

习题 4.4.

4-(2)

$$y = (1 + \frac{1}{x})^x = e^{x \ln(1 + \frac{1}{x})}$$

$$y' = e^{x \ln(1 + \frac{1}{x})} \cdot [\ln(1 + \frac{1}{x}) + x \cdot \frac{1}{1 + \frac{1}{x}} \cdot (-\frac{1}{x^2})] = e^{x \ln(1 + \frac{1}{x})} [\ln(1 + \frac{1}{x}) - \frac{1}{x+1}]$$

$$\text{令 } g(x) = \ln(1 + \frac{1}{x}) - \frac{1}{x+1} \quad \text{则 } g'(x) = -\frac{1}{x(x+1)} + \frac{1}{(x+1)^2} = \frac{-1}{x(x+1)^2} < 0$$

$g(x)$ 在 $(0, 1)$ 上单调递减 $g(x) > g(1) = \ln 2 - \frac{1}{2} > 0$

故 $y' > 0$, 故 y 在 $(0, 1)$ 上单调递增. 无极值

4-(4)

$$y' = 6x^2 - 6x - 12 = 6(x-2)(x+1)$$

故 y 在 $(-\infty, -1)$, $(2, +\infty)$ 单调递增

在 $(-1, 2)$ 单调递减

x	$(-\infty, -1)$	-1	$(-1, 2)$	2	$(2, +\infty)$
y'	+	0	-	0	+
y	$\nearrow$	极大	$\searrow$	极小	$\nearrow$

在 -1 处取得极大值 $y = 8$, 在 2 处取得极小值 $y = 5$.

习题 4.5

1-(1)

$$y' = 6x - 3x^2 \quad y'' = 6 - 6x = 6(1-x) \quad x > 1 \Rightarrow y'' < 0 \quad y \text{ 上凸} \quad x < 1 \Rightarrow y'' > 0 \quad y \text{ 下凸} \quad x = 1 \text{ 为拐点.}$$

即上凸区间 $(1, +\infty)$, 下凸区间 $(-\infty, 1)$, 拐点 $x = 1$

1-(4)

$$y' = e^x + x \cdot (-e^x) = e^x(1-x)$$

同理 (1): 上凸区间 $(1, +\infty)$, 下凸区间 $(-\infty, 1)$, 拐点 $x = 1$

5-(2)

$$\text{令 } f(x) = x^p \quad f'(x) = p x^{p-1} \quad f''(x) = p(p-1) x^{p-2} > 0$$

故 f 在 $\mathbb{R}$ 上凸的, 故 $f(\frac{1}{n}x_1 + \frac{1}{n}x_2 + \dots + \frac{1}{n}x_n) \leq \frac{1}{n}f(x_1) + \frac{1}{n}f(x_2) + \dots + \frac{1}{n}f(x_n)$

$$\text{即 } \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right)^p \leq \frac{x_1^p + x_2^p + \dots + x_n^p}{n}$$

7.

$$\forall x \in (a, b), \exists \lambda \in (0, 1), \text{ s.t. } x = \lambda a + (1-\lambda)b$$

$$\text{由 } f(x) \text{ 下凸} \Rightarrow f(x) = f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b) \leq [\lambda + (1-\lambda)] \max\{f(a), f(b)\}$$

$$\text{故 } \forall x \in [a, b] \quad f(x) \leq \max\{f(a), f(b)\} \quad \text{即 } \max_{a \leq x \leq b} f(x) = \max\{f(a), f(b)\}$$

8.

由 $f(x)$ 下凸, 可知 $f'(x)$ 在 (a, b) 单调递增.

故 $a \leq x < x_0$ 时 $f'(x) < 0$, $f(x)$ 单减 $f(x) > f(x_0)$ $\Rightarrow f(x_0)$ 为 f 在 $[a, b]$ 上最小值点.

$x_0 < x \leq b$ 时 $f'(x) > 0$, $f(x)$ 单增 $f(x) > f(x_0)$

习题 4.6.

1-(1)

y 的定义域: $(-\infty, 0] \cup (1, +\infty)$.

$\lim_{x \rightarrow 1^+} f(x) = +\infty$ 故 $x=1$ 为垂直渐近线

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \sqrt{\frac{x}{x-1}} = 1 \quad \lim_{x \rightarrow +\infty} (f(x) - x) = \lim_{x \rightarrow +\infty} \frac{\frac{x}{x+1}}{\sqrt{\frac{x}{x-1}} + 1} = \frac{1}{2}.$$

$y = x + \frac{1}{2}$ 为一条斜渐近线

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = \lim_{x \rightarrow -\infty} -\sqrt{\frac{x}{x-1}} = -1 \quad \lim_{x \rightarrow -\infty} (f(x) + x) = \lim_{x \rightarrow -\infty} \frac{1 + \sqrt{\frac{x}{x-1}}}{\frac{-x}{x-1}} = -2.$$

$y = -x - 2$ 为一条斜渐近线

无水平渐近线

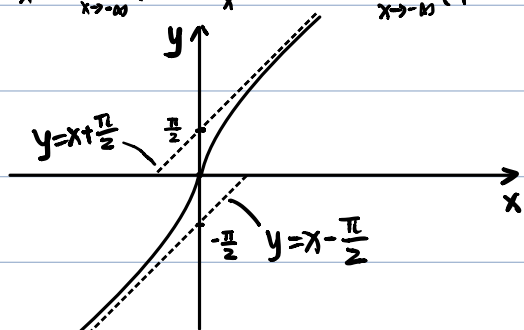
2-(4)

$$\lim_{x \rightarrow +\infty} \arctan x = \frac{\pi}{2} \quad \lim_{x \rightarrow -\infty} \arctan x = -\frac{\pi}{2} \quad y' = 1 + \frac{1}{1+x^2} > 0 \text{ 单增}$$

$$f(-x) = -f(x) \Rightarrow \text{奇函数.}$$

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x} = \lim_{x \rightarrow +\infty} \left(1 + \frac{\arctan x}{x}\right) = 1 \quad \lim_{x \rightarrow +\infty} (f(x) - x) = \lim_{x \rightarrow +\infty} (\arctan x) = \frac{\pi}{2} \quad y = x + \frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{x} = \lim_{x \rightarrow -\infty} \left(1 + \frac{\arctan x}{x}\right) = 1 \quad \lim_{x \rightarrow -\infty} (f(x) - x) = \lim_{x \rightarrow -\infty} (\arctan x) = -\frac{\pi}{2} \quad y = x - \frac{\pi}{2}$$



习题 4.2.

2-(2) 由洛必达法则:

$$\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x \rightarrow 0} \frac{e^x - e^{\sin x} \cos x}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{e^x - e^{\sin x} \cos^2 x + e^{\sin x} \sin x}{\sin x} = \lim_{x \rightarrow 0} \left(\frac{e^x - e^{\sin x} \cos^2 x + e^{\sin x} \cdot 2 \cos x \cdot \sin x}{-\cos x} + e^{\sin x} \cdot \cos x \right) = 1$$

3. 由洛必达法则

$$\lim_{h \rightarrow 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a-h)}{2h} = \frac{1}{2} \lim_{h \rightarrow 0} \frac{f'(a+h) - f'(a-h)}{h} = \frac{f'(a) - f'(a-h)}{h} = f''(a)$$

4.

$$\lim_{x \rightarrow 0} (f(\sin x) - 1) = 0 \quad \lim_{x \rightarrow 0} \ln(f(x)) = 0$$

$$\text{故由洛必达法则: } \lim_{x \rightarrow 0} \frac{f(\sin x) - 1}{\ln f(x)} = \lim_{x \rightarrow 0} \frac{f'(\sin x) \cdot \cos x}{\frac{1}{f(x)} \cdot f'(x)} = \frac{1 \times 1}{1 \times 1} = 1$$

2-(10)

$$\lim_{x \rightarrow +\infty} \frac{\ln(x)}{x^2} = \lim_{x \rightarrow +\infty} \frac{\frac{1}{x}}{2x} = 0$$

2-(13)

$$\begin{aligned} \lim_{x \rightarrow 1} (x-1) \tan \frac{\pi x}{2} &= \lim_{x \rightarrow 1} \frac{\tan \frac{\pi x}{2}}{\frac{1}{x-1}} = \lim_{x \rightarrow 1} \frac{\frac{1}{\cos^2(\frac{\pi x}{2})} \cdot \frac{\pi}{2}}{\frac{-1}{(x-1)^2}} = -\frac{\pi}{2} \lim_{x \rightarrow 1} \frac{(x-1)^2}{\cos^2(\frac{\pi x}{2})} = -\frac{\pi}{2} \lim_{x \rightarrow 1} \frac{2(x-1)}{-\pi \cos(\frac{\pi x}{2}) \sin(\frac{\pi x}{2})} = \frac{1}{2} \lim_{x \rightarrow 1} \frac{2}{-\frac{\pi}{2} \sin(\frac{\pi}{2})} \\ &= -\frac{2}{\pi} \end{aligned}$$

2-(17)

$$\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\pi}{x} - 1 \right)^{\tan x} = \lim_{x \rightarrow \frac{\pi}{2}} e^{\frac{\ln(\frac{\pi}{x} - 1)}{\frac{1}{\tan x}}} = e^{\frac{4}{\pi}}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln(\frac{\pi}{x} - 1)}{\frac{1}{\tan x}} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{x}{\pi - x} \cdot (-\frac{\pi}{x^2})}{\frac{-1}{\tan^2 x} \cdot \frac{1}{\cos^2 x}} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\pi \sin^2 x}{x(\pi - x)} = \frac{4}{\pi}$$

习题 4.3

3-(12)

$$\begin{aligned} y &= f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \frac{f'''(x_0)}{3!} (x - x_0)^3 \\ &= \sin x_0 + \cos x_0 (x - x_0) - \frac{\sin x_0}{2} (x - x_0)^2 - \frac{\cos x_0}{6} (x - x_0)^3 \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} (x - \frac{\pi}{4}) - \frac{\sqrt{2}}{4} (x - \frac{\pi}{4})^2 - \frac{\sqrt{2}}{12} (x - \frac{\pi}{4})^3 \end{aligned}$$

3-(4)

$$y = \tan\left(\frac{\pi}{4}\right) + \frac{1}{\cos^2\left(\frac{\pi}{4}\right)}\left(x - \frac{\pi}{4}\right) + \frac{2 \cdot \sin\left(\frac{\pi}{4}\right)}{\cos^3\left(\frac{\pi}{4}\right)}\left(x - \frac{\pi}{4}\right)^2$$

$$= 1 + 2\left(x - x_0\right) + 4\left(x - \frac{\pi}{4}\right)^2$$

9.

利用 f 的带拉格朗日余项的2阶泰勒公式,有:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(\xi)}{6}(x - x_0)^3$$

$$f(0) = f\left(\frac{1}{2}\right) - \frac{1}{2}f'\left(\frac{1}{2}\right) + \frac{f''(\xi)}{2} \cdot \frac{1}{4} - \frac{f'''(\xi)}{6} \cdot \frac{1}{8}$$

$$f(1) = f\left(\frac{1}{2}\right) + \frac{1}{2}f'\left(\frac{1}{2}\right) + \frac{f''(\xi)}{2} \cdot \frac{1}{4} + \frac{f'''(\xi)}{6} \cdot \frac{1}{8}$$

$$f(1) - f(0) = f'\left(\frac{1}{2}\right) + \frac{1}{4}f''(\xi) \quad f'''(\xi) = 12$$

10.

由 f 的带拉格朗日余项的一阶泰勒公式,有 $\exists \xi_1, \xi_2 \in (a, b)$, s.t

$$f(a) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right) \cdot \frac{a-b}{2} + \frac{f''(\xi_1)}{2}\left(\frac{a-b}{2}\right)^2 \quad \text{--- (1)}$$

$$f(b) = f\left(\frac{a+b}{2}\right) - f'\left(\frac{a+b}{2}\right) \cdot \frac{a-b}{2} + \frac{f''(\xi_2)}{2}\left(\frac{a-b}{2}\right)^2 \quad \text{--- (2)}$$

$$\text{(1)+(2) 得: } f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) = \frac{(a-b)^2}{4} \frac{f''(\xi_1) + f''(\xi_2)}{2}$$

$$\text{由达布定理, } \exists \xi \in (a, b), \text{ s.t } f'(\xi) = \frac{f''(\xi_1) + f''(\xi_2)}{2}.$$

11.

由 f 的带拉格朗日余项的一阶泰勒公式,有 $\exists \xi_1, \xi_2 \in (a, b)$, s.t

$$f\left(\frac{a+b}{2}\right) = f(a) - f'(a) \cdot \frac{a-b}{2} + \frac{f''(\xi_1)}{2}\left(\frac{a-b}{2}\right)^2$$

$$= f(b) + f'(b) \cdot \frac{a-b}{2} + \frac{f''(\xi_2)}{2}\left(\frac{a-b}{2}\right)^2$$

$$\therefore |f(b) - f(a)| \frac{4}{(b-a)^2} \leq \left| \frac{f''(\xi_1) - f''(\xi_2)}{2} \right| \leq \frac{|f''(\xi_1)| + |f''(\xi_2)|}{2} \leq \max\{|f''(\xi_1)|, |f''(\xi_2)|\}$$

即 $\exists \xi \in (a, b)$, $|f''(\xi)| = \max\{|f''(\xi_1)|, |f''(\xi_2)|\}$, 使得不等式成立.