t) dt &= \int_0^x (x-t) d(\int_0^t f(s) ds) \\ &= (x-t) \int_0^t f(s) ds \Big|_0^x - \int_0^x \int_0^t f(s) ds d(x-t) \\ &= \int_0^x (\int_0^t f(s) ds) dt \end{aligned}$$

$$8. f'(x) = e^{-x^4} \cdot 2x$$

$$\begin{aligned} \int_0^1 x f'(x) dx &= \frac{1}{2} \int_0^1 f'(x) dx^2 \\ &= \frac{1}{2} x^2 f'(x) \Big|_0^1 - \frac{1}{2} \int_0^1 x^2 f''(x) dx \\ &= 0 - 0 - \int_0^1 x^3 e^{-x^4} dx \\ &= -\frac{1}{4} \int_0^1 e^{-u} du \\ &\stackrel{\text{令 } u=x^4}{=} -\frac{1}{4} \int_0^1 e^{-u} du \\ &= \frac{1}{4} e^{-u} \Big|_0^1 = \frac{1}{4e} - \frac{1}{4} \end{aligned}$$

习题 5.7

2. (1) $y^2 - 2y = 2y^2 - 8y + 6$

$y = 3 \pm \sqrt{3}$

$S = \left| \int_{3-\sqrt{3}}^{3+\sqrt{3}} (y^2 - 2y) dy - \int_{3-\sqrt{3}}^{3+\sqrt{3}} (2y^2 - 8y + 6) dy \right|$

$= \int_{3-\sqrt{3}}^{3+\sqrt{3}} (-y^2 + 6y - 6) dy$

$= -\frac{1}{3}y^3 + 3y^2 - 6y \Big|_{3-\sqrt{3}}^{3+\sqrt{3}}$

$= 4\sqrt{3}$

(4) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

$S = 4 \int_0^a y v x dx \quad (y \geq 0)$

$= 4 \int_{\frac{\pi}{2}}^0 a \sin^3 t da \cos^3 t$

$= 12a^2 \int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t dt$

$= 12a^2 \int_0^{\frac{\pi}{2}} (\sin^4 t - \sin^6 t) dt$

$= 12a^2 \left(\frac{\sin^5 x \cos x}{5} \Big|_0^{\frac{\pi}{2}} + \frac{1}{6} \int_0^{\frac{\pi}{2}} \sin^4 t dt \right)$

$= 2a^2 \left(-\frac{\sin^3 x \cos x}{4} \Big|_0^{\frac{\pi}{2}} + \frac{3}{4} \int_0^{\frac{\pi}{2}} \sin^2 t dt \right)$

$= \frac{3}{4} a^2 (x - \sin x \cos x) \Big|_0^{\frac{\pi}{2}}$

$= \frac{3\pi a^2}{8}$

3. (1) $L = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + \left(\int_{-\frac{\pi}{2}}^x |\cos t| dt \right)^2} dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + \cos^2 x} dx$

$= \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \cos \frac{x}{\sqrt{2}} \right| dx$

$= 2\sqrt{2} \sin \frac{x}{\sqrt{2}} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4$

(2) $L = \int_0^1 \sqrt{(\arctan t)' ^2 + (\ln \sqrt{1+t^2})' ^2} dt$

$= \int_0^1 \sqrt{\frac{1}{1+t^2} + \frac{t^2}{(1+t^2)^2}} dt$

$= \int_0^1 \frac{dt}{\sqrt{1+t^2}}$

$= \ln |\sqrt{1+t^2} + t| \Big|_0^1 = \ln(1 + \sqrt{2})$

(5) $L = \int_0^{2\pi} \sqrt{|\rho(\theta)|^2 + |\rho'(\theta)|^2} d\theta$

$= a \int_0^{2\pi} \sqrt{\theta^2 + 1} d\theta$

$= a \left(\frac{\theta \sqrt{\theta^2 + 1}}{2} + \frac{1}{2} \ln |\theta + \sqrt{\theta^2 + 1}| \right) \Big|_0^{2\pi}$

$= \pi \sqrt{4\pi^2 + 1} a + \frac{1}{2} a \ln(2\pi + \sqrt{4\pi^2 + 1})$

$$7. (1) x^2 = x^3$$

$$x = 0, 1$$

$$\begin{aligned} V &= \left| \int_0^1 \pi (x^3)^2 dx - \int_0^1 \pi (x^2)^2 dx \right| \\ &= \left| \int_0^1 \pi (x^6 - x^4) dx \right| \\ &= \pi \left| \left(\frac{x^7}{7} - \frac{x^5}{5} \right) \right|_0^1 = \frac{2\pi}{35} \end{aligned}$$

$$\begin{aligned} (5) V &= \int_{-\pi}^0 \pi (y(x))^2 dx \\ &= \int_{-\pi}^0 \pi (a \sin^2 t)^2 da \cos^2 t \\ &= \int_0^\pi 3\pi a^3 \sin^4 t \cos^2 t dt \\ &= 3\pi a^3 \int_0^\pi (\sin^2 t - \sin^4 t) dt \\ &= 3\pi a^3 \left(\frac{\sin^3 t \cos t}{3} \Big|_0^\pi + \frac{1}{5} \int_0^\pi \sin^2 t dt \right) \\ &= -\frac{16}{105} \pi a^3 \cos t \Big|_0^\pi = \frac{32}{105} \pi a^3 \end{aligned}$$

$$\begin{aligned} 8. (1) S &= \int_0^2 2\pi x \sqrt{1 + (f'(x))^2} dx \\ &= \pi \int_0^2 \sqrt{4x+1} dx \\ \text{令 } x &= \frac{t^2-1}{4} \\ t > 0 \\ &= \frac{\pi t^3}{6} \Big|_1^3 = \frac{13}{2} \pi \end{aligned}$$

$$(3) y = \frac{2}{3} \sqrt{9-x^2}$$

$$\begin{aligned} V_x &= \int_{-3}^3 \pi (y(x))^2 dx \\ &= \frac{4}{9} \pi \int_{-3}^3 (9-x^2) dx \\ &= \frac{4}{9} \pi \left(9x - \frac{x^3}{3} \right) \Big|_{-3}^3 \\ &= 16\pi \end{aligned}$$

$$x = \frac{2}{3} \sqrt{4-y^2}$$

$$\begin{aligned} V_y &= \int_{-2}^2 \pi (x(y))^2 dy \\ &= \frac{9}{4} \pi \int_{-2}^2 (4-y^2) dy \\ &= \frac{9}{4} \pi \left(4y - \frac{y^3}{3} \right) \Big|_{-2}^2 \\ &= 24\pi \end{aligned}$$

$$\begin{aligned} (2) V &= \int_{-\pi}^{\pi} 2\pi (x(t) - a\pi) \sqrt{(x'(t))^2 + (y'(t))^2} dt \\ &= 2\pi a^2 \int_{-\pi}^{\pi} (t - \sin t - \pi) \sqrt{2 - 2\cos t} dt \\ &= 4\pi a^2 \int_{-\pi}^{\pi} (t - \sin t - \pi) \sin \frac{t}{2} dt \\ &= -8\pi a^2 \left((t - \sin t - \pi) \cos \frac{t}{2} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (1 - \cos t) \cos \frac{t}{2} dt \right) \\ \text{令 } \sin \frac{t}{2} &= u \\ &= 8\pi a^2 + 32\pi a^2 \int_0^1 u^2 du \\ &= 8\pi a^2 + \frac{32}{3} \pi a^2 u^3 \Big|_0^1 = 8\pi a^2 + \frac{32}{3} \pi a^2 \end{aligned}$$

第5章总复习题

$$\begin{aligned} 9. \text{原式} &= \lim_{x \rightarrow 0^+} \frac{\int_0^x (f(t+x) - f(t-x)) dt - \int_0^x (f(t+x) - f(t-x)) dt}{4x^2} \\ &= \lim_{x \rightarrow 0^+} \frac{f(2x) - f(-2x)}{8x} \\ &= \frac{1}{4} \lim_{x \rightarrow 0^+} \frac{f(2x) - f(0)}{2x} + \frac{1}{4} \lim_{x \rightarrow 0^+} \frac{f(0) - f(-2x)}{2x} \\ &= \frac{1}{2} f'(0) \end{aligned}$$

$$10. g(a) = \int_0^a f(x) dx$$

$$f(a) = g'(a) = a + \frac{a}{2} \cos a + \frac{1-\pi}{2} \sin a$$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

$$12. \text{令 } t = \frac{1}{x}, \varphi(x) = \int_0^x f(t) \cdot \frac{1}{x} dt \quad (x \neq 0)$$

$$\varphi'(x) = \frac{f(x)}{x} \quad (x \neq 0)$$

$$\varphi'(0) = \lim_{x \rightarrow 0} \frac{\varphi(x) - \varphi(0)}{x} = \lim_{x \rightarrow 0} \varphi'(x) \quad \text{洛必达法则}$$

故 $\varphi'(x)$ 在 $x=0$ 处连续

$$\varphi'(x) = \begin{cases} \frac{f(x)}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

习题 5.7

$$\begin{aligned} 4. (2) k &= \frac{|x''(t)y'(t) - x'(t)y''(t)|}{(|x'(t)|^2 + |y'(t)|^2)^{\frac{3}{2}}} \\ &= \frac{|a^2 \sin^2 t - a^2 \cos^2 t + a^2 \cos^2 t|}{|a^2 - 2a^2 \cos^2 t + a^2 \cos^2 t + a^2 \sin^2 t|^{\frac{3}{2}}} \\ &= \frac{1 - \cos^2 t}{a(2 - 2\cos^2 t)^{\frac{3}{2}}} = \frac{1}{2a(2 - 2\cos^2 t)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} (3) k &= \frac{|y''(x)|}{(1 + |y'(x)|^2)^{\frac{3}{2}}} \\ &= \frac{e^x}{(1 + e^{2x})^{\frac{3}{2}}} \end{aligned}$$

$$5. (1) x = a(1 + \cos \theta) \cos \theta$$

$$y = a(1 + \cos \theta) \sin \theta$$

$$\begin{aligned} k &= \frac{|x''(\theta)y'(\theta) - x'(\theta)y''(\theta)|}{(|x'(\theta)|^2 + |y'(\theta)|^2)^{\frac{3}{2}}} \\ &= \frac{|a \cos^2 \theta + 2a \cos \theta \cos^2 \theta + a \sin^2 \theta + 3a \sin \theta \sin^2 \theta + 2a \sin^2 \theta|}{a(2 + 2\cos^2 \theta + 2\cos^2 \theta + 2\sin^2 \theta + 2\sin^2 \theta)^{\frac{3}{2}}} \\ &= \frac{|2 + 4\cos^2 \theta + 4\cos^2 \theta - 2\cos^2 \theta - 4\cos^2 \theta|}{a(2 + 2\cos^2 \theta)^{\frac{3}{2}}} \\ &= \frac{|1 + \cos^2 \theta + \cos^2 \theta - 2\cos^2 \theta|}{a(2 + 2\cos^2 \theta)^{\frac{3}{2}}} \end{aligned}$$

$$6. k = \frac{|\ln'' x|}{(1 + |\ln' x|^2)^{\frac{3}{2}}}$$

$$= \frac{\frac{1}{x^2}}{(1 + \frac{1}{x^2})^{\frac{3}{2}}}$$

$$= \frac{1}{(x^{\frac{2}{3}} + x^{-\frac{2}{3}})^{\frac{3}{2}}}$$

$x = \frac{\sqrt{2}}{2}$ 时, 取最大值

$$(\frac{\sqrt{2}}{2}, -\frac{1}{2} \ln 2)$$

$$x=1 \text{ 时, } k = \frac{\sqrt{2}}{4}, R = \frac{1}{k} = 2\sqrt{2}$$

$$y''' = 1, \text{ 切线: } y = x - 1$$

设圆心为 (x_0, y_0)

$$\begin{cases} y_0 = x_0 - 1 \\ (x_0 - 1)^2 + y_0^2 = R^2 \end{cases}$$

$$\text{故 } x_0 = 3, y_0 = 2$$

$$(x - 3)^2 + (y - 2)^2 = 8$$

$$12. F = \int_0^b \rho g h \cdot 2\sqrt{3b-h^2} dh$$

$$\underline{h = b \sin t} \int_0^{\frac{\pi}{2}} \rho g b \sin t \cdot 12 \cos t \cdot b \cos t dt$$

$$= 432 \rho g \int_0^{\frac{\pi}{2}} (\sin t - \sin^3 t) dt$$

$$= 432 \rho g \left(\frac{\sin^2 t \cos t}{2} \Big|_0^{\frac{\pi}{2}} + \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin t dt \right)$$

$$= -144 \rho g \cos t \Big|_0^{\frac{\pi}{2}} = 144 \rho g$$

$$1b. V(h) = \int_{-R}^{R-h} \pi (\sqrt{R^2 - x^2})^2 dx$$

$$= \left(\pi R^2 x - \frac{\pi}{3} x^3 \right) \Big|_{-R}^{R-h}$$

$$= \frac{\pi}{3} (4R^3 - R^2 h - 2Rh^2 + h^3)$$

$$W = \int_0^{2R} \left(\rho g \cdot \frac{4}{3} \pi R^3 - \rho g V(h) \right) dh$$

$$= \rho g \int_0^{2R} (R^2 h + 2Rh^2 - h^3) dh$$

$$= \rho g \left(\frac{1}{2} R^2 h^2 + \frac{2}{3} R h^3 - \frac{1}{4} h^4 \right) \Big|_0^{2R}$$

$$= \frac{40}{3} R^4$$

习题 6.1

$$\begin{aligned}
 2. (1) \forall A > 1, \int_1^A \frac{1}{x(x+1)} dx &= \int_1^A \left(\frac{1}{x} - \frac{1}{x+1} \right) dx \\
 &= (\ln x - \ln(x+1)) \Big|_1^A \\
 &= \ln \frac{A}{A+1}
 \end{aligned}$$

$$\int_1^{+\infty} \frac{1}{x(x+1)} dx = \lim_{A \rightarrow +\infty} \int_1^A \frac{1}{x(x+1)} dx = \ln 2$$

$$\begin{aligned}
 13) \forall A > 0, \int_0^A e^{-x} \sin x dx &= -e^{-x} \cos x \Big|_0^A - \int_0^A e^{-x} \cos x dx \\
 &= -e^{-x} \cos x \Big|_0^A - e^{-x} \sin x \Big|_0^A - \int_0^A e^{-x} \sin x dx \\
 &= -e^{-A} (\cos A + \sin A) + 1 - \int_0^A e^{-x} \sin x dx
 \end{aligned}$$

$$\text{故 } \int_0^A e^{-x} \sin x dx = \frac{1}{2} - \frac{e^{-A}}{2} (\cos A + \sin A)$$

$$\int_0^{+\infty} e^{-x} \sin x dx = \lim_{A \rightarrow +\infty} \int_0^A e^{-x} \sin x dx = \frac{1}{2}$$

17) $x=0$ 为瑕点

$$\forall \varepsilon \in (0, 1), \int_{\varepsilon}^1 \frac{e^{-\frac{1}{x}}}{x^2} dx \stackrel{\text{令 } u = -\frac{1}{x}}{=} \int_{-\frac{1}{\varepsilon}}^{-1} e^u du = e^u \Big|_{-\frac{1}{\varepsilon}}^{-1} = \frac{1}{e} - e^{-\frac{1}{\varepsilon}}$$

$$\int_0^1 \frac{e^{-\frac{1}{x}}}{x^2} dx = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \frac{e^{-\frac{1}{x}}}{x^2} dx = \frac{1}{e}$$

18) $x=e$ 为瑕点

$$\begin{aligned}
 \forall \varepsilon \in (0, e-1), \int_1^{e-\varepsilon} \frac{dx}{x \sqrt{1-(\ln x)^2}} &\stackrel{\text{令 } u = \ln x}{=} \int_0^{\ln(e-\varepsilon)} \frac{du}{\sqrt{1-u^2}} \\
 &= \arcsin u \Big|_0^{\ln(e-\varepsilon)} = \arcsin(\ln(e-\varepsilon))
 \end{aligned}$$

$$\int_1^e \frac{dx}{x \sqrt{1-(\ln x)^2}} = \lim_{\varepsilon \rightarrow 0^+} \int_1^{e-\varepsilon} \frac{dx}{x \sqrt{1-(\ln x)^2}} = \frac{\pi}{2}$$

$$3. (1) \int_1^{+\infty} \frac{1}{x^2 \sqrt{1+x^2}} dx = \lim_{A \rightarrow +\infty} \int_1^A \frac{1}{x^2 \sqrt{1+x^2}} dx \stackrel{\text{令 } x = \tan t}{=} \lim_{t \in [\frac{\pi}{4}, \frac{\pi}{2})} \int_{\frac{\pi}{4}}^{\arctan A} \frac{\cos t}{\sin^3 t} dt$$

$$= \lim_{A \rightarrow +\infty} \int_{\frac{\pi}{4}}^{\arctan A} \frac{d(\sin t)}{\sin^3 t}$$

$$= \lim_{A \rightarrow +\infty} \left(-\frac{1}{\sin t} \right) \Big|_{\frac{\pi}{4}}^{\arctan A}$$

$$= \lim_{A \rightarrow +\infty} \left(\sqrt{2} - \frac{\sqrt{A^2+1}}{A} \right) = \sqrt{2} - 1$$

$$(b) \int_0^1 \frac{1}{(2+x)\sqrt{1-x}} dx = \lim_{\varepsilon \rightarrow 0^+} \int_0^{1-\varepsilon} \frac{1}{(2+x)\sqrt{1-x}} dx \stackrel{\text{令 } t = \sqrt{1-x}}{=} \lim_{t \rightarrow 0^+} \int_{\frac{\sqrt{1-\varepsilon}}{2}}^{\frac{\sqrt{1-0}}{2}} \frac{-2}{(2+\frac{1-t^2}{2})t} dt$$

$$= \lim_{\varepsilon \rightarrow 0^+} \int_{\frac{\sqrt{1-\varepsilon}}{2}}^{\frac{1}{2}} \left(\frac{1}{\sqrt{1-\varepsilon} + t} + \frac{1}{\sqrt{1-\varepsilon} - t} \right) dt$$

$$= \lim_{\varepsilon \rightarrow 0^+} \left(\frac{1}{2} \ln \frac{\sqrt{1-\varepsilon} + t}{\sqrt{1-\varepsilon} - t} \right) \Big|_{\frac{\sqrt{1-\varepsilon}}{2}}^{\frac{1}{2}}$$

$$= \lim_{\varepsilon \rightarrow 0^+} \left(\frac{1}{2} \ln(2 + \sqrt{1-\varepsilon}) - \frac{1}{2} \ln \frac{\sqrt{1-\varepsilon} + \frac{\sqrt{1-\varepsilon}}{2}}{\sqrt{1-\varepsilon} - \frac{\sqrt{1-\varepsilon}}{2}} \right) = \frac{1}{2} \ln(2 + \sqrt{1-\varepsilon})$$

4.11) 瑕积分. $x = -2, 2$ 为瑕点

$$\begin{aligned} \int_{-2}^2 \frac{2x}{x^2-4} dx &= \lim_{\varepsilon \rightarrow 0^+} \int_{-2-\varepsilon}^0 \frac{2x}{x^2-4} dx + \lim_{\varepsilon \rightarrow 0^+} \int_0^{2-\varepsilon} \frac{2x}{x^2-4} dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{-2-\varepsilon}^0 \left(\frac{1}{x-2} + \frac{1}{x+2} \right) dx + \lim_{\varepsilon \rightarrow 0^+} \int_0^{2-\varepsilon} \left(\frac{1}{x-2} + \frac{1}{x+2} \right) dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \left(\ln|x-2| + \ln|x+2| \right) \Big|_{-2-\varepsilon}^0 + \lim_{\varepsilon \rightarrow 0^+} \left(\ln|x-2| + \ln|x+2| \right) \Big|_0^{2-\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} (2\ln 2 - \ln \varepsilon - \ln(4-\varepsilon)) + \lim_{\varepsilon \rightarrow 0^+} (\ln \varepsilon + \ln(4-\varepsilon) - 2\ln 2) \\ &= \lim_{\varepsilon \rightarrow 0^+} 1 - \ln \varepsilon + \lim_{\varepsilon \rightarrow 0^+} \ln \varepsilon \end{aligned}$$

发散

13) 混合. $x=1$ 为瑕点

$$\begin{aligned} \int_1^{+\infty} \frac{1}{x\sqrt{x^2-1}} dx &= \lim_{\varepsilon \rightarrow 0^+} \int_{1+\varepsilon}^{\sqrt{2}} \frac{1}{x\sqrt{x^2-1}} dx + \lim_{A \rightarrow +\infty} \int_{\sqrt{2}}^A \frac{1}{x\sqrt{x^2-1}} dx \\ &\stackrel{\text{令 } x = \frac{1}{\cos t}}{\text{令 } x = \frac{1}{\cos t}} \lim_{\varepsilon \rightarrow 0^+} \int_{\arccos \frac{1}{1+\varepsilon}}^{\frac{\pi}{4}} \frac{\frac{\sin t}{\cos^2 t}}{\frac{1}{\cos t} \cdot \frac{\sin t}{\cos t}} dt + \lim_{A \rightarrow +\infty} \int_{\frac{\pi}{4}}^{\arccos \frac{1}{A}} \frac{\frac{\sin t}{\cos^2 t}}{\frac{1}{\cos t} \cdot \frac{\sin t}{\cos t}} dt \\ &= \lim_{\varepsilon \rightarrow 0^+} t \Big|_{\arccos \frac{1}{1+\varepsilon}}^{\frac{\pi}{4}} + \lim_{A \rightarrow +\infty} t \Big|_{\frac{\pi}{4}}^{\arccos \frac{1}{A}} \\ &= \lim_{\varepsilon \rightarrow 0^+} \left(\frac{\pi}{4} - \arccos \frac{1}{1+\varepsilon} \right) + \lim_{A \rightarrow +\infty} \left(\arccos \frac{1}{A} - \frac{\pi}{4} \right) \\ &= \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{2} \end{aligned}$$

习题 6.2

4.12) $0 < \frac{\arctan x}{x^2} \leq \frac{\arctan 1}{x^2} = \frac{\pi}{4x^2} \quad (x \geq 1)$

由 $\int_1^{+\infty} \frac{\pi}{4x^2} dx$ 收敛, $\int_1^{+\infty} \frac{\arctan x}{x^2} dx$ 收敛

16) $\int_0^1 \frac{1}{\sqrt{x-x^2}} dx = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{x-x^2}} dx + \int_{\frac{1}{2}}^1 \frac{1}{\sqrt{x-x^2}} dx$

$\lim_{x \rightarrow 0^+} x^{\frac{1}{2}} \cdot \frac{1}{\sqrt{x-x^2}} = \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{1-x}} = 1 \quad \frac{1}{2} < 1$, 故 $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{x-x^2}} dx$ 收敛

$\lim_{x \rightarrow 1^-} (1-x)^{\frac{1}{2}} \cdot \frac{1}{\sqrt{x-x^2}} = \lim_{x \rightarrow 1^-} \frac{1}{\sqrt{x}} = 1 \quad \frac{1}{2} < 1$, 故 $\int_{\frac{1}{2}}^1 \frac{1}{\sqrt{x-x^2}} dx$ 收敛

故 $\int_0^1 \frac{1}{\sqrt{x-x^2}} dx$ 收敛

110) $\lim_{x \rightarrow 0^+} x^{\frac{1}{2}} \cdot \frac{\sqrt{x}}{\sin x} = \lim_{x \rightarrow 0^+} \frac{x}{\sin x} = 1 \quad \frac{1}{2} < 1$, 故 $\int_0^1 \frac{\sqrt{x}}{\sin x} dx$ 收敛

$\lim_{x \rightarrow 0^+} \frac{\frac{\sqrt{x}}{e^{\sin x} - 1}}{\frac{\sqrt{x}}{\sin x}} = \lim_{x \rightarrow 0^+} \frac{\sin x}{e^{\sin x} - 1} = 1$ 由 $\int_0^1 \frac{\sqrt{x}}{\sin x} dx$ 收敛, $\int_0^1 \frac{\sqrt{x}}{e^{\sin x} - 1} dx$ 收敛

$$5.12) \int_0^{+\infty} \frac{\ln(1+x^2)}{x^p} dx = \int_0^1 \frac{\ln(1+x^2)}{x^p} dx + \int_1^{+\infty} \frac{\ln(1+x^2)}{x^p} dx$$

$$\lim_{x \rightarrow 0^+} x^{p-2} \cdot \frac{\ln(1+x^2)}{x^p} = \lim_{x \rightarrow 0^+} \frac{\ln(1+x^2)}{x^2} = 1 \quad \text{故 } p < 3 \Leftrightarrow \int_0^1 \frac{\ln(1+x^2)}{x^p} dx \text{ 收敛}$$

当 $p > 1$ 时, 取 $q = \frac{p+1}{2}$

$$\lim_{x \rightarrow +\infty} x^q \cdot \frac{\ln(1+x^2)}{x^p} = \lim_{x \rightarrow +\infty} \frac{\ln(1+x^2)}{x^{p-q}} = \lim_{x \rightarrow +\infty} \frac{2x^2}{(p-q)x^{p-q}(1+x^2)} = 0$$

由 $q > 1$, $\int_1^{+\infty} \frac{\ln(1+x^2)}{x^p} dx$ 收敛

当 $p \leq 1$ 时, $\frac{\ln(1+x^2)}{x^p} \geq \frac{\ln 2}{x^p} (x \geq 1)$

由 $\int_1^{+\infty} \ln 2 \cdot \frac{1}{x^p} dx$ 发散, $\int_1^{+\infty} \frac{\ln(1+x^2)}{x^p} dx$ 发散

故 $1 < p < 3 \Leftrightarrow \int_0^{+\infty} \frac{\ln(1+x^2)}{x^p} dx$ 收敛

$$17) \lim_{x \rightarrow 0^+} x^{\frac{1}{2}} \cdot |\ln \sin x|^p = \lim_{x \rightarrow 0^+} \left| \frac{\ln \sin x}{x^{-\frac{1}{2}}} \right|^p = \lim_{x \rightarrow 0^+} \left| 2^{\frac{1}{2}} x^{\frac{1}{2}} \cos x \cdot \frac{x}{\sin x} \right|^p = 0$$

由 $\frac{1}{2} < 1$, $\int_0^{\frac{\pi}{2}} |\ln \sin x|^p dx$ 收敛

$$115) \int_0^1 x^{p-1} (1-x)^{q-1} \ln x dx = \int_0^{\frac{1}{2}} x^{p-1} (1-x)^{q-1} \ln x dx + \int_{\frac{1}{2}}^1 x^{p-1} (1-x)^{q-1} \ln x dx$$

$$\lim_{x \rightarrow 1^-} (1-x)^{q-1} \cdot (x^{p-1} (1-x)^{q-1} \ln x) = \lim_{x \rightarrow 1^-} \frac{-\ln x}{1-x} = \lim_{x \rightarrow 1^-} \frac{1}{x} = 1$$

故 $q > -1 \Leftrightarrow \int_{\frac{1}{2}}^1 x^{p-1} (1-x)^{q-1} \ln x dx$ 收敛

$$-x^{p-1} (1-x)^{q-1} \ln x \geq x^{p-1} (1-x)^{q-1} \ln 2 \geq 0 \quad (0 \leq x \leq \frac{1}{2})$$

当 $p \leq 0$ 时, $\int_0^{\frac{1}{2}} x^{p-1} (1-x)^{q-1} \ln 2 dx$ 发散, 故 $\int_0^{\frac{1}{2}} -x^{p-1} (1-x)^{q-1} \ln x dx$ 发散

$$\text{当 } p > 0 \text{ 时, } \lim_{x \rightarrow 0^+} x^{1-\frac{p}{2}} \cdot (-x^{p-1} (1-x)^{q-1} \ln x) = \lim_{x \rightarrow 0^+} \frac{-\ln x}{x^{-\frac{p}{2}}} = \lim_{x \rightarrow 0^+} \frac{p}{2} x^{\frac{p}{2}} = 0$$

故 $\int_0^{\frac{1}{2}} x^{p-1} (1-x)^{q-1} \ln x dx$ 收敛

综上, $p > 0, q > -1 \Leftrightarrow \int_0^1 x^{p-1} (1-x)^{q-1} \ln x dx$ 收敛

$$6.11) \int_a^{+\infty} \frac{f^2(x) + g^2(x)}{2} dx = \frac{1}{2} \int_a^{+\infty} f^2(x) dx + \frac{1}{2} \int_a^{+\infty} g^2(x) dx \text{ 收敛}$$

$$\frac{f^2(x) + g^2(x)}{2} \geq \sqrt{f^2(x)g^2(x)} = |f(x)g(x)|$$

故 $\int_a^{+\infty} |f(x)g(x)| dx$ 收敛, 即 $\int_a^{+\infty} f(x)g(x) dx$ 绝对收敛

$$(2) \int_a^{+\infty} (f(x) + g(x))^2 dx = \int_a^{+\infty} f^2(x) dx + \int_a^{+\infty} g^2(x) dx + 2 \int_a^{+\infty} f(x)g(x) dx$$

故 $\int_a^{+\infty} (f(x) + g(x))^2 dx$ 收敛

7. 由 $\int_0^{+\infty} g(x) dx, \int_0^{+\infty} h(x) dx$ 收敛

$\forall \varepsilon > 0, \exists M_1 > 0$. 当 $A_2 > A_1 > M_1$ 时, $|\int_{A_1}^{A_2} g(x) dx| < \varepsilon$

$\exists M_2 > 0$. 当 $A_2 > A_1 > M_2$ 时, $|\int_{A_1}^{A_2} h(x) dx| < \varepsilon$

取 $M = \max\{M_1, M_2\}$. 当 $A_2 > A_1 > M$ 时, 由 $g(x) \leq f(x) \leq h(x)$

$$\int_{A_1}^{A_2} g(x) dx \leq \int_{A_1}^{A_2} f(x) dx \leq \int_{A_1}^{A_2} h(x) dx$$

故 $-\varepsilon < \int_{A_1}^{A_2} f(x) dx < \varepsilon$. 所以 $\int_0^{+\infty} f(x) dx$ 收敛

9. (2) $\forall A > 0, \int_0^A |x \cos x^3| dx = \int_0^{A^3} |u^{\frac{1}{3}} \cos u| du^{\frac{1}{3}} = \int_0^{A^3} \frac{|\cos u|}{3u^{\frac{2}{3}}} du \leq \int_0^{A^3} \frac{1}{3u^{\frac{2}{3}}} du$

由 $\int_0^{+\infty} \frac{1}{3u^{\frac{2}{3}}} du$ 收敛, $\int_0^{+\infty} |x \cos x^3| dx$ 收敛, 即 $\int_0^{+\infty} x \cos x^3 dx$ 绝对收敛

(4) $\int_0^{+\infty} \left| \frac{x \sin x}{x+1} \right| dx = \int_0^{+\infty} \frac{\sqrt{x} |\sin x|}{\sqrt{x}+1} dx$

$$\frac{\sqrt{x} |\sin x|}{\sqrt{x}+1} \geq \frac{\sqrt{x} \sin^2 x}{\sqrt{x}+1} = \frac{\sqrt{x}}{2(\sqrt{x}+1)} - \frac{\sqrt{x} \cos 2x}{2(\sqrt{x}+1)}$$

$$\lim_{x \rightarrow +\infty} \frac{\sqrt{x}}{2(\sqrt{x}+1)} = \frac{1}{2} \quad \frac{1}{2} < 1, \text{ 故 } \int_0^{+\infty} \frac{\sqrt{x}}{2(\sqrt{x}+1)} dx \text{ 发散}$$

$$\lim_{x \rightarrow +\infty} \frac{\frac{\sqrt{x} \cos 2x}{2(\sqrt{x}+1)}}{\frac{\cos 2x}{\sqrt{x}}} = \lim_{x \rightarrow +\infty} \frac{x}{2(\sqrt{x}+1)} = \frac{1}{2} \quad \text{由于 } \int_1^{+\infty} \frac{\cos 2x}{\sqrt{x}} dx \text{ 收敛, } \int_0^{+\infty} \frac{\sqrt{x} \cos 2x}{2(\sqrt{x}+1)} dx \text{ 收敛}$$

故 $\int_0^{+\infty} \frac{\sqrt{x} \sin^2 x}{\sqrt{x}+1} dx$ 发散, 从而 $\int_0^{+\infty} \left| \frac{\sqrt{x} \sin x}{\sqrt{x}+1} \right| dx$ 发散

$$\forall A > 0, \left| \int_0^A \sin x dx \right| = |1 - \cos A| \leq 2 \quad \text{且 } \lim_{x \rightarrow +\infty} \frac{\sqrt{x}}{\sqrt{x}+1} = 0$$

由 Dirichlet 定理, $\int_0^{+\infty} \frac{\sqrt{x} \sin x}{\sqrt{x}+1} dx$ 收敛

故 $\int_0^{+\infty} \frac{\sqrt{x} \sin x}{\sqrt{x}+1} dx$ 条件收敛

第6章总复习题

2. (1) $\int_0^{+\infty} \frac{\sin^2 x}{x^p} dx = \int_0^1 \frac{\sin^2 x}{x^p} dx + \int_1^{+\infty} \frac{\sin^2 x}{x^p} dx$

$$\lim_{x \rightarrow 0^+} x^{p-2} \cdot \frac{\sin^2 x}{x^p} = \lim_{x \rightarrow 0^+} \frac{\sin^2 x}{x^2} = 1$$

故 $\int_0^1 \frac{\sin^2 x}{x^p} dx$ 收敛 $\Leftrightarrow p < 3$

$$\frac{\sin^2 x}{x^p} \leq \frac{1}{x^p}, \quad p > 1 \text{ 时, } \int_1^{+\infty} \frac{1}{x^p} dx \text{ 收敛}$$

故 $p > 1$ 时, $\int_1^{+\infty} \frac{\sin^2 x}{x^p} dx$ 收敛

$$\frac{\sin^2 x}{x^p} = \frac{1}{2x^p} - \frac{\cos 2x}{2x^p}, \quad 0 < p \leq 1 \text{ 时, } \int_1^{+\infty} \frac{1}{2x^p} dx \text{ 发散, } \int_1^{+\infty} \frac{\cos 2x}{2x^p} dx \text{ 收敛}$$

故 $0 < p \leq 1$ 时, $\int_1^{+\infty} \frac{\sin^2 x}{x^p} dx$ 发散

$p \leq 0$ 时, $\frac{\sin^2 x}{x^p} \geq \sin^2 x$, $\int_1^{+\infty} \sin^2 x dx$ 发散

故 $p \leq 0$ 时, $\int_1^{+\infty} \frac{\sin^2 x}{x^p} dx$ 发散

综上, $\int_0^{+\infty} \frac{\sin^2 x}{x^p} dx$ 收敛 $\Leftrightarrow 1 < p < 3$

3. $\forall \varepsilon > 0$, 取 $M_1 = \frac{1}{\arcsin \frac{\varepsilon}{2}}$

$$\begin{aligned} \text{当 } A_2 > A_1 > M_1 \text{ 时, } \left| \int_{A_1}^{A_2} \sin x \sin \frac{1}{x} dx \right| & \stackrel{\exists \xi \in [A_1, A_2]}{=} \left| \sin \frac{1}{\xi} \int_{A_1}^{A_2} \sin x dx \right| \\ & = \left| \sin \frac{1}{\xi} (\cos A_1 - \cos A_2) \right| \\ & \leq 2 \sin \frac{1}{\xi} < 2 \sin \frac{1}{M_1} = \varepsilon \end{aligned}$$

故 $\int_1^{+\infty} \sin x \sin \frac{1}{x} dx$ 收敛

取 $\varepsilon_0 = \frac{1}{2\pi} > 0$, $\forall M > 1$

由 $\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$, $\exists x_0 > 0$, 当 $0 < x < x_0$ 时, $\frac{\sin x}{x} > \frac{1}{2}$

设 $k = \left[\frac{M + \frac{1}{2x_0}}{\pi} \right] + 1$, 取 $A_1 = k\pi$, $A_2 = 2k\pi$

$$\begin{aligned} \left| \int_{A_1}^{A_2} \sin x \sin \frac{1}{x} dx \right| & \stackrel{\exists \xi \in [A_1, A_2]}{=} \left| \sin \frac{1}{\xi} \int_{k\pi}^{2k\pi} \sin x dx \right| \\ & = k \sin \frac{1}{\xi} \int_0^\pi \sin x dx \\ & = 2k \sin \frac{1}{\xi} \geq 2k \sin \frac{1}{2k\pi} = \frac{\sin \frac{1}{2k\pi}}{\frac{1}{2k\pi}} \cdot \frac{1}{\pi} > \frac{1}{2\pi} = \varepsilon_0. \end{aligned}$$

故 $\int_1^{+\infty} \sin x \sin \frac{1}{x} dx$ 发散

故 $\int_1^{+\infty} \sin x \sin \frac{1}{x} dx$ 条件收敛

5. 设 $F(A) = \int_0^A f(x) dx$

$\int_0^{+\infty} f(x) dx$ 收敛, 即 $\lim_{A \rightarrow +\infty} F(A)$ 存在

故 $\forall \varepsilon > 0$, $\exists M > 0$, 当 $A_1, A_2 > M$ 时, $|F(A_1) - F(A_2)| < \varepsilon$

$$|F(A)| = |F'(A)| = \left| \lim_{\varepsilon \rightarrow 0^+} \frac{F(A+\varepsilon) - F(A)}{\varepsilon} \right|$$

当 $A > M$ 时, $|F(A+\varepsilon) - F(A)| < \varepsilon$

故 $|F'(A)| < \lim_{\varepsilon \rightarrow 0^+} \varepsilon$

即 $\lim_{x \rightarrow +\infty} f(x) = 0$

习题 7.1

$$2. y_1'' + a_1(x)y_1' + a_2(x)y_1 = f(x)$$

$$y_2'' + a_1(x)y_2' + a_2(x)y_2 = g(x)$$

$$(k_1 y_1 + k_2 y_2)'' + a_1(x)(k_1 y_1 + k_2 y_2)' + a_2(x)(k_1 y_1 + k_2 y_2)$$

$$= k_1 y_1'' + k_2 y_2'' + a_1(x)(k_1 y_1' + k_2 y_2') + a_2(x)(k_1 y_1 + k_2 y_2)$$

$$= k_1 (y_1'' + a_1(x)y_1' + a_2(x)y_1) + k_2 (y_2'' + a_1(x)y_2' + a_2(x)y_2)$$

$$= k_1 f(x) + k_2 g(x)$$

故 $k_1 y_1 + k_2 y_2$ 为 $y'' + a_1(x)y' + a_2(x)y = k_1 f(x) + k_2 g(x)$ 的解

$$3. (3) y' = e^{x^2} \cdot e^{-x^2} + 2xe^{x^2} (1 + \int_0^x e^{-t^2} dt)$$

$$= 1 + 2xy$$

$$\text{故 } y' - 2xy = 1$$

是

$$14) y' = -C_1 w \sin wx + C_2 w \cos wx - \frac{1}{1+w^2} e^{-x}$$

$$y'' = -C_1 w^2 \cos wx - C_2 w^2 \sin wx + \frac{1}{1+w^2} e^{-x}$$

$$y'' + w^2 y = \frac{1}{1+w^2} e^{-x} + \frac{w^2}{1+w^2} e^{-x} = e^{-x}$$

是

$$15) y' = -\frac{1}{x^2} + \frac{1}{2}$$

$$1 - \frac{y}{x} = 1 - \frac{1}{x^2} - \frac{1}{2} = -\frac{1}{x^2} + \frac{1}{2}$$

$$\text{故 } \frac{dy}{dx} = 1 - \frac{y}{x}$$

$$y|_{x=1} = \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

是

$$16) y' - 1 = C y e^y$$

$$(1 - C e^y) y' = 1$$

$$x - y + 1 = 1 - C e^y$$

$$\text{故 } (x - y + 1) y' = 1$$

是

$$4.12) y' = k e^{kx}$$

$$y'' = k^2 e^{kx}$$

$$k^2 e^{kx} - 3k e^{kx} - 4e^{kx} = 0$$

$$(k-4)(k+1)e^{kx} = 0$$

$$k = 4 \text{ 或 } -1$$

$$5. y' = 3Cx^2$$

$$3y - xy' = 3Cx^3 - x \cdot 3Cx^2 = 0$$

$$A(1, 1) \text{ 代入 } C = 1, y = x^3$$

$$B(1, -\frac{1}{3}) \text{ 代入 } C = -\frac{1}{3}, y = -\frac{1}{3}x^3$$

习题 7.2

$$1.18) \frac{y'}{y^2+1} = \frac{x}{x+1}$$

$$\int \frac{y'}{y^2+1} dx = \int \frac{x}{x+1} dx$$

$$\int \frac{1}{y^2+1} dy = \int (1 - \frac{1}{x+1}) dx$$

$$\arctan y = x - \ln|x+1| + C$$

$$110) yy' = \frac{e^x}{1+e^x}$$

$$\int_1^x y(t)y'(t)dt = \int_1^x \frac{e^t}{1+e^t} dt$$

$$\int_1^x y(t)dy(t) = \int_1^x \frac{1}{1+e^t} de^t$$

$$\frac{y^2(t)}{2} \Big|_1^x = \ln|1+e^t| \Big|_1^x$$

$$\frac{y^2}{2} - \frac{1}{2} = \ln|1+e^x| - \ln 2$$

$$2.14) (ye^{\int_0^x t dt})' = e^{\int_0^x t dt} x^3$$

$$ye^{\int_0^x t dt} = \int_0^x e^{\int_0^s t dt} s^3 ds$$

$$= \int_0^x e^{\frac{s^2}{2}} s^3 ds$$

$$= e^{\frac{s^2}{2}} s^2 \Big|_0^x - \int_0^x e^{\frac{s^2}{2}} ds^2$$

$$= e^{\frac{x^2}{2}} x^2 - (2e^{\frac{x^2}{2}} - 2)$$

$$\text{故 } ye^{\frac{x^2}{2}} = (x^2 - 2)e^{\frac{x^2}{2}} + 2$$

$$17) y' - 2xy = \frac{e^{x^2}}{(x+1)^2}$$

$$(ye^{\int_0^x -2t dt})' = e^{\int_0^x -2t dt} \cdot \frac{e^{x^2}}{(x+1)^2}$$

$$ye^{-x^2} - 5 = \int_0^x e^{\int_0^s -2t dt} \frac{e^{s^2}}{(s+1)^2} ds$$

$$= \int_0^x \frac{1}{(s+1)^2} ds$$

$$= -\frac{1}{s+1} \Big|_0^x = 1 - \frac{1}{x+1}$$

$$\text{故 } ye^{-x^2} - 5 = \frac{x}{x+1}$$

$$3. (5) \text{ 令 } u = xy, u' = \frac{1}{x} \ln u$$

$$\frac{u'}{u \ln u} = \frac{1}{x}$$

$$\int \frac{1}{x} dx = \int \frac{u'}{u \ln u} dx$$

$$= \int \frac{1}{\ln u} d \ln u$$

$$= \ln \ln u + C$$

$$\ln x = \ln \ln xy + C$$

$$\text{故 } x = C \ln xy$$

$$17) \text{ 令 } y = ux, u'x + u = \frac{u^2 + 1}{2}$$

$$\frac{2u'}{(u-1)^2} = \frac{1}{x}$$

$$\int \frac{1}{x} dx = \int \frac{2u'}{(u-1)^2} dx$$

$$= \int \frac{2}{(u-1)^2} du$$

$$= -\frac{2}{u-1}$$

$$\text{故 } \ln x + \frac{2x}{y-x} = C$$

$$11) \text{ 令 } u = \frac{y+3}{x+1}, u'x + u + u' = \frac{u-1}{u+1}$$

$$-\frac{(u+1)u'}{u^2+1} = \frac{1}{x+1}$$

$$\int \frac{1}{x+1} dx = \int -\frac{(u+1)u'}{u^2+1} dx$$

$$= -\int \left(\frac{u}{u^2+1} + \frac{1}{u^2+1} \right) du$$

$$= -\int \frac{1}{2(u^2+1)} du^2 - \arctan u + C$$

$$= -\frac{1}{2} \ln(u^2+1) - \arctan u + C$$

$$\text{故 } \ln(x+1) + \frac{1}{2} \ln \left(\frac{(y+3)^2}{(x+1)^2} + 1 \right) + \arctan \frac{y+3}{x+1} + C = 0$$

$$4. \begin{cases} y' = \frac{3y}{x} \\ |y|-1=1 \end{cases} \quad \frac{y'}{y} = \frac{3}{x}$$

$$\Rightarrow \int_{-1}^x \frac{y'}{y} dt = \int_{-1}^x \frac{3}{t} dt$$

$$\int_1^y \frac{1}{u} du = 3 \ln |t| \Big|_{-1}^x$$

$$\ln |y| - \ln 1 = 3(\ln |x| - \ln 1)$$

$$\text{故 } y = -x^3$$

$$1.15) \frac{\cos y}{\sin y} dy = \frac{\sin x}{\cos x} dx$$

$$\int \frac{d \sin y}{\sin y} = - \int \frac{d \cos x}{\cos x}$$

$$\ln |\sin y| = - \ln |\cos x| + C$$

$$\text{故 } \cos x \sin y = C$$

$$1.11) \frac{2dx}{x^2-1} = \frac{\sin y}{\cos y} dy$$

$$\int_{\frac{1}{2}}^x \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt = - \int_{\frac{1}{2}}^y \cos y d \cos y$$

$$\ln \frac{1-x}{1+x} - \ln \frac{1}{2} = - \ln \cos y + \ln \frac{1}{2}$$

$$\text{故 } \frac{1-x}{1+x} \cos y = \frac{1}{2}$$

$$3.11) \text{ 令 } u = z - x + y, u' + 1 = u^2$$

$$\int \frac{u'}{u^2-1} dx = \int 1 dx$$

$$\int_{\frac{1}{2}}^x \left(\frac{1}{u-1} - \frac{1}{u+1} \right) du = x + C$$

$$\frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| = x + C$$

$$\text{故 } \frac{1-x+y}{z-x+y} = Ce^{2x}$$

$$1.14) \frac{y'}{y^2} + \frac{2x}{y} = 2x^3$$

$$\text{令 } u = \frac{1}{y}, -u' + 2xu = 2x^3$$

$$(ue^{\int_0^x -2tdt})' = e^{\int_0^x -2tdt} \cdot 2x^3$$

$$ue^{\int_0^x -2tdt} + C = \int e^{\int_0^x -2tdt} \cdot 2x^3 dx$$

$$= \int -x^2 de^{-x^2}$$

$$= -x^2 e^{-x^2} - e^{-x^2}$$

$$\text{故 } \frac{1}{y} + Ce^{x^2+x^2+1} = 0$$

$$1.8) \text{ 令 } x = a \cos \theta, y = a \sin \theta$$

$$a \cos \theta da \sin \theta - a \sin \theta da \cos \theta = a da \cos \theta$$

$$a^2 \cos^2 \theta d\theta + a^2 \sin^2 \theta d\theta = a^2 d \cos \theta$$

$$\int a^2 d\theta = \int a^2 d \cos \theta$$

$$\theta = \cos \theta + C$$

$$\text{故 } \arctan \frac{y}{x} = \frac{x}{\sqrt{x^2+y^2}} + C$$

习题 7.3

$$12) \text{ 令 } p(x) = y'(x), xp' + p^2 - p = 0$$

$$\frac{p'}{p^2-p} = \frac{1}{x}$$

$$\int_1^x \frac{p'(t)}{p(t)^2-p(t)} dt = \int_1^x \frac{1}{t} dt$$

$$\ln x = \int_1^x \left(\frac{1}{p(t)} + \frac{1}{1-p(t)} \right) dp(t)$$

$$= \ln \frac{p}{1-p}$$

$$y' = 1 - \frac{1}{x+1}$$

$$\text{故 } y = x - \ln(x+1)$$

$$17) \text{ 令 } p(x) = y'(x), xp' - p \ln p + p = 0$$

$$\frac{p'}{p(\ln p - 1)} = \frac{1}{x}$$

$$\int \frac{p'}{p(\ln p - 1)} dx = \int \frac{1}{x} dx$$

$$\ln x + C = \int \frac{1}{\ln p - 1} d \ln p$$

$$= \ln |\ln p - 1|$$

$$y' = e^{Cx+1}$$

$$\text{故 } y = \frac{1}{C_1} e^{Cx+1} + C_2$$

第7章总复习题

1. 13) 令 $u = \sin y$, $(1-x^2)u' + xu = 1$

$$(ue^{\int_0^x \frac{1}{1-t^2} dt})' = e^{\int_0^x \frac{1}{1-t^2} dt} \cdot \frac{1}{1-x^2}$$

$$ue^{-\frac{1}{2}\ln|1-x^2|} + C = \int e^{-\frac{1}{2}\ln|1-x^2|} \cdot \frac{1}{1-x^2} dx$$

$$= \int (1-x^2)^{-\frac{3}{2}} dx$$

$$\underline{\text{令 } x = \sin t} \int \frac{1}{\cos^3 t} d(\sin t)$$

$$= \int \frac{1}{\cos^3 t} dt$$

$$= \tan t = \frac{x}{\sqrt{1-x^2}}$$

$$\sin y \cdot \frac{1}{\sqrt{1-x^2}} + C = \frac{x}{\sqrt{1-x^2}}$$

$$\text{故 } \sin y = x - C\sqrt{1-x^2}$$

17) $(x^4 + y^2)y' = 4x^3y$

$$\left(\frac{y}{x^4 - y^2}\right)' = \frac{y'x^4 + y'y^2 - 4x^3y}{(x^4 - y^2)^2} = 0$$

$$\frac{y}{x^4 - y^2} = C$$

$$\text{故 } Cy^2 + y = Cx^4$$

15) 令 $u = \frac{y}{1+x}$, $u' = \frac{y'(1+x) - y}{(1+x)^2}$

$$(1+x)u' = y^2 = (1+x)^2 u^2$$

$$\frac{u'}{u^2} = 1+x$$

$$\int (1+x) dx = \int \frac{u'}{u^2} dx$$

$$= \int \frac{1}{u^2} du$$

$$= -\frac{1}{u} + C$$

$$x + \frac{x^2}{2} = -\frac{1+x}{y} + C$$

$$\text{故 } \frac{1+x}{y} + x + \frac{x^2}{2} = C$$

验证

$$\begin{aligned}
 y' &= -\varphi_1'(x) \int_x^{\infty} \frac{\varphi_2(t)f(t)}{W(\varphi_1, \varphi_2)(t)} dt - \frac{\varphi_1(x)\varphi_2'(x)f(x)}{W(\varphi_1, \varphi_2)(x)} + \varphi_2'(x) \int_x^{\infty} \frac{\varphi_1(t)f(t)}{W(\varphi_1, \varphi_2)(t)} dt + \frac{\varphi_1(x)\varphi_2'(x)f(x)}{W(\varphi_1, \varphi_2)(x)} + C_1\varphi_1'(x) \\
 &\quad + C_2\varphi_2'(x) \\
 &= -\varphi_1'(x) \int_x^{\infty} \frac{\varphi_2(t)f(t)}{W(\varphi_1, \varphi_2)(t)} dt + \varphi_2'(x) \int_x^{\infty} \frac{\varphi_1(t)f(t)}{W(\varphi_1, \varphi_2)(t)} dt + C_1\varphi_1'(x) + C_2\varphi_2'(x)
 \end{aligned}$$

$$\begin{aligned}
 y'' &= -\varphi_1''(x) \int_x^{\infty} \frac{\varphi_2(t)f(t)}{W(\varphi_1, \varphi_2)(t)} dt - \frac{\varphi_1'(x)\varphi_2'(x)f(x)}{W(\varphi_1, \varphi_2)(x)} + \varphi_2''(x) \int_x^{\infty} \frac{\varphi_1(t)f(t)}{W(\varphi_1, \varphi_2)(t)} dt + \frac{\varphi_1'(x)\varphi_2''(x)f(x)}{W(\varphi_1, \varphi_2)(x)} + C_1\varphi_1''(x) + C_2\varphi_2''(x) \\
 &= -\varphi_1''(x) \int_x^{\infty} \frac{\varphi_2(t)f(t)}{W(\varphi_1, \varphi_2)(t)} dt + \varphi_2''(x) \int_x^{\infty} \frac{\varphi_1(t)f(t)}{W(\varphi_1, \varphi_2)(t)} dt + C_1\varphi_1''(x) + C_2\varphi_2''(x) + f(x)
 \end{aligned}$$

$$\begin{aligned}
 y'' + p(x)y' + q(x)y &= (C_1 - \int_x^{\infty} \frac{\varphi_2(t)f(t)}{W(\varphi_1, \varphi_2)(t)} dt)(\varphi_1''(x) + p(x)\varphi_1'(x) + q(x)\varphi_1(x)) + (C_2 + \int_x^{\infty} \frac{\varphi_1(t)f(t)}{W(\varphi_1, \varphi_2)(t)} dt)(\varphi_2''(x) \\
 &\quad + p(x)\varphi_2'(x) + q(x)\varphi_2(x)) + f(x) \\
 &= 0 + 0 + f(x) = f(x)
 \end{aligned}$$

习题 7.4

$$1. \text{ 假设 } (k_1 y_1(x) + k_2 y_2(x))^{(m)} = k_1 y_1^{(m)}(x) + k_2 y_2^{(m)}(x)$$

$$\begin{aligned}
 (k_1 y_1(x) + k_2 y_2(x))^{(m+1)} &= (k_1 y_1^{(m)}(x) + k_2 y_2^{(m)}(x))' \\
 &= k_1 y_1^{(m+1)}(x) + k_2 y_2^{(m+1)}(x)
 \end{aligned}$$

$$\text{故 } (k_1 y_1(x) + k_2 y_2(x))^{(n)} = k_1 y_1^{(n)}(x) + k_2 y_2^{(n)}(x)$$

将 $y = k_1 y_1(x) + k_2 y_2(x)$ 代入方程

$$\begin{aligned}
 \text{左式} &= k_1 y_1^{(n)}(x) + k_2 y_2^{(n)}(x) + a_{n-1}(x)k_1 y_1^{(n-1)}(x) + a_{n-1}(x)k_2 y_2^{(n-1)}(x) + \dots + a_0(x)k_1 y_1(x) + a_0(x)k_2 y_2(x) \\
 &= k_1 (y_1^{(n)}(x) + a_{n-1}(x)y_1^{(n-1)}(x) + \dots + a_0(x)y_1(x)) + k_2 (y_2^{(n)}(x) + a_{n-1}(x)y_2^{(n-1)}(x) + \dots + a_0(x)y_2(x)) \\
 &= k_1 f(x) + k_2 g(x)
 \end{aligned}$$

得证

$$2. \frac{d\varphi(x+T)}{dx} + \varphi(x+T)$$

$$= \frac{d\varphi(x+T)}{d(x+T)} + \varphi(x+T)$$

$$= f(x+T) = f(x)$$

$$\varphi(0+T) = \varphi(0)$$

$$\varphi(x), \varphi(x+T) \text{ 均满足 } \begin{cases} \frac{dy}{dx} + y = f(x) \\ y(0) = \varphi(0) \end{cases}$$

由解的唯一性 $\varphi(x) = \varphi(x+T)$

得证

$$b. \text{ 设 } y(x) = C_1 e^x + C_2 e^{-x}$$

$$y'(x) = C_1 e^x - C_2 e^{-x}$$

$$y(x) = a \cdot \frac{e^x + e^{-x}}{2} + b \cdot \frac{e^x - e^{-x}}{2}$$

$$= \frac{a+b}{2} \cdot e^x + \frac{a-b}{2} \cdot e^{-x}$$

$$\begin{cases} y(0) = C_1 + C_2 = 1 \\ y'(0) = C_1 - C_2 = 0 \end{cases} \quad \begin{cases} y(0) = C_1 + C_2 = 0 \\ y'(0) = C_1 - C_2 = 1 \end{cases}$$

$$\Rightarrow C_1 = C_2 = \frac{1}{2}$$

$$\Rightarrow C_1 = \frac{1}{2}, C_2 = -\frac{1}{2}$$

$$y(x) = \frac{e^x + e^{-x}}{2}$$

$$y(x) = \frac{e^x - e^{-x}}{2}$$

8. $(x^2 + x^3) - (x^2 + x) = x^3 - x$ 也是齐次微分方程的解

$$\text{通解} = x^2 + x + C_1 + C_2 x + C_3 (x^3 - x)$$

第7章总复习题

$$3. W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \quad a_1(x)W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ a_1 y_1^{(n-1)} & a_1 y_2^{(n-1)} & \cdots & a_1 y_n^{(n-1)} \end{vmatrix}$$

$$W' = \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \cdots + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

$$W' + a_1(x)W = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \vdots & \vdots \\ y_1^{(n)} + a_1 y_1^{(n-1)} & \cdots & y_n^{(n)} + a_1 y_n^{(n-1)} \end{vmatrix}$$

$$\text{由 } y_i^{(n)} + a_1 y_i^{(n-1)} + \cdots + a_n y_i = 0, W' + a_1(x)W = 0$$

$$5. \text{ 若 } \exists x_0 \in (a, b), \varphi(x_0) = \varphi'(x_0) = 0$$

$$\text{则 } y = \varphi(x), y \equiv 0 \text{ 均满足 } \begin{cases} y'' + a_1(x)y' + a_2(x)y = 0 \\ y(x_0) = 0, y'(x_0) = 0 \end{cases}$$

由解的唯一性, $\varphi(x) \equiv 0$, 矛盾

习题7.4

5. (2) $\frac{dy}{dx}=1 \quad \frac{d^2y}{dx^2}=0$

$$\frac{1}{2}x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 0 - x + x = 0$$

设 $y = C_1 x + C_2$

$$y' = C_1 \quad y'' = 0$$

代入, $\frac{1}{2}x^2 \cdot 0 - x \cdot C_1 + C_2 = 0$

$$-C_1 x + C_2 = 0 \quad C_1 = 0 \quad C_2 = 0$$

故 $y = 0$

14) 设 $y = C_1 x^2 + C_2 x^{-1}$

$$y' = 2C_1 x - C_2 x^{-2}$$

$$y'' = 2C_1 + 2C_2 x^{-3}$$

$$x^2 y'' - 2y = 2C_1 x^2 + 2C_2 x^{-1} - 2C_1 x^2 + 2C_2 x^{-1} = 0$$

代入, $2C_1 x^2 + 2C_2 x^{-1} - 2C_1 x^2 + 2C_2 x^{-1} = 0$

解得 $C_1 = \frac{4}{3} \quad C_2 = -\frac{4}{3}x^3$

$$C_1 x^2 = \frac{4}{3}x^2 \quad C_2 x^{-1} = -\frac{4}{3}x^2$$

故 $y = \frac{4}{3}x^2 - \frac{4}{3}x^2 = 0$

$$(x^2)' = 2x \quad (x^2)'' = 2$$

$$x^2 \frac{d^2y}{dx^2} - 2y = 2x^2 - 2x^2 = 0$$

$$(\frac{1}{x})' = -\frac{1}{x^2} \quad (\frac{1}{x})'' = \frac{2}{x^3}$$

$$x^2 \frac{d^2y}{dx^2} - 2y = \frac{2}{x} - \frac{2}{x} = 0$$

习题7.5

2. (2) $\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0$

$\lambda = 3$ (2重)

设 $y = C_1 e^{3x} + C_2 x e^{3x}$

$$y' = 3C_1 e^{3x} + 3C_2 x e^{3x} + C_2 e^{3x}$$

$$y(0) = C_1 = 1 \quad y'(0) = 3C_1 + C_2 = 1$$

$$C_1 = 1 \quad C_2 = -2$$

故 $y = e^{3x} - 2x e^{3x}$

13) $\lambda^3 - 3\lambda^2 - 4\lambda = \lambda(\lambda - 4)(\lambda + 1) = 0$

$\lambda = 0, 4, -1$

设 $y = C_1 + C_2 e^{4x} + C_3 e^{-x}$

$$y' = 4C_2 e^{4x} - C_3 e^{-x}$$

$$y'' = 16C_2 e^{4x} + C_3 e^{-x}$$

$$y(0) = C_1 + C_2 + C_3 = 1 \quad y'(0) = 4C_2 - C_3 = 1$$

$$y''(0) = 16C_2 + C_3 = 1$$

$$C_1 = \frac{3}{2} \quad C_2 = \frac{1}{10} \quad C_3 = -\frac{3}{5}$$

故 $y = \frac{3}{2} + \frac{1}{10}e^{4x} - \frac{3}{5}e^{-x}$

$$7. f(x) = \sin x + \int_0^x t f(t) dt - x \int_0^x f(t) dt$$

$$f'(x) = \cos x + x f(x) - x f(x) - \int_0^x f(t) dt = \cos x - \int_0^x f(t) dt$$

$$f''(x) = -\sin x - f(x)$$

$$f''(x) + f(x) = -\sin x$$

$$\lambda^2 + 1 = 0 \quad \lambda = \pm i$$

$$\varphi_1(x) = \cos x \quad \varphi_2(x) = \sin x$$

$$\text{Let } \varphi_*(x) = x(A \cos x + B \sin x)$$

$$\varphi'_*(x) = A \cos x + B \sin x - Ax \sin x + Bx \cos x$$

$$\varphi''_*(x) = -A \sin x + B \cos x - A \sin x - Ax \cos x + B \cos x - Bx \sin x$$

$$\varphi''_*(x) + \varphi_*(x) = 2B \cos x - 2A \sin x = -\sin x$$

$$A = \frac{1}{2} \quad B = 0$$

$$\text{Hence } \varphi_*(x) = \frac{1}{2} x \cos x$$

$$\text{Let } f(x) = \frac{1}{2} x \cos x + C_1 \sin x + C_2 \cos x$$

$$f(0) = C_2 = 0$$

$$\text{Hence } f(x) = \frac{1}{2} x \cos x + C_1 \sin x$$