

高阶导数

$$\begin{aligned} \text{Ex. } y &= \frac{1}{x^2 - x - 2}, \text{ 求 } y^{(n)}. \\ \text{解: } y &= \frac{1}{(x+1)(x-2)} = \frac{1}{3} \left(\frac{1}{x-2} - \frac{1}{x+1} \right). \\ y^{(n)} &= \frac{1}{3} \left(\frac{1}{x-2} \right)^{(n)} - \frac{1}{3} \left(\frac{1}{x+1} \right)^{(n)} \\ &= \frac{1}{3} (-1)^n n! (x-2)^{-(n+1)} - \frac{1}{3} (-1)^n n! (x+1)^{-(n+1)}. \quad \square \end{aligned}$$

Ex. $y = x^2 e^{-x}$, 求 $y^{(n)}$.

解: $y = x^2 e^{-x}$.

$$y^{(n)} = x^2 e^{-x} (-1)^n + C_1^1 2x e^{-x} (-1)^{n-1} + C_2^2 x e^{-x} (-1)^{n-2} + 0 + \dots + 0$$

适用于高阶之后 $f^{(n)}(x) = 0$

高中學校典例 (1) $f(x) = \frac{1}{x^2} = x^{-2}$ $f'(x) = -2x^{-3} = -\frac{2}{x^3}$
 (2) $f(x) = x^2 \ln x$ $f'(x) = 2x \ln x + x$
 (3) 內心定理 $P(1, 2), M = C$

1. $f(x) = \ln(x+1)$
 $f'(x) = \frac{1}{x+1}$
 $f''(x) = -\frac{1}{(x+1)^2}$
 $f'''(x) = \frac{2}{(x+1)^3}$
 $f^{(4)}(x) = -\frac{6}{(x+1)^4}$
 $f^{(5)}(x) = \frac{24}{(x+1)^5}$
 $f^{(6)}(x) = -\frac{120}{(x+1)^6}$
 $f^{(7)}(x) = \frac{840}{(x+1)^7}$
 $f^{(8)}(x) = -\frac{5040}{(x+1)^8}$
 $f^{(9)}(x) = \frac{40320}{(x+1)^9}$
 $f^{(10)}(x) = -\frac{362880}{(x+1)^{10}}$
 $f^{(11)}(x) = \frac{4790016}{(x+1)^{11}}$
 $f^{(12)}(x) = -\frac{64515840}{(x+1)^{12}}$
 $f^{(13)}(x) = \frac{1032243200}{(x+1)^{13}}$
 $f^{(14)}(x) = -\frac{18080371200}{(x+1)^{14}}$
 $f^{(15)}(x) = \frac{377472000000}{(x+1)^{15}}$
 $f^{(16)}(x) = -\frac{8464320000000}{(x+1)^{16}}$
 $f^{(17)}(x) = \frac{201158400000000}{(x+1)^{17}}$
 $f^{(18)}(x) = -\frac{5028998400000000}{(x+1)^{18}}$
 $f^{(19)}(x) = \frac{130722969600000000}{(x+1)^{19}}$
 $f^{(20)}(x) = -\frac{3217754880000000000}{(x+1)^{20}}$
 $f^{(21)}(x) = \frac{81661527040000000000}{(x+1)^{21}}$
 $f^{(22)}(x) = -\frac{2041538688000000000000}{(x+1)^{22}}$
 $f^{(23)}(x) = \frac{51038467200000000000000}{(x+1)^{23}}$
 $f^{(24)}(x) = -\frac{1275959168000000000000000}{(x+1)^{24}}$
 $f^{(25)}(x) = \frac{31898979200000000000000000}{(x+1)^{25}}$
 $f^{(26)}(x) = -\frac{797473478400000000000000000}{(x+1)^{26}}$
 $f^{(27)}(x) = \frac{19936836960000000000000000000}{(x+1)^{27}}$
 $f^{(28)}(x) = -\frac{498420929280000000000000000000}{(x+1)^{28}}$
 $f^{(29)}(x) = \frac{12460523232000000000000000000000}{(x+1)^{29}}$
 $f^{(30)}(x) = -\frac{311513080800000000000000000000000}{(x+1)^{30}}$
 $f^{(31)}(x) = \frac{7787827020800000000000000000000000}{(x+1)^{31}}$
 $f^{(32)}(x) = -\frac{194695680512000000000000000000000000}{(x+1)^{32}}$
 $f^{(33)}(x) = \frac{4867392012800000000000000000000000000}{(x+1)^{33}}$
 $f^{(34)}(x) = -\frac{121684800320000000000000000000000000000}{(x+1)^{34}}$
 $f^{(35)}(x) = \frac{3042120008960000000000000000000000000000}{(x+1)^{35}}$
 $f^{(36)}(x) = -\frac{76053000224000000000000000000000000000000}{(x+1)^{36}}$
 $f^{(37)}(x) = \frac{1901325005696000000000000000000000000000000}{(x+1)^{37}}$
 $f^{(38)}(x) = -\frac{47533125143040000000000000000000000000000000}{(x+1)^{38}}$
 $f^{(39)}(x) = \frac{1188328128576000000000000000000000000000000000}{(x+1)^{39}}$
 $f^{(40)}(x) = -\frac{29708203214400000000000000000000000000000000000}{(x+1)^{40}}$
 $f^{(41)}(x) = \frac{742705181360000000000000000000000000000000000000}{(x+1)^{41}}$
 $f^{(42)}(x) = -\frac{18567629534080000000000000000000000000000000000000}{(x+1)^{42}}$
 $f^{(43)}(x) = \frac{466690738304000000000000000000000000000000000000000}{(x+1)^{43}}$
 $f^{(44)}(x) = -\frac{11667268457600}{(x+1)^{44}}$
 $f^{(45)}(x) = \frac{2916817114400}{(x+1)^{45}}$
 $f^{(46)}(x) = -\frac{729204278600}{(x+1)^{46}}$
 $f^{(47)}(x) = \frac{182301069600}{(x+1)^{47}}$
 $f^{(48)}(x) = -\frac{455752674000}{(x+1)^{48}}$
 $f^{(49)}(x) = \frac{113938168000}{(x+1)^{49}}$
 $f^{(50)}(x) = -\frac{2848454200}{(x+1)^{50}}$
 $f^{(51)}(x) = \frac{71211355000}{(x+1)^{51}}$
 $f^{(52)}(x) = -\frac{178028388000}{(x+1)^{52}}$
 $f^{(53)}(x) = \frac{4450709700}{(x+1)^{53}}$
 $f^{(54)}(x) = -\frac{1112677425000}{(x+1)^{54}}$
 $f^{(55)}(x) = \frac{27816935600}{(x+1)^{55}}$
 $f^{(56)}(x) = -\frac{69542339000}{(x+1)^{56}}$
 $f^{(57)}(x) = \frac{173855847500}{(x+1)^{57}}$
 $f^{(58)}(x) = -\frac{434639618000}{(x+1)^{58}}$
 $f^{(59)}(x) = \frac{1086599045000}{(x+1)^{59}}$
 $f^{(60)}(x) = -\frac{27164976100}{(x+1)^{60}}$
 $f^{(61)}(x) = \frac{679124402000}{(x+1)^{61}}$
 $f^{(62)}(x) = -\frac{1697811005000}{(x+1)^{$

$$\begin{aligned} \text{Ex. } \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, & \text{求 } y'(x), y''(x). \\ \text{解: } y'(x) = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t} \\ y''(x) = \frac{d(\frac{dy}{dx})}{dx} = \frac{d}{dt} \left(\frac{\sin t}{1 - \cos t} \right) \cdot \frac{dt}{dx} = \frac{\frac{\cos t(1 - \cos t) - \sin^2 t}{(1 - \cos t)^2}}{\frac{a(1 - \cos t)}{dt}} \\ = \frac{-1}{a(1 - \cos t)^2} \square \end{aligned}$$
$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{dY(\varphi^{-1}(x))}{dx} \\ &= Y'(\varphi^{-1}(x))[\varphi^{-1}(x)]' \quad (\text{复合函数求导}) \\ &= \frac{y''(t)\varphi'(t) - y'(t)\varphi''(t)}{(\varphi'(t))^2} \Big|_{t=\varphi^{-1}(x)} \cdot \frac{1}{\varphi'(\varphi^{-1}(x))} \quad (\text{四则运算求导, 反函数求导}) \\ &= \frac{y''(t)\varphi'(t) - y'(t)\varphi''(t)}{(\varphi'(t))^3} \Big|_{t=\varphi^{-1}(x)} \rightarrow \text{注意到此处的} 3 \text{次方} \end{aligned}$$

参数函数求导

Def. (导数, 左、右导数)

$$(1) f'(x_0) \triangleq \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$(2) f'_-(x_0) \triangleq \lim_{\Delta x \rightarrow 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$(3) f'_+(x_0) \triangleq \lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

注: 取出来的是导数的函数, 如果说函数连续, 则: $f'_-(x_0) \neq f'_+(x_0)$ 只能用定义法

左导数与导函数的左极限

左导数 $f'(x_0) \triangleq \lim_{\Delta x \rightarrow 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

导函数的左极限 $\lim_{x \rightarrow x_0^-} f'(x)$

例：已知函数 $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$ ，求0处的左导数和导函数在0处的左极限。

$\lim_{x \rightarrow 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \rightarrow 0} x \sin(1/x) = 0$

$f'(x) = 2x \sin(1/x) - \cos(1/x)$ ，在0处没有左极限

两点：

- ① 区分左导数与导函数的左极限
- ② $x \cdot \sin x, x \rightarrow 0$ 与 $x \rightarrow \infty$ 有本质区别

对数求导与指数求导

Ex. $f(x) = f_1(x)f_2(x)\cdots f_n(x)$, 求 $f'(x)$.

解: $\ln|f(x)| = \ln|f_1(x)| + \ln|f_2(x)| + \cdots + \ln|f_n(x)|$,

两边对 x 求导, 得

$$\frac{f'(x)}{f(x)} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \cdots + \frac{f_n'(x)}{f_n(x)}.$$

$$f'(x) = f(x) \left(\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \cdots + \frac{f_n'(x)}{f_n(x)} \right). \quad \square$$

Remark. 多个因子连乘的函数求导时先取对数再两端求导可简化计算.

注: 连乘函数与幂指函数有本质差别

$y = f(x)^{g(x)}$ 取 $\ln$ 后取绝对值

但是 $y = f(x)^{g(x)}$ 幂指函数黑头底数为正

$\therefore$ 放心大胆 $\ln y = g(x) \cdot \ln f(x)$

另: 区分 $y = x^{\sqrt{1+x}}$ 与 $y = \sqrt{1+x}^x$

Ex. 求下列函数的导数.

(1) $y = x\sqrt{\frac{1-x}{1+x}}$; (2) $y = e^{e^x}$; (3) $y = (x + \sqrt{1+x^2})^n$

解: $\ln y = \ln |x| + \frac{1}{2} \ln |1-x| - \frac{1}{2} \ln |1+x|$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} - \frac{1}{2(1-x)} - \frac{1}{2(1+x)}$$
$$\frac{dy}{dx} = y \left(\frac{1}{x} - \frac{1}{2(1-x)} - \frac{1}{2(1+x)} \right) = \frac{1-x-x^2}{x(1-x^2)}$$

隐函数求导

Ex. $x^2 + xy + y^2 = 1$ 确定了隐函数 $y = y(x)$, 求 $y'(x)$. $\rightarrow y'(x)$

解: 视 $x^2 + xy + y^2 = 1$ 中 $y = y(x)$, 两边对 x 求导, 得

$$2x + y + xy' + 2yy' = 0, \quad y' = -\frac{2x + y}{x + 2y}.$$

于是

$$\begin{aligned} y'' &= -\frac{(2x + y)'(x + 2y) - (2x + y)(x + 2y)'}{(x + 2y)^2} \\ &= -\frac{(2 + y')(x + 2y) - (2x + y)(1 + 2y')}{(x + 2y)^2} \\ &= \frac{3(xy' - y)}{(x + 2y)^2} = \frac{-6}{(x + 2y)^3} \quad \square \end{aligned}$$