

习题 4.3.

4-(1)

$$y = 1 + \frac{2x}{1-x+x^2} = 1 + 2x + \frac{2x^4-2x^3}{1-x+x^2} = 1 + 2x + 2x^2 - \frac{2x^4}{1-x+x^2} = 1 + 2x + 2x^2 - 2x^4 + \frac{2x^6-2x^5}{1-x+x^2}$$

$$= 1 + 2x + 2x^2 - 2x^4 + o((x-x_0)^4)$$

即展开至 4 次 $y = 1 + 2x + 2x^2 - 2x^4$

4-(b)

$$y = \frac{x-2}{(x-2)^2-4} = 0 - \frac{1}{4}(x-2) + \frac{\frac{1}{4}(x-2)^2}{(x-2)^2-4} = 0 - \frac{1}{4}(x-2) - \frac{1}{4^2}(x-2)^2 + \frac{\frac{1}{4^3}(x-2)^3}{(x-2)^2-4} = \dots$$

$$= 0 - \frac{1}{4}(x-2) - \frac{1}{4^2}(x-2)^2 - \frac{1}{4^3}(x-2)^3 - \dots - \frac{1}{4^n}(x-2)^n + \frac{\frac{1}{4^{n+1}}(x-2)^{n+1}}{(x-2)^2-4}$$

即展开到 $2n$ 次: $y = -\frac{1}{4}(x-2) - \frac{1}{4^2}(x-2)^2 - \frac{1}{4^3}(x-2)^3 - \dots - \frac{1}{4^n}(x-2)^n$

4-(8)

$$y = \frac{1}{2} \ln(1+x) - \frac{1}{2} \ln(1-x)$$

$$\ln(1+x) = \frac{1}{1+x} \quad \ln^{(n)}(1+x) = (-1)^{n-1} \cdot (n-1)! \cdot \frac{1}{(1+x)^n} \quad \ln'(1-x) = -\frac{1}{1-x} = \frac{1}{x-1} \quad \ln^{(n)}(1-x) = (-1)^{n-1} \cdot (n-1)! \cdot \frac{1}{(x-1)^n}$$

$$\text{故 } y = \frac{1}{2} \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n-1}}{n} x^n + o(x^n) \right] - \frac{1}{2} \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots - \frac{x^n}{n} - o(x^n) \right]$$

$$= \begin{cases} x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^n}{n} + o(x^n) & n \text{ 为奇} \\ x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{n-1}}{n-1} + o(x^n) & n \text{ 为偶} \end{cases}$$

$$\text{故 } y \text{ 的 } n \text{ 阶泰勒多项式: } y = \begin{cases} x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^n}{n} & n \text{ 为奇} \\ x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{n-1}}{n-1} & n \text{ 为偶} \end{cases}$$

4-(9)

$$\lim_{x \rightarrow 0} e^{-\frac{1}{x^2}} = 0 \quad \text{故 } y \text{ 在 } x=0 \text{ 处连续.}$$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\lim_{x \rightarrow 0} \frac{f(x)}{x^n} = \lim_{x \rightarrow 0} \frac{\frac{1}{x^n}}{e^{-\frac{1}{x^2}}} = \lim_{x \rightarrow 0} \frac{\frac{-n}{x^{n+1}}}{e^{-\frac{1}{x^2}} \cdot \frac{-2}{x^3}} = \frac{n}{2} \lim_{x \rightarrow 0} \frac{\frac{1}{x^{n+2}}}{e^{-\frac{1}{x^2}}} = \frac{n}{2} \cdot \frac{n-2}{2} \lim_{x \rightarrow 0} \frac{\frac{1}{x^{n+4}}}{e^{-\frac{1}{x^2}}}$$

$$\text{若 } n \text{ 为偶数: } \lim_{x \rightarrow 0} \frac{f(x)}{x^n} = \frac{n!!}{2^{\frac{n}{2}}} \lim_{x \rightarrow 0} \frac{1}{e^{\frac{1}{x^2}}} = 0.$$

$$\text{故 } f(x) = o(x^n)$$

$$\text{若 } n \text{ 为奇数: } \lim_{x \rightarrow 0} \frac{f(x)}{x^n} = \frac{n!!}{2^{\frac{n+1}{2}}} \lim_{x \rightarrow 0} \frac{x}{e^{\frac{1}{x^2}}} = 0$$

$$\text{故 } f(x) \text{ 的 } n \text{ 阶泰勒展开为 } 0.$$

5-1)

$$\text{令 } \frac{1}{x} = t \quad \ln(1+t) = t - \frac{t^2}{2} + o(t^2)$$

$$\lim_{x \rightarrow \infty} \left[x - x^2 \cdot \ln\left(1 + \frac{1}{x}\right) \right] = \lim_{t \rightarrow 0} \left[\frac{1}{t} - \frac{1}{t^2} \ln(1+t) \right] = \lim_{t \rightarrow 0} \left\{ \frac{1}{t} - \frac{1}{t^2} \left[t - \frac{t^2}{2} + o(t^2) \right] \right\} = \frac{1}{2}$$

5-2)

$$\lim_{x \rightarrow 0} \frac{\frac{x^2}{2} + 1 - \sqrt{1+x^2}}{(\cos x - e^x) \sin x^2} = \lim_{x \rightarrow 0} \frac{\frac{x^2}{2} + 1 - [1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4)]}{\left\{ \left[1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4) \right] - \left[1 + x^2 + \frac{x^4}{2} + o(x^4) \right] \right\} [x^2 + o(x^4)]} = \lim_{x \rightarrow 0} \frac{\frac{1}{8}x^4 + o(x^4)}{-\frac{3}{2}x^4 + o(x^4)} = -\frac{1}{12}$$

7-2)

$$\ln(1+x) = 1+x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{(1+\xi)^4}$$

$$\approx 1+x - \frac{x^2}{2} + \frac{x^3}{3} = \ln(1+x_1)$$

$$\text{公式误差 } |\ln(1+x_0) - \ln(1+x_1)| < \frac{0.02^3}{4} = 2 \times 10^{-6}$$

$$\ln(1+x_1) = 0.02 - \frac{0.02^2}{2} + \frac{0.02^3}{3} \approx 0.0198 + 2.7 \times 10^{-6} = 0.0198027 = \ln(1+x_2)$$

$$\text{计算误差 } |\ln(1+x_1) - \ln(1+x_2)| < 5 \times 10^{-7}$$

$$\ln(1+x_2) = 0.01980$$

$$\text{近似误差 } |\ln(1+x_2) - \ln(1+x_3)| < 5 \times 10^{-6}$$

$$|\ln(1+x_0) - \ln(1+x_3)| < 2 \times 10^{-6} + 5 \times 10^{-7} + 5 \times 10^{-6} < 10^{-5}$$

故近似值为 0.01980.

第4章总复习.

9. 对于 $\forall x_0 \in (0, 1)$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi_1)}{2} (x-x_0)^2$$

$$\text{令 } x=1 \quad f(1) = f(x_0) + f'(x_0)(1-x_0) + \frac{f''(\xi_1)}{2} (1-x_0)^2 \quad x_0 < \xi_1 < 1$$

$$\text{令 } x=0 \quad f(0) = f(x_0) + f'(x_0)(0-x_0) + \frac{f''(\xi_2)}{2} (x_0)^2 \quad 0 < \xi_2 < x_0$$

$$\text{两式相减 } |f'(x_0)| = \left| \frac{f''(\xi_1)}{2} (1-x_0)^2 - \frac{f''(\xi_2)}{2} (x_0)^2 \right| \leq \frac{|f''(\xi_1)|}{2} (1-x_0)^2 + \frac{|f''(\xi_2)|}{2} (x_0)^2 \leq \frac{|M|}{2} \frac{(2x_0-1)^2+1}{2} < \frac{|M|}{2}$$

$$\text{而 } f(1) = f(0) + f'(0) + \frac{f''(\xi_3)}{2} \Rightarrow |f'(0)| \leq \frac{|M|}{2}. \quad \text{同理 } |f'(1)| \leq \frac{|M|}{2}.$$

综上 $\forall x_0 \in [0, 1]$, 有 $|f'(x_0)| \leq \frac{|M|}{2}$

11. 设 $x=x_0$ 时 $f(x)$ 取得最小值 -1 , 由 Fermat 定理 $f'(x_0)=0$

$$\text{故 } f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2}(x-x_0)^2.$$

$$= -1 + \frac{f''(\xi)}{2}(x-x_0)^2$$

$$\text{令 } x=0 \quad 0=f(0) = -1 + \frac{f''(\xi_1)}{2} x_0^2 \quad f''(\xi_1) = \frac{2}{x_0^2}$$

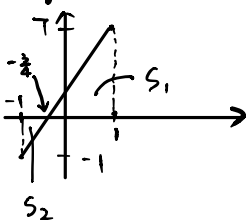
$$\text{令 } x=1 \quad 0=f(1) = -1 + \frac{f''(\xi_2)}{2} (x_0-1)^2 \quad f''(\xi_2) = \frac{2}{(x_0-1)^2}$$

$$\text{故若 } x_0 = \frac{1}{2} \quad f''(\xi_1) = f''(\xi_2) = 8 \quad \text{若 } x_0 < \frac{1}{2} \quad f''(\xi_1) > 8 \quad \text{若 } x_0 > \frac{1}{2} \quad f''(\xi_2) > 8$$

综上所述: $\exists \xi \in (0,1)$, 使得 $f''(\xi) \geq 8$.

习题 5.1

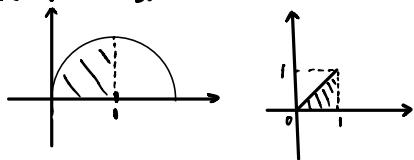
1-(3)



$$\int_{-1}^1 (3+4x) dx = S_1 - S_2 = \frac{1}{2} \left(\frac{7}{4} \times 1 - \frac{1}{4} \times 1 \right) = 6.$$

1-(4)

$$\text{原式} = \int_0^1 \sqrt{2x-x^2} \cdot dx - \int_0^1 x dx = \frac{\pi}{4} - \frac{1}{2}$$



$$y^2 + (x-1)^2 = 1$$

2.

由 $f(x) < g(x)$, $\exists \delta \in \mathbb{R}$, $\forall |T| < \delta$, s.t. $U(f, T) < L(g, T)$

$$\int_a^b f(x) dx \leq U(f, T) < L(g, T) \leq \int_a^b g(x) dx \quad \text{若 } f(x) = g(x), \text{ 则 } \int_a^b f(x) dx = \int_a^b g(x) dx \quad \text{若 } \int_a^b f(x) dx < \int_a^b g(x) dx \text{ 则 } f(x) \leq g(x) \text{ 不恒成立。}$$

$$4. \forall \varepsilon > 0, \exists |\tau| < \delta, U(f, \tau) - L(f, \tau) < \varepsilon^2$$

则:

$$U\left(\frac{1}{f}, \tau\right) - L\left(\frac{1}{f}, \tau\right) = \sum_{i=1}^n \sup_{\xi, \eta \in [x_{i-1}, x_i]} \left(\frac{1}{f(\xi)} - \frac{1}{f(\eta)}\right) (x_i - x_{i-1}) < \frac{1}{d^2} (U(f, \tau) - L(f, \tau)) < \varepsilon$$

$$\text{故 } \frac{1}{f(x)} \in R[a, b]$$

6.

$$\forall x_1, x_2 \in [a, b] \text{ 且 } x_1 \geq x_2,$$

$$F(x_1) - F(x_2) = \sup_{t \in [a, x_1]} f(t) - \sup_{t \in [a, x_2]} f(t) \geq 0$$

故 $F(x)$ 在 $[a, b]$ 单调不减.

$$\text{故由定理 5.1.5, } f \in R[a, b]$$

5.1.4 设 f 在 $[a, b]$ 上有界, 且只有有限个不连续点, 则 $f \in R[a, b]$

不妨设 f 在 $[a, b]$ 上只有间断点 $x=b$, 由题设, f 在 $[a, b]$ 有界:

$$|f(x)| \leq M$$

$$\forall \varepsilon > 0, \text{ 取 } c \in (a, b) \text{ 使得 } b-c < \frac{\varepsilon}{4M}. \text{ 而 } f \text{ 在 } [a, c] \text{ 上连续故在 } [a, c] \text{ 上可积}$$

因此存在 $[a, c]$ 的一个分割 $T_1: a=x_0 < x_1 < \dots < x_n=c$, 使得.

$$U(f, T_1) - L(f, T_1) < \varepsilon/2.$$

则区间 $[a, b]$ 的分割: $T_2: a=x_0, x_1, \dots, x_{n-1}, c, b$, 由于

$$U(f, T_2) - L(f, T_2) < \frac{\varepsilon}{2} + \sup_{\xi \in [b, c]} f(\xi) (b-c) - \inf_{\eta \in [b, c]} f(\eta) (b-c) \leq \varepsilon/2 + 2M \cdot \frac{\varepsilon}{4M} = \varepsilon$$

$$\text{故 } f \in R[a, b]$$