

$$A \in \mathbb{R}^{m \times n} \quad \text{rank}(A) = r$$

$$R(A) \in \mathbb{R}^m \quad \dim = r$$

$$N(A^T) \in \mathbb{R}^m \quad \dim = m - r$$

+
||
m

$$R(A^T) \in \mathbb{R}^n \quad \dim = r$$

$$N(A) \in \mathbb{R}^n \quad \dim = n - r$$

+
||
n

$$\begin{cases} R(A^T) + N(A) = \mathbb{R}^n \\ R(A^T) \cap N(A) = \{0\} \end{cases}$$

$$N(A) = \{x \mid Ax = 0\}$$

$$\forall x_1, x_2 \in \{x \mid Ax = b\}$$

$$Ax_1 = b$$

$$Ax_2 = b$$

$$\downarrow$$

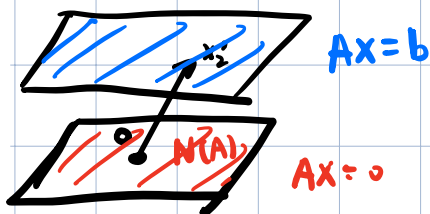
$$A(x_1 - x_2) = 0$$

= y

$$x_2 = x_1 + y$$

$$\{x \mid Ax = b\} = \{x_1 + y \mid y \in N(A)\}$$

$$Ax = b \text{ 通解} = Ax = b \text{ 的一个特解} + N(A) \text{ 的线性组合 (任意个)}$$



$$[A \mid b] \xrightarrow{E} \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\{x \mid Ax = b\} = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} + \underbrace{\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_{\text{线性组合}}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\{Ax = b\} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ -5 \\ 1 \end{pmatrix} \right\}$$

判定定理:

若 $r(A) = r([A \mid b])$ 则有解

若 $r(A) = n$ 则唯一解

若 $r(A) + 1 = r([A \mid b])$ 则无解

Chp. III 内积与正交性

$Ax = b$ 无解?

有误差

$$\tilde{A} \tilde{x} = \tilde{b} \text{ 真. 有解}$$

$\text{dist}(A, \tilde{A})$ 很小

$\text{dist}(b, \tilde{b})$ 很小

求 x st. $Ax - b$ 最小

最小 $\leftrightarrow$ 垂直

$$R(A) = \{Ax \mid x \in \mathbb{R}^n\}$$

定义(内积): 向量 $a, b \in \mathbb{R}^n$ 的内积

$$a^T b = a_1 b_1 + \dots + a_n b_n$$

若 $a^T b = 0$ 则称 a, b 正交, 记 $a \perp b$

$$\sqrt{a^T a} = \|a\| \text{ 称为 } a \text{ 的长度}$$

$$\arccos \frac{a^T b}{\|a\| \|b\|}$$

$\in [-1, 1]$ 定义:

Cauchy-Schwarz 不等式

$$|a^T b| \leq \|a\| \|b\|$$

$$\Leftrightarrow (a^T b)^2 \leq \|a\|^2 \|b\|^2$$

HW: 1, 2, 4, 6, 9, 12, 17

11.1 Review:

$Ax=b$ 解集:

$$[A \ b] \xrightarrow{E} R = \left[\begin{array}{c|c} \begin{matrix} f_1 & p_1 & f_2 & \dots & p_r \end{matrix} \\ \hline 0 \end{array} \middle| \begin{matrix} E_1 b \\ \vdots \\ E_r b \\ 0 \end{matrix} \right]$$

$$\left\{ p \begin{bmatrix} E_1 b \\ 0 \end{bmatrix} + p_{k_1} \begin{bmatrix} f_{k_1} \\ e_{k_1} \end{bmatrix} + \dots + p_{k_r} \begin{bmatrix} f_{k_r} \\ e_{k_r} \end{bmatrix} \mid \begin{matrix} \text{定义: } \angle(a, b) = \arccos \frac{a^T b}{\|a\| \|b\|} \text{ 称为} \\ k_1 \dots k_r \in \mathbb{R} \end{matrix} \right\} \text{ } a, b \text{ 的夹角}$$

$$\|Ax-b\|_{\min}$$

$$A \in \mathbb{R}^{m \times n}$$

定义: $\forall a, b \in \mathbb{R}^n$, a 与 b 的

$$\min_{x \in \mathbb{R}^n} \|Ax-b\| \text{ 为 } x \text{ 为 } a^T b = a_1 b_1 + \dots + a_n b_n$$

$$\text{性质: } 1 > a^T b = b^T a$$

$$2 > (k_1 a_1 + k_2 a_2)^T b = (k_1 a_1^T + k_2 a_2^T) b$$

$$3 > a^T a \geq 0 \quad a^T a = 0 \Leftrightarrow a = 0$$

定义: $\|a\| = \sqrt{a^T a}$ 称为 a 的长度

长度为1的向量称为单位向量

$\frac{a}{\|a\|}$ 称为 a 的单位化(向量)

$\|a-b\|$ 称为 a, b 的距离

内积 $\rightarrow$ 长度 $\rightarrow$ 距离

若 $\dim(R(A)) = 1$
 $R(A) = \text{span}\{a\}$

$$\|Ax-b\| \rightarrow \|\lambda a-b\| \quad \lambda \in \mathbb{R}$$

$$\min_{\lambda} \|\lambda a-b\|^2 = (\lambda a-b)^T (\lambda a-b)$$

$\lambda \rightarrow x$:

$$= (a^T a) x^2 - 2(a^T b) x + (b^T b)$$

$$= (a^T a) \left(x - \frac{a^T b}{a^T a}\right)^2 + b^T b - \frac{(a^T b)^2}{a^T a}$$

$x_0 = \frac{a^T b}{a^T a}$ 时 长度最小

$$\|x_0 a - b\|^2 = b^T b - \frac{(a^T b)^2}{a^T a} \geq 0$$

$\therefore$ 定理 (Cauchy-Schwarz):

$$|a^T b| \leq \|a\| \|b\|$$

定义: $\angle(a, b) = \arccos \frac{a^T b}{\|a\| \|b\|}$ 称为 a, b 的夹角

$a^T b = 0$ 时, 称为 a 与 b 垂直(或正交)

记作 $a \perp b$

$$\text{结论: } \|a+b\| \leq \|a\| + \|b\|$$

$$a \perp b \Leftrightarrow \|a+b\|^2 = \|a\|^2 + \|b\|^2$$

$$a^T (b - x_0 a) = 0 \Leftrightarrow a \perp b - x_0 a$$

定义: 其中向量两两正交的向量组称为正交向量组, 其中向量都是单位向量的正交向量组称为正交单位向量组

子空间的一组基,若是正交向量组,则称为其一组正交基

正交单位向量组, --- 标准正交基

命题: 不含零向量的正交向量组线性无关,

Proof: 记为 $a_1, \dots, a_n \in \mathbb{R}^m$
 $k_1 a_1 + \dots + k_n a_n = 0$

$$0 = a_1^T (k_1 a_1 + \dots + k_n a_n) = k_1 a_1^T a_1 + \dots + k_n a_1^T a_n = k_1 a_1^T a_1$$

$$\vdots$$

$$0 = a_n^T (\dots) \Rightarrow k_n = 0 \quad \#$$

命题: $M \neq \{0\}$, 存在一组正交基, 从而存在一组标准正交基

Proof: 设 $a_1, \dots, a_r$ 为 M 的一组基

$$a_1 \vee$$

$$\text{span}\{a_1, a_2\} = \text{span}\{a_1, b_2\}$$

$$b_2 = k_1 a_1 + k_2 a_2$$

$$0 = a_1^T b_2 = k_1 a_1^T a_1 + k_2 a_1^T a_2 = k_1 (a_1^T a_1 - \frac{a_1^T a_2 a_2^T a_1}{a_1^T a_1})$$

$$k_1 = -\frac{a_1^T a_2}{a_1^T a_1}, k_2$$

1位 $\vee$

归纳: 设对 p 维子空间都存在一组正交基, 观察 $p+1$ 维子空间

设 $a_1, \dots, a_p, a_{p+1}$ 为一组基

$\text{span}\{a_1, \dots, a_p\}$ 是 p 维子空间有正交基 $q_1, \dots, q_p$

$$\text{span}\{a_1, \dots, a_{p+1}\} = \text{span}\{q_1, \dots, q_p, a_{p+1}\}$$

$$q_{p+1} = k_1 q_1 + \dots + k_p q_p + k_{p+1} a_{p+1}$$

$$0 = q_j^T q_{p+1} = k_1 q_j^T q_1 + \dots + k_p q_j^T q_p + k_{p+1} q_j^T a_{p+1}$$

$$\therefore k_j = -\frac{q_j^T a_{p+1}}{q_j^T q_j} \cdot k_{p+1}$$

取 $k_{p+1} = 1$ 则

$$q_{p+1} = -\frac{q_1^T a_{p+1}}{q_1^T q_1} q_1 - \dots - \frac{q_p^T a_{p+1}}{q_p^T q_p} q_p + a_{p+1}$$

则 $q_1, \dots, q_{p+1}$ 是一组正交基

基 $a_1, \dots, a_r$

$$q_1 = a_1$$

$$q_2 = a_2 - \frac{a_2^T q_1}{q_1^T q_1} q_1$$

$$q_3 = a_3 - \frac{a_3^T q_1}{q_1^T q_1} q_1 - \frac{a_3^T q_2}{q_2^T q_2} q_2$$

$$\vdots$$

$$q_r = a_r - \frac{a_r^T q_1}{q_1^T q_1} q_1 - \dots - \frac{a_r^T q_{r-1}}{q_{r-1}^T q_{r-1}} q_{r-1}$$

$$\tilde{q}_1 = \frac{q_1}{\|q_1\|}$$

$$\tilde{q}_2 = \frac{q_2}{\|q_2\|}$$

$$\Rightarrow \begin{cases} q_1 = a_1 \\ q_2 = (I - \frac{q_1 q_1^T}{q_1^T q_1}) a_2 \\ q_3 = (I - \frac{q_1 q_1^T}{q_1^T q_1} - \frac{q_2 q_2^T}{q_2^T q_2}) a_3 \\ \vdots \\ q_r = (I - \frac{q_1 q_1^T}{q_1^T q_1} - \dots - \frac{q_{r-1} q_{r-1}^T}{q_{r-1}^T q_{r-1}}) a_r \end{cases}$$

$\mathbb{R}^n$ $a_1, \dots, a_n$ 是基 $\Leftrightarrow A = [a_1, \dots, a_n]$ 可逆

$q_1, \dots, q_n$ 是标准正交基 $\Leftrightarrow Q = [q_1, \dots, q_n]$?

$\rightarrow$ 正交

定义: $Q \in \mathbb{R}^{n \times n}$, $Q^T Q = I_n$, 则称 Q 是正交矩阵

性质: Q 正交 ($Q^T Q = I$)

$$\Leftrightarrow Q^T = Q^{-1}$$

$$\Leftrightarrow Q Q^T = I, Q^{-1} \text{ 正交}$$

$$\Leftrightarrow Q \text{ 的列是一组标准正交基}$$

$$\Leftrightarrow Q^T \dots$$

$$[a_1, \dots, a_r] = [q_1, \dots, q_r] \begin{bmatrix} * & * & * \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$A = Q B$$

HW: 1-4, 7-9, 11-13, 15, 16(1), 17, 19-21

11.6

内积 $\rightarrow$ 长度 $\rightarrow$ 夹角

正交 $\rightarrow$ 正交基, 标准正交基

Gram-Schmidt 正交化

$a_1, \dots, a_r$ 为基

(CGS)

Classic
Gram
Schmidt

$$\tilde{q}_1 = a_1 \quad q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|}$$

$$\tilde{q}_2 = a_2 - (q_1^T a_2) q_1 \quad q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|}$$

$$\tilde{q}_3 = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2 \quad q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|}$$

$\vdots$

$$\tilde{q}_r = a_r - (q_1^T a_r) q_1 - \dots - (q_{r-1}^T a_r) q_{r-1} \quad q_r = \frac{\tilde{q}_r}{\|\tilde{q}_r\|}$$

基 $a_1, \dots, a_r$

(MGS)

Modified
Gram
Schmidt

$$q_1^{(0)} = a_1$$

$$q_2^{(0)} = a_2 - \frac{q_1^{(0)T} a_2}{q_1^{(0)T} q_1^{(0)}} q_1^{(0)}$$

$$q_3^{(0)} = a_3 - \frac{q_1^{(0)T} a_3}{q_1^{(0)T} q_1^{(0)}} q_1^{(0)} - \frac{q_2^{(0)T} a_3}{q_2^{(0)T} q_2^{(0)}} q_2^{(0)}$$

$$q_r^{(0)} = a_r - \dots - \frac{q_1^{(0)T} a_r}{q_1^{(0)T} q_1^{(0)}} q_1^{(0)} - \dots - \frac{q_{r-1}^{(0)T} a_r}{q_{r-1}^{(0)T} q_{r-1}^{(0)}} q_{r-1}^{(0)}$$

$$q_3^{(1)} = q_3^{(0)} - \frac{q_2^{(1)T} q_3^{(0)}}{q_2^{(1)T} q_2^{(1)}} q_2^{(1)}$$

$$q_r^{(1)} = q_r^{(0)} - \frac{q_2^{(1)T} q_r^{(0)}}{q_2^{(1)T} q_2^{(1)}} q_2^{(1)} - \dots - \frac{q_{r-1}^{(1)T} q_r^{(0)}}{q_{r-1}^{(1)T} q_{r-1}^{(1)}} q_{r-1}^{(1)}$$

$\mathbb{R}^n$

$a_1, \dots, a_n$ 基 $\Leftrightarrow (a_1, \dots, a_n)$ 可逆

$q_1, \dots, q_n$ 标准正交基 $\Leftrightarrow (q_1, \dots, q_n)$ 正交

命题: Q 正交, 即 $Q^T Q = I_n$

$\Leftrightarrow Q$ 保距 (保长) 即 $\|Qx\| = \|x\| \quad \forall x$

$\Leftrightarrow Q$ 保内积, 即 $(Qx)^T (Qy) = x^T y \quad \forall x, y$

Proof:

$$1) \Rightarrow 2): (Qx)^T (Qx) = x^T Q^T Q x = x^T I x = x^T x$$

$$2) \Rightarrow 3): x^T y = \frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2}$$

$$(Qx)^T (Qy) = \frac{\|Q(x+y)\|^2 - \|Qx\|^2 - \|Qy\|^2}{2}$$

3) $\Rightarrow$ 1):

$$3 \Leftrightarrow (Qe_i)^T (Qe_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$\Downarrow$

#

$$\mathbb{R}^2: Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \text{ 正交 } = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$$(\cos \theta \quad \sin \theta) \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} = 0 \Rightarrow \theta - \varphi = \frac{\pi}{2} \cdot \frac{3}{2} \pi \quad (\text{非} \pi)$$

$$(1) \quad \theta = \varphi + \frac{\pi}{2} \quad Q = \begin{pmatrix} -\sin \varphi & \cos \varphi \\ \cos \varphi & \sin \varphi \end{pmatrix} \quad \text{反射}$$

$$(2) \quad \theta = \varphi + \frac{3}{2} \pi \quad Q = \begin{pmatrix} \sin \varphi & \cos \varphi \\ -\cos \varphi & \sin \varphi \end{pmatrix} \quad \text{旋转}$$

高维旋转:

$$\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \cos \theta & -\sin \theta \\ & & \sin \theta & \cos \theta \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \quad \text{Givens 变换}$$

反射:

$$Hx = x - 2 \frac{(x-v)(v^T x)}{v^T v}$$

$$\|Hx\| = \|x\|$$

$$Hx - x = -2 \frac{v^T x}{v^T v} v$$

$$\therefore Hx = (I - 2 \frac{vv^T}{v^T v}) x$$

$$\therefore H = I - 2 \frac{vv^T}{v^T v} = I - 2v_0 v_0^T$$

$\dim N-1$ Householder 变换

命题: 给定 $x, y, \|x\| = \|y\|$, 则存在 Householder 变换 H , s.t. $Hx = y$

(3列满秩):

$$\begin{bmatrix} a_1 & \dots & a_r \end{bmatrix}_{m \times r} = \begin{bmatrix} q_1 & \dots & q_r \end{bmatrix}_{m \times r} \begin{bmatrix} \|q_1\| & q_1^T a_2 & q_1^T a_3 & \dots & q_1^T a_r \\ 0 & \|q_2\| & q_2^T a_3 & \dots & q_2^T a_r \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \|q_{r-1}\| & q_{r-1}^T a_r \\ 0 & \dots & 0 & 0 & \|q_r\| \end{bmatrix}_{r \times r}$$

定义: $m \times n$ 矩阵 Q , 若 $Q^T Q = I_n$, 则称 Q 为 $m \times n$ 列正交矩阵

($m \geq n$) 定理 (QR 分解): 对 $m \times n$ 矩阵 A

(简化 QR 分解) $\Rightarrow$ 存在列正交矩阵 Q 和对角元非负的上三角矩阵 R , 使得 $A = QR$ $\begin{matrix} m \times n & m \times n & n \times n \end{matrix}$

(QR 分解) $\Rightarrow$ 存在正交矩阵 Q , 使得 $A = Q \begin{bmatrix} R \\ 0 \end{bmatrix}$ $\begin{matrix} m \times n & m \times m & m \times n \end{matrix}$

$$\text{注: } Q = [q_1 \quad q_2] \quad [a_1 \quad a_2] \begin{bmatrix} R \\ 0 \end{bmatrix} = QR$$

Proof: 列变换视角

对 n 归纳 $m \neq 1$ $\checkmark$

设 $m \times (n-1)$ $\checkmark$ 观察 $m \times n$

$$A = (a_1 \dots a_{n-1} \ a_n) \\ = (\tilde{Q}_1 \tilde{R}_1 \ a_n)$$

若 $a_n \in R(\tilde{Q}_1)$

若 $a_n \in R(\tilde{Q}_1)$

$$A = [\tilde{Q}_1 \tilde{R}_1 \ a_n] \begin{matrix} a_n = \tilde{Q}_1 \tilde{R}_1 + \tilde{Q}_1 \tilde{a}_n \\ \tilde{Q}_1 \perp \tilde{Q}_1 \end{matrix} \quad \begin{matrix} A = [\tilde{Q}_1 \tilde{R}_1 \ \tilde{Q}_1 \tilde{a}_n] \\ = \tilde{Q}_1 [\tilde{R}_1 \ \tilde{a}_n] \\ = [\tilde{Q}_1 \ q_n] \begin{bmatrix} \tilde{R}_1 & \tilde{a}_n \\ 0 & 0 \end{bmatrix} \\ \tilde{Q}_1 \quad R \end{matrix}$$

$$= [\tilde{Q}_1 \ q_n] \begin{bmatrix} \tilde{R}_1 & \tilde{a}_n \\ 0 & 1 \end{bmatrix}$$

Proof 2: 正交行变换视角

$$Q^T A = \begin{bmatrix} \nabla \\ 0 \end{bmatrix}$$

$$\Leftrightarrow Q_1^T \dots Q_n^T Q^T A = \begin{bmatrix} \nabla \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \times & \times & \times \\ \vdots & \vdots & \vdots \\ \times & \times & \times \end{bmatrix} \xrightarrow{H_1} \begin{bmatrix} + & + & + \\ 0 & + & + \\ \vdots & \vdots & \vdots \\ 0 & + & + \end{bmatrix} \xrightarrow{H_2} \begin{bmatrix} + & + & + \\ t_2 & t_2 & t_2 \\ 0 & + & + \\ \vdots & \vdots & \vdots \\ 0 & + & + \end{bmatrix} \xrightarrow{H_3} \begin{bmatrix} + & + & + \\ t_2 & t_2 & t_2 \\ 0 & + & + \\ \vdots & \vdots & \vdots \\ 0 & + & + \end{bmatrix}$$

证: 对 m 归纳

$m \neq 1$ $\checkmark$

设 $(m-1) \times p$ ($p \leq m-1$) $\checkmark$

观察: $m \times n$:

$$\exists H \text{ s.t. } H a_1 = \|a_1\| e_1$$

$$H_1 A = [H_1 a_1 \dots H_1 a_n] = \begin{bmatrix} \|a_1\| & * & * \\ 0 & A_2 \end{bmatrix}$$

$$= \begin{bmatrix} \|a_1\| & * \\ 0 & \tilde{Q}_1^T A_2 \end{bmatrix} = \begin{bmatrix} 1 & \tilde{Q}_1^T A_2 \\ 0 & R \end{bmatrix} \begin{bmatrix} \|a_1\| & * \\ 0 & R \end{bmatrix}$$

$$\text{若 } a_1 = a_2 = \dots = a_{j-1} = 0 \quad a_j \neq 0$$

$$A = \begin{bmatrix} 0 & A_2 \end{bmatrix} \quad A_2 = \tilde{Q}_1^T R \quad \tilde{Q}_1^T \begin{bmatrix} 0 & R \end{bmatrix} = A$$

$\Rightarrow$ 若 A 可逆, 则 QR 分解唯一

$$\text{证: } A = Q_1 R_1 = Q_2 R_2$$

$$Q_1^T Q_1 = R_2 R_1^{-1} \Rightarrow \text{对角} \Rightarrow \text{对角元为 } 1$$

HW, 1-2. 4-8. 10-12. 15. 17-19.

11. 8.

Review: 正交矩阵、QR分解

Gram-Schmidt 正交化

$$\tilde{q}_1 = a_1 \quad q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|}$$

$(I - q_1 q_1^T) a_2$
新向量向正交补正交投影

$$\tilde{q}_2 = a_2 - (q_1^T a_2) q_1 \quad \text{若 } \tilde{q}_2 \neq 0 \quad q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|}$$

原向量

$$(I - [q_1 \ q_2] \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix}) a_3$$

$\Rightarrow$ CGS:

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} = \begin{bmatrix} \tilde{q}_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} q_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \frac{1}{\|\tilde{q}_1\|} & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} q_1 & \tilde{q}_2 & a_3 & \dots & a_n \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & \frac{1}{\|\tilde{q}_1\|} & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} q_1 & q_2 & a_3 & \dots & a_n \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & \frac{1}{\|\tilde{q}_1\|} & & & \\ & & \frac{1}{\|\tilde{q}_2\|} & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \rightarrow \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & \frac{1}{\|\tilde{q}_1\|} & & & \\ & & \frac{1}{\|\tilde{q}_2\|} & & \\ & & & \ddots & \\ & & & & \frac{1}{\|\tilde{q}_n\|} \end{bmatrix}$$

MGS:

$$q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \frac{a_1}{\|a_1\|}$$

$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{a_2 - q_1^T a_2 q_1}{\|a_2 - q_1^T a_2 q_1\|}$$

$$\vdots \quad q_n = \frac{\tilde{q}_n}{\|\tilde{q}_n\|} = \frac{a_n - q_1^T a_n q_1 - \dots - q_{n-1}^T a_n q_{n-1}}{\|a_n - q_1^T a_n q_1 - \dots - q_{n-1}^T a_n q_{n-1}\|}$$

$$\begin{array}{c|cc} a_1 & a_2 & a_3 \\ \hline \tilde{q}_1 & a_2 & a_3 \\ q_1 & \tilde{q}_2 & a_3 \\ \hline q_1 & \tilde{q}_2 & a_3 \\ q_1 & q_2 & a_3 \\ \hline q_1 & q_2 & \frac{1}{\|\tilde{q}_3\|} \\ q_1 & q_2 & q_3 \end{array} \quad \begin{array}{l} \tilde{q}_2 \perp \text{span}(q_1) \\ \tilde{q}_3 \perp \text{span}(q_1, q_2) \\ \tilde{q}_r \perp \text{span}(q_1, q_2, \dots, q_{r-1}) \end{array}$$

定义: $\mathbb{R}^n$ 的子空间 M, N , 若 $\forall a \in M, b \in N$, 满足

$a \perp b$, 则称 $M \perp N$ 正交, 记作 $M \perp N$

$\text{span}(a) \perp M$ 正交, 简记为 $a \perp M$

定义(正交补): 对 $\mathbb{R}^n$ 的子空间 M , 称 $\mathbb{R}^n$ 的子集 $\{a \in \mathbb{R}^n \mid a \perp M\} = M^\perp$ 为 M 的正交补

性质: $M \perp N \Rightarrow N \subset M^\perp$

$$M = \text{span}\{q_1, \dots, q_r\} \quad \dim M = r$$

$$Q^T X = [q_1, \dots, q_r]^T X = 0$$

$$M^\perp = N(Q^T) \quad \dim M^\perp = n-r$$

$$(M^\perp)^\perp = ?$$

$$q_1, \dots, q_r, q_{r+1}, \dots, q_n$$

$$\therefore \dim(\text{span}\{q_{r+1}, \dots, q_n\}) = \dim M^\perp$$

$$\text{span}\{q_{r+1}, \dots, q_n\} \perp M \subseteq M^\perp$$

$$\therefore \text{span}\{q_{r+1}, \dots, q_n\} = M^\perp$$

命题: 1. $M^\perp$ 是子空间

$$2. M \cap M^\perp = \{0\}$$

$$3. \dim M + \dim M^\perp = n$$

$$4. (M^\perp)^\perp = M$$

$$5. \forall a \in \mathbb{R}^n \text{ 存在唯一分解 } a = a_1 + a_2, \quad a_1 \in M, a_2 \in M^\perp$$

$$R(A), N(A), R(A^T), N(A^T)$$

$$\text{命题: } (1) R(A) = N(A^T)^\perp \quad N(A) = R(A^T)^\perp$$

$$(2) R(ATA) = R(AT) \quad N(ATA) = N(A)$$

$$R(AAT) = R(A) \quad N(AAT) = N(AT)$$

$$\text{证: } (1). \dim N(A) = n-r \quad \dim R(A^T) = r$$

$$N(A) \perp R(A^T): \forall x \in N(A), Ax = 0$$

$$\forall y \in R(A^T), \exists \hat{y} \text{ s.t. } y = A^T \hat{y}$$

$$x^T y = x^T A^T \hat{y} = (Ax)^T \hat{y} = 0 \quad \therefore N(A) \subseteq R(A^T)^\perp$$

$$\text{又: } \dim N(A) = \dim R(A^T)^\perp$$

$$\therefore N(A) = R(A^T)^\perp$$

$$\Rightarrow N(A) \subseteq N(ATA):$$

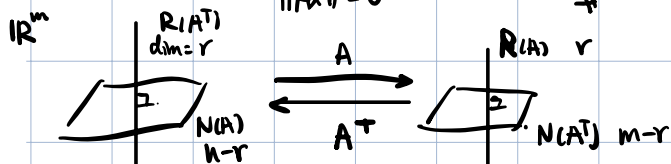
$$Ax = 0 \Rightarrow A^T Ax = 0$$

$$N(ATA) \subseteq N(A):$$

$$A^T Ax = 0$$

$$x^T A^T Ax = 0$$

$$\|Ax\| = 0 \Rightarrow Ax = 0 \quad \#$$



$$R(A^T) \xrightarrow{A} R(A) \quad R(A) \xrightarrow{A^T} R(A^T)$$

$$N(A) \xrightarrow{A} \{0\} \quad N(A^T) \xrightarrow{A^T} \{0\}$$

$$\text{下证 } \tilde{A}: R(A^T) \rightarrow R(A) \text{ 为双射}$$

$$\text{证: } \# \quad Ax_1 = Ax_2$$

$$\Rightarrow x_1 = x_2$$

$$A(x_1 - x_2) = 0$$

$$Ay = 0 \Rightarrow y = 0$$

$$y \in R(A^T)$$

$$\text{满: } \forall x \in R(A) \quad (\exists y \in R(A^T), Ay = x)$$

$$x = A\hat{x} = A\hat{x}_1 + A\hat{x}_2 = A\hat{x}_1$$

$$\hat{x} = \hat{x}_1 + \hat{x}_2$$

$$\hat{x}_1 \in R(A^T), \hat{x}_2 \in N(A) \quad \#$$

定义: 对 $\mathbb{R}^n$ 的子空间 M , 若 $a \in M$, 则存在唯一的 $a_1 \in M, a_2 \in M^\perp$, s.t. $a = a_1 + a_2$

故: $P_M: \mathbb{R}^n \rightarrow \mathbb{R}^n$ 是一个映射, 称为子空间 M 上的正交投影(变换)

向量 a 的像 $P_M(a)$ 称为 a 在 M 上的正交投影

$$I = P_M + P_{M^\perp}$$

$$\underbrace{q_1, \dots, q_r}_M \quad \underbrace{q_{r+1}, \dots, q_n}_{M^\perp}$$

$$\forall a \in \mathbb{R}^n, a = \underbrace{k_1 q_1 + \dots + k_r q_r}_r + k_{r+1} q_{r+1} + \dots + k_n q_n$$

$$a_1 = k_1 q_1 + \dots + k_r q_r$$

$$P_M(a) = a_1$$

$$Q = [Q_1 \ Q_2] \quad k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$P_M = [P_M(e_1) \ \dots \ P_M(e_n)]$$

$$= (P_M(Q Q^T e_1) \ \dots)$$

$$= (Q_1 [Q_1^T e_1] \ \dots)$$

$$= (Q_1 Q_1^T e_1 \ \dots \ Q_1 Q_1^T e_n)$$

$$\text{HW: } 1, 4-8 \quad = Q_1 Q_1^T$$

f : 子空间 $\mapsto$ 正交投影

双射

11.13 Review:

$$M^\perp = \{a \in \mathbb{R}^n \mid a \perp M\}$$

$$\underbrace{q_1 \dots q_r}_M \quad \underbrace{q_{r+1} \dots q_n}_{M^\perp}$$

$$N(A) = R(A^T)^\perp \quad N(A^T) = R(A)^\perp$$

正交投影: $a = \underbrace{a_1}_M + \underbrace{a_2}_{M^\perp}$

$$P_M: \mathbb{R}^n \rightarrow \mathbb{R}^n \quad a \rightarrow a_1 \quad P_M = [q_1 \dots q_r] [q_1 \dots q_r]^T$$

$Q \quad Q^T$

$$\min \|b - Ax\| = \min \|b - y\|$$

$\downarrow$
 $y \in R(A)$

$$= \min \|(I - QRQ^T + QRQ^T)b - y\|$$

$$(QRQ^T = P_{R(A)})$$

$$= \|(\underbrace{I - QRQ^T}_{R(A)^\perp} b + \underbrace{QRQ^T}_{R(A)} b - y \|$$

$$= \min \sqrt{((I - QRQ^T)b)^2 + (QRQ^T b - y)^2}$$

$= 0$

$$= \underline{(I - QRQ^T)b} \quad \Rightarrow \underline{y = QRQ^T b}$$

$$\text{子空间 } M \rightarrow QRQ^T = P_M$$

命题: P 是正交投影矩阵 $\Leftrightarrow P^2 = P = P^T$

Proof: $\Rightarrow \checkmark$

$$\Leftarrow \text{只需 } P = P_{R(P)}$$

$$P^T(I - P) = 0$$

$$\forall x \in \mathbb{R}^n \quad (I - P)x \in N(P^T)$$

$$x = \underbrace{Px}_{R(P)} + \underbrace{(I - P)x}_{N(P^T)} \quad \text{唯一分解}$$

$$Px = P_{R(P)} x$$

子空间 $M \rightarrow$ 基 $A \rightarrow$ 标准正交基 Q

$\rightarrow$ 正交投影 QQ^T

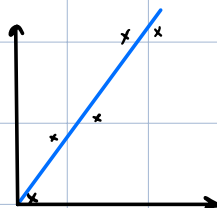
$$A^T A = R^T Q^T Q R = R^T R$$

$$A = QR \quad \begin{matrix} \uparrow \text{列满秩} & \uparrow \text{可逆} \end{matrix} \quad Q = AR^{-1}$$

$$\begin{aligned} QQ^T &= AR^{-1}(R^{-1})^T A^T \\ &= A(R^T R)^{-1} A^T \\ &= \underline{A(A^T A)^{-1} A^T} \end{aligned}$$

拟设 $y = kx + b$
(Ansatz)

模型



(x_i, y_i) 采样
 $y = \tilde{k}x + \tilde{b}$ 拟合

$$\text{令 } y = k_1 f_1(x) + k_2 f_2(x) + \dots + k_n f_n(x)$$

$$\begin{cases} f_1(x_1)k_1 + f_2(x_1)k_2 + \dots + f_n(x_1)k_n = y_1 \\ f_1(x_2)k_1 + f_2(x_2)k_2 + \dots + f_n(x_2)k_n = y_2 \\ \vdots \\ f_1(x_m)k_1 + f_2(x_m)k_2 + \dots + f_n(x_m)k_n = y_m \end{cases}$$

$$\begin{pmatrix} f_1(x_1) & f_2(x_1) & \dots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \dots & f_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(x_m) & f_2(x_m) & \dots & f_n(x_m) \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

$$\min \|y - Fk\| = \|P_{R(F)} y\|$$

$$Fk = P_{R(F)} y \quad \text{※ 定理 (Fk=y)}$$

$$k = k_0 + \xi$$

↑ ↑
特解 N(F) 中的向量

若下列满秩

$$P_{R(F)} = F(F^T F)^{-1} F^T$$

$$Fk = F(F^T F)^{-1} F^T y$$

$$k = (F^T F)^{-1} F^T y \Leftrightarrow (F^T F)k = F^T y$$

$$Ax=b \text{ 可能无解} \rightarrow \|b - Ax_0\| = \|b - Ax\| \min$$

称 x_0 是 $Ax=b$ 的最小二乘解 ↓ 剩余 (残差)

命题: x_0 是 $Ax=b$ 的最小二乘解 $\Leftrightarrow$

$$A^T A x_0 = A^T b$$

$$\Rightarrow Ax_0 = P_{R(A)} b$$

$$\text{证明 } A^T b = A^T P_{R(A)} b$$

$$= (P_{R(A)}^T A)^T b$$

(P 对称)

$$= (P_{R(A)} A)^T b$$

$$(A \in \mathbb{R}^m) \\ = A^T b$$

$$\Leftarrow \text{已知 } A^T A x_0 = A^T b \Rightarrow Ax_0 = P_{R(A)} b$$

$$A^T (b - Ax_0) = 0$$

$$b - Ax_0 \in N(A^T) \perp R(A)$$

$$b = \underbrace{b - Ax_0}_{N(A^T)} + \underbrace{Ax_0}_{R(A)}$$

$$\therefore P_{R(A)} b = Ax_0$$

注: 最小二乘解唯一 $\Leftrightarrow N(A^T A) = N(A) = \{0\}$

正则化方法: $A^T A x = A^T b$

$$1 > C = A^T A$$

$$d = A^T b$$

$$2 > L D L^T = C$$

$\rightarrow$ 可逆对称方阵

$$3 > L \xi = d$$

$$4 > D \eta = \xi$$

$$5 > L^T x = \eta$$

正交化方法:

计算量小

$$1 > A = QR$$

$$2 > d = Q^T b$$

$$3 > Rx = d$$

$$HW: 9, 10, 12(奇), 13-15, 17, 19, 24$$

11, 15 chp 4 §3 例 1

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = a_1$$

$$S_\Delta = S(a_1, a_2) = \frac{1}{2} \|a_1\| \left\| \left(I - \frac{a_1 a_1^T}{a_1^T a_1} \right) a_2 \right\|$$

$$S^2 = \frac{1}{4} (a_1^T a_1) (a_2^T (I - \frac{a_1 a_1^T}{a_1^T a_1}) a_2)$$

$$= \frac{1}{4} a_1^T a_1 (a_2^T (I - \frac{a_1 a_1^T}{a_1^T a_1}) a_2)$$

$$= \frac{1}{4} (a_1^T a_1 a_2^T a_2 - a_2^T a_1 a_1^T a_2)$$

$$= \frac{1}{4} (a_{11}^2 + a_{21}^2)(a_{12}^2 + a_{22}^2) - (a_{11} a_{12} + a_{21} a_{22})^2$$

$$\Rightarrow S = \frac{1}{2} |a_{11} a_{22} - a_{21} a_{12}|$$

有向面积 (符号面积):

$$S(a_1, a_2) = a_{11} a_{22} - a_{21} a_{12}$$

"S(A)"

$$B: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$x \mapsto Bx \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$S(Ba_1, Ba_2) = S \left(\begin{bmatrix} a_{11}b_{11} + a_{21}b_{12} \\ a_{11}b_{21} + a_{21}b_{22} \end{bmatrix}, \begin{bmatrix} a_{21}b_{11} + a_{22}b_{12} \\ a_{21}b_{21} + a_{22}b_{22} \end{bmatrix} \right)$$

$$= (a_{11}b_{11} + a_{21}b_{12})(a_{21}b_{21} + a_{22}b_{22}) - (a_{21}b_{11} + a_{22}b_{12})(a_{11}b_{21} + a_{12}b_{22})$$

$$= (b_{11}a_{11} - b_{12}a_{21})(a_{11}a_{22} - a_{12}a_{21})$$

$$S(Ba_1, Ba_2) = S(b_1, b_2) \cdot S(a_1, a_2)$$

$$\Leftrightarrow S(BA) = S(B) S(A)$$

$$\delta(a_1 + \tilde{a}_1, a_2) = \delta(a_1, a_2) + \delta(\tilde{a}_1, a_2)$$

$$\delta(ka_1, a_2) = k\delta(a_1, a_2)$$

$$\delta([a_1, a_2] [k, 1]) = \delta([k, 1]) \cdot \delta(a_1, a_2)$$

定义(行列式): 定义在方阵上的函数 $\delta: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$
(n 个向量组)

若满足如下条件:

1° 列(行)线性:

加法 & 数乘

$$\delta(\dots, ka_i + ka_i, \dots) = k\delta(\dots, a_i, \dots) + k\delta(\dots, a_i, \dots)$$

2° 列反对称:

$$\delta(\dots, a_i, \dots, a_j, \dots) = -\delta(\dots, a_j, \dots, a_i, \dots)$$

3° 单位化条件:

$$\delta(I_n) = 1$$

则称 δ 是 n 阶方阵上的一个行列式函数

将证明: n 阶方阵的行列式函数存在 & 唯一:

记为 $\det(A)$, 称为 A 的行列式

($|A|$) 不推荐

$$\delta(a_1 + a_2, a_1 + a_2) = \delta(a_1, a_1) + \delta(a_1, a_2) + \delta(a_2, a_1) + \delta(a_2, a_2) = \delta(a_1, a_1) + \delta(a_2, a_2) = 0$$

$$\delta(a_1, a_2) = 0 \Leftrightarrow \delta(m, m) = 0$$

性质:

$$1. A \text{ 有 2 列相等} \Rightarrow \det(A) = 0$$

$$2. A \text{ 不满秩} \Leftrightarrow \det(A) = 0$$

Proof: 不妨设 a_n 被其他列表示, $\mathbb{R}^n$

$$a_n = k_1 a_1 + \dots + k_{n-1} a_{n-1}$$

$$\det A = \det(a_1, \dots, a_{n-1}, k_1 a_1 + \dots + k_{n-1} a_{n-1}) = 0$$

$$\delta(A) = -\delta(A P_{ij})$$

$$\delta(A E_{ij, k}) = \delta([a_1, \dots, a_n] \begin{bmatrix} & & k \\ & & \\ & 1 & \\ & & \end{bmatrix})$$

$$= \delta(a_1, \dots, a_i, \dots, a_j + ka_i, \dots, a_n)$$

$$= \delta(a_1, \dots, a_n) + k\delta(a_1, \dots, a_i, \dots, a_i, \dots, a_n) = \delta(A)$$

$$3. \det(A P_{ij}) = -\det(A)$$

$$\det(A E_{ij, k}) = \det(A)$$

$$\det(A E_{ii, k}) = k \det(A)$$

$$\text{令 } A = I \quad \det(P_{ij}) = -1$$

$$\Rightarrow \det(E_{ii, k}) = k$$

$$\det(E_{ij, k}) = 1$$

$$4. \det(AB) = \det(A) \cdot \det(B)$$

$$\text{若 } B \text{ 可逆} \Rightarrow \det(B) = \det(E_1 \dots E_t)$$

$$\det(AB) = \det(A E_1 \dots E_t) \det(E_t)$$

$$= \det(A) \det(E_1) \det(E_2) \dots \det(E_t)$$

$$= \det(A) \det(E_1 \dots E_t) = \det(A) \cdot \det(B)$$

若 B 不可逆

$$\det(AB) = 0 = \det(B) = \det(A) \cdot \det(B)$$

$\Rightarrow AB$ 不可逆

※

$$5. \det(A) \neq 0 \Leftrightarrow A \text{ 可逆}$$

$$\det(A) = 0 \Leftrightarrow A \text{ 不可逆}$$

$$6. \det(A+B) \times$$

$$\det(kA) = k^n \det(A)$$

$$\det(A^T) = \det(A)$$

$$A \text{ 不可逆} \quad \det(A^T) = 0 = \det(A)$$

A 可逆

$$\det(A^T) = \det(E_1^T \dots E_t^T) = \det(E_1^T) \dots \det(E_t^T)$$

$$= \det(E_1) \dots \det(E_t) = \det(E_1) \dots \det(E_t) = \det(E_1 \dots E_t) = \det(A)$$

$$\Rightarrow \text{行列线性, 行反对称}$$

命题: 若函数 $\delta: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ 满足

$$(1) \det(AB) = \det(A) \det(B)$$

$$(2) \det(A) = 0 \text{ 若 } A \text{ 不可逆}$$

$$(3) \det(P_{ij}) = -1; \det(E_{ii, k}) = k; \det(E_{ij, k}) = 1 \quad (i \neq j)$$

则 $\delta = \det$.

$$\begin{vmatrix} u_{11} & \dots & u_{1n} \\ & \ddots & \\ & & u_{nn} \end{vmatrix} = u_{11} \dots u_{nn}$$

$$\begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = 1 \dots 1$$

$$\text{正交} \Rightarrow \det(Q) = \pm 1 \quad \det \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = 1$$

$$\downarrow \quad \det(Q^T Q) = \det(Q)^2 = 1 \quad \det \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} = -1$$

HW: 1, 2, 4-7, 12-13, 17, 19-20, 22-23, 25, 27-28

11, 20

Review:

行列式定义, 唯一性

列多线性. 列反对称. 单位化条件 $\rightarrow$ 行列式

行... , 列... , ...

乘积性质 特殊点值

$$\det \begin{pmatrix} 1 & \dots & 1 \\ & & \\ & & \end{pmatrix} = k$$

↑
按列
展开

$$\det A = \det A^T$$

$$\det(AB) = \det A \cdot \det B$$

多项式插值:

$$(x_i, y_i) \quad i=0,1,2,3$$

$$y = a_0 x^3 + a_1 x^2 + a_2 x + a_3$$

$$\begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{pmatrix} \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{pmatrix} \quad \text{Vandermonde 矩阵}$$

$$= (x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \cdot \dots$$

$$\det V = \begin{vmatrix} 1 & x_0 & x_0^2 & 0 \\ 1 & x_1 & x_1^2 & x_1^3 - x_1^2 x_0 \\ 1 & x_2 & x_2^2 & x_2^3 - x_2^2 x_0 \\ 1 & x_3 & x_3^2 & x_3^3 - x_3^2 x_0 \end{vmatrix} = \begin{vmatrix} 1 & x_0 & 0 & 0 \\ 1 & x_1 & x_1(x_1 - x_0) & x_1^2(x_1 - x_0) \\ 1 & x_2 & x_2(x_2 - x_0) & x_2^2(x_2 - x_0) \\ 1 & x_3 & x_3(x_3 - x_0) & x_3^2(x_3 - x_0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x_1 - x_0 & x_1(x_1 - x_0) & x_1^2(x_1 - x_0) \\ 1 & x_2 - x_0 & x_2(x_2 - x_0) & x_2^2(x_2 - x_0) \\ 1 & x_3 - x_0 & x_3(x_3 - x_0) & x_3^2(x_3 - x_0) \end{vmatrix}$$

$$= 1 \cdot \det V = (x_1 - x_0)(x_2 - x_0)(x_3 - x_0) \dots$$

反之

$$\Rightarrow (x_1 - x_0)(x_2 - x_0)(x_3 - x_0) \dots$$

$$= \prod_{0 \leq i < j \leq 3} (x_i - x_j)$$

Results: x_i 两两不同 $\rightarrow$ 唯一 n 阶多项式

$i=0, \dots, n \quad \exists x_i = x_j, y_i \neq y_j \rightarrow$ 无多项式

$y_i = y_j \rightarrow \infty$ 多项式

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a^4 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a & a^3 \end{vmatrix}$$

$$= (a-1)(a^2-1)(a^2-a) + (a^4-a^3)(a-1)$$

$$\begin{vmatrix} 1 & a & a^2 \\ a & a^2 & a^3 \\ a^2 & a^3 & a^4 \end{vmatrix} = 0 \quad (\text{rank}=1) \quad \text{不满秩}$$

存在性:

- 阶:

$$\det A_1 = a_{11}$$

$$\det A_2 = a_{11}a_{22} - a_{12}a_{21}$$

= 阶:

$$\det A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^2 a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} - a_{31}a_{13}a_{22}$$

定义: 方阵 A 划去 i 行 j 列得到的低一阶矩阵记作 $A(i,j)$;

$M_{ij} = \det A(i,j)$ 称为 a_{ij} 关于 A 的余子式;

$C_{ij} = (-1)^{i+j} \det A(i,j)$ 代数余子式;

定理: (行列式按行/列展开) 行列式的存在性

对 n 阶方阵 $A, n \geq 2$, 则

$$a_{11}C_{11} + a_{21}C_{21} + \dots + a_{n1}C_{n1} \text{ 是行列式函数}$$

Proof: 记 $b(A) = a_{11}C_{11} + a_{21}C_{21} + \dots + a_{n1}C_{n1}$

行多线性:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ a_{21} & & & \\ \vdots & & & \\ \tilde{a}_{n1} & & & \tilde{a}_{nn} \end{bmatrix}$$

$$\Delta \left(\begin{bmatrix} k a_{11} + \tilde{k} \tilde{a}_{11} & k a_{12} + \tilde{k} \tilde{a}_{12} & \dots & k a_{1n} + \tilde{k} \tilde{a}_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \right) = k \Delta(A) + \tilde{k} \Delta(\tilde{A})$$

$$= (k a_{11} + \tilde{k} \tilde{a}_{11}) C_{11} + a_{21} \tilde{C}_{21} + \dots + a_{n1} \tilde{C}_{n1}$$

$$= \dots \quad \#$$

行反对称:

不妨考虑第1,2行 $\Leftrightarrow$ 两行一样 $\therefore \delta = 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

$$\delta(A) = a_{11}C_{11} + a_{21}M_{21}(-1)^{1+2} + a_{31} \cdot 0 + \dots + 0$$

$$= a_{11}M_{11} - a_{11}M_{11} = 0 \quad \#$$

单位化条件:

$$\delta(I) = a_{11} \cdot \delta(I_{n-1}) = a_{11}a_{22} \cdot \dots \cdot a_{nn} = 1 \quad \#$$

按第j行展开: $\det(A) = a_{j1}C_{j1} + a_{j2}C_{j2} + \dots + a_{jn}C_{jn} = \tilde{a}_j^T \tilde{C}_j$

按第i列展开: $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in} = a_i^T \tilde{C}_i$ 注1:

$$\det(P) = (-1)^{i-1} \quad \leftarrow \text{冒泡 } (i-1) \text{ 次}$$

$$AP = [a_i \ a_1 \ \dots \ a_{i-1} \ a_{i+1} \ \dots \ a_n]$$

$$\det(AP) = a_{i1}M_{i1}(-1)^{i+1} + a_{i2}M_{i2}(-1)^{i+2} + \dots + a_{in}M_{in}(-1)^{i+n} = (-1)^{i-1} (a_{i1}C_{i1} + \dots + \sim) = \det P \cdot \det A \quad \checkmark$$

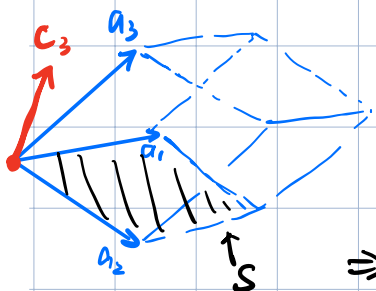
$$a_2^T C_1 = a_{12}C_{11} + a_{22}C_{21} + \dots + a_{n2}C_{n1}$$

$$= \det \begin{bmatrix} a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = 0$$

$$\begin{cases} a_i^T C_j = 0 & i \neq j \\ a_i^T C_i = \det A \end{cases}$$

$$A^T C = [a_i^T C_j] = \begin{bmatrix} \det A & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \det A & 0 \\ 0 & 0 & 0 & \det A \end{bmatrix} = \det(A) \cdot I_n$$

命题: $\det A \neq 0 \Rightarrow A^T = \frac{C^T}{\det(A)}$ A的补矩阵 (伴随矩阵) (附属矩阵)



$$\begin{aligned} a_3^T \frac{C_3}{\|C_3\|} \cdot S \\ = \det A = a_3^T C_3 \\ \Rightarrow S = \|C_3\| \end{aligned}$$

11.22

行列式

- 1> 列线性, 列反对称, 单位
- 2> 行 ... 行 ... 单位
- 3> 乘积性质
- 4> 按行/列展开
- 5> 完全展开

$$A^T C = \det(A) \cdot I$$

推论 (Cramer 法则)

对矩阵 A, $Ax=b$ 存在唯一解

$$\Leftrightarrow \det(A) \neq 0, \text{ 有解时, } x_i = \frac{\det(B_i)}{\det(A)}$$

其中 B_i 是 A 中第 i 列换为 b

$$B_i = A + (b - a_i) e_i^T$$

$$x = A^{-1} b = \frac{C^T b}{\det(A)}$$

$$x_i = e_i^T x = \frac{e_i^T C^T b}{\det(A)} = \frac{C_i^T b}{\det A} = \frac{\det(B_i)}{\det A}$$

$$\text{证: } \det(B_i) = \det(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)$$

$$= \det(\dots x_1 a_1 \dots + x_n a_n \dots)$$

$$= 0 + \dots + 0 + x_i \det(A) + 0 + \dots + 0 = x_i \det(A)$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \det(a_{11}e_1 + a_{21}e_2 + a_{31}e_3, a_{12}e_1 + a_{22}e_2 + a_{32}e_3,$$

$$a_{13}e_1 + a_{23}e_2 + a_{33}e_3)$$

$$= a_{11}a_{22}a_{33} \det(e_1, e_1, e_1) + a_{11}a_{23}a_{32} \det(e_1, e_1, e_2)$$

... 共 $3^3 = 27$ 个 $\rightarrow$ 求式子

$$\det(e_i, e_i, e_j) = 0$$

n^n 个

$$\Rightarrow a_{11}a_{22}a_{33} \det(e_1, e_2, e_3)$$

$$+ a_{11}a_{23}a_{32} \det(e_1, e_3, e_2)$$

$$+ a_{21}a_{12}a_{33} \det(e_2, e_1, e_3)$$

+ ...

$n!$ 个

$$\det(A) = \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_n=1}^n a_{i_1,1} a_{i_2,2} \dots a_{i_n,n} \det(e_{i_1} \dots e_{i_n}) \quad \text{Chp. V. 特征值和特征向量}$$

$$= \sum_{(i_1, i_2, \dots, i_n) \text{ 是排列}} a_{i_1,1} a_{i_2,2} \dots a_{i_n,n} \det(e_{i_1} \dots e_{i_n})$$

排列: $\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \rightarrow \begin{matrix} 3 \\ 1 \\ 2 \end{matrix}$ 双交换 $\tau \mapsto [e_{\tau(1)} \dots e_{\tau(n)}] = P_\tau$

$$\text{sgn}(\tau) = \det(P_\tau)$$

$$= \sum_{\text{排列 } \tau} \text{sgn}(\tau) a_{\tau(1),1} a_{\tau(2),2} \dots a_{\tau(n),n}$$

$$n! \sim O(10^n) \quad (\text{stirling 近似})$$

列线性

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ \vdots & \vdots & & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \dots & \tilde{a}_{nn} \end{bmatrix} \quad \hat{A} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \dots & \hat{a}_{1n} \\ \vdots & \vdots & & \vdots \\ \hat{a}_{n1} & \hat{a}_{n2} & \dots & \hat{a}_{nn} \end{bmatrix}$$

$$a_{i\tau} = k \tilde{a}_{i\tau} + l \hat{a}_{i\tau}$$

$$\delta(A) = \sum_{\tau} \text{sgn}(\tau) a_{\tau(1),1} a_{\tau(2),2} \dots a_{\tau(n),n}$$

$$= \sum_{\tau} \text{sgn}(\tau) (k \tilde{a}_{\tau(1),1} + l \hat{a}_{\tau(1),1}) a_{\tau(2),2} \dots a_{\tau(n),n}$$

$$= k \left(\sum_{\tau} \text{sgn}(\tau) \tilde{a}_{\tau(1),1} a_{\tau(2),2} \dots a_{\tau(n),n} \right) + l \left(\sum_{\tau} \text{sgn}(\tau) \hat{a}_{\tau(1),1} a_{\tau(2),2} \dots a_{\tau(n),n} \right)$$

$$\hat{a}_{\tau(1),1} \dots a_{\tau(n),n} = k \delta(A') + l \delta(A'')$$

列反对称:

单位化条件

1, 2 交换

$$\delta(A) = \sum_{\tau} \text{sgn}(\tau) a_{\tau(1),1} a_{\tau(2),2} \dots a_{\tau(n),n}$$

$$= \sum_{\tau} \text{sgn}(\tau) a_{\tau(2),1} a_{\tau(1),2} a_{\tau(3),3} \dots a_{\tau(n),n}$$

$$= \sum_{\tau} \text{sgn}(\tau) a'_{\tau(1),1} a'_{\tau(2),2} \dots a'_{\tau(n),n}$$

$$= - \sum_{\tau} \text{sgn}(\tau) a'_{\tau(1),1} a'_{\tau(2),2} \dots a'_{\tau(n),n}$$

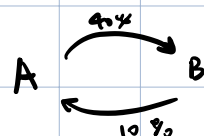
$$= -\delta(A')$$

$$\textcircled{1} x_0, Ax_0, A^2x_0, \dots, A^n x_0$$

$$1 > \lim_{n \rightarrow \infty} A^n x_0 \quad \lim_{n \rightarrow \infty} A^n$$

$$2 > A^k$$

$$\textcircled{2} x = x(t) \quad \frac{dx}{dt} = Ax$$



$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 0.6x + 0.1y \\ 0.4y + 0.4x \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{特征: } Ax_{\infty} = x_{\infty}$$

$$(I - A)x_{\infty} = 0$$

$$\begin{bmatrix} 0.4 & -0.1 \\ -0.4 & 0.1 \end{bmatrix} x_{\infty} = 0 \quad x_{\infty} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} k$$

$$w_0 = k_1 x_{\infty} + k_2 y \quad \text{设 } x_{\infty}, y \text{ 为一组基}$$

$$Aw_0 = k_1 x_{\infty} + k_2 Ay$$

$$A^2 w_0 = k_1 x_{\infty} + k_2 A^2 y \quad Ay = l_1 x_{\infty} + l_2 y$$

$$Aw_0 = (k_1 + k_2 l_1) x_{\infty} + k_2 l_2 y$$

$$A^2 w_0 = (k_1 + k_2 l_1) x_{\infty} + k_2 l_2 (l_1 x_{\infty} + l_2 y)$$

$$= (k_1 + k_2 l_1 + k_2 l_2 l_1) x_{\infty} + k_2 l_2^2 y$$

$$\text{若 } l_1 = 0, \text{ 则 } Ay = l_2 y$$

$$A^k w_0 = k_1 A^k x_{\infty} + k_2 A^k y$$

$$= k x_{\infty} + k_2 l_2^k y$$

$$(l_2 I - A)y = 0 \Leftrightarrow \det(l_2 I - A) = 0$$

$$Ax = \lambda x$$

$$\Leftrightarrow \det(\lambda I - A) = 0$$

$$(\lambda I - A) \cdot x = 0$$

代数基本定理: 复系数多项式至少有一个复数根
推论: n 次多项式有 n 个 (计重数) 复根;

HW: 4.3 1-2.4-5.8.11.14.16.18.21

5.1 1.

Review:

$$Ax = \lambda x \quad Ax \parallel x$$

$$\text{复数集 } \mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$$

$$(a_1+b_1i) \pm (a_2+b_2i) = (a_1 \pm a_2) + (b_1 \pm b_2)i$$

$$(a_1+b_1i)(a_2+b_2i) = a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1)$$

代数学基本定理: 复系数一元 n 次多项式在 $\mathbb{C}$ 内至少有一个根;

推论: 一元 n 次多项式在 $\mathbb{C}$ 内有 n 个根 (根可相同)

推论: 实系数一元 n 次多项式的根集关于共轭封闭

$$\overline{a+bi} = a-bi \quad \overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2} \quad \Leftrightarrow (x_0 \text{ 是根}) \Rightarrow \overline{x_0} \text{ 是根}$$

$$\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2} \quad \frac{1}{\overline{z}} = \frac{1}{z}$$

$$\text{Proof: } a_1 x_0^n + a_2 x_0^{n-1} + \dots + a_n = 0$$

$$0 = \overline{0} = \overline{a_1 x_0^n + a_2 x_0^{n-1} + \dots + a_n} = \overline{a_1} \cdot \overline{x_0^n} + \overline{a_2} \cdot \overline{x_0^{n-1}} + \dots + \overline{a_n} = 0$$

推论: 实系数一元奇数次多项式至少有一个实数根

Proof: 复数根成对出现

Vieta定理: ~

定义: 对方阵 $A \in \mathbb{C}^{n \times n}$, 若存在 $\lambda \in \mathbb{C}, x \in \mathbb{C}^n$,

$x \neq 0$ s.t. $Ax = \lambda x$ 则称 λ 是 A 的一个特征值, x 是 A

的一个特征向量; (λ, x) 是 A 的一个特征对

特别地, 若 $\lambda \in \mathbb{R}, x \in \mathbb{R}^n$... 在 $\mathbb{R}$ 上的特征值; 在 $\mathbb{R}$ 上的

特征向量, ... 在 $\mathbb{R}$ 上的特征对

注 > (1) 必须是方阵才有特征值和特征向量

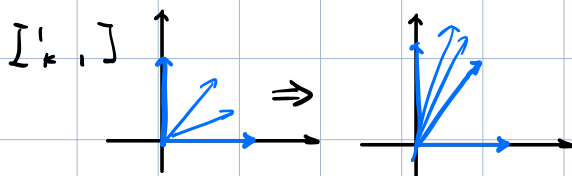
$$(2) x \neq 0$$

$$(k \neq 0)$$

$$(3) x \text{ 是特征向量} \Leftrightarrow kx \text{ 是 } \sim$$

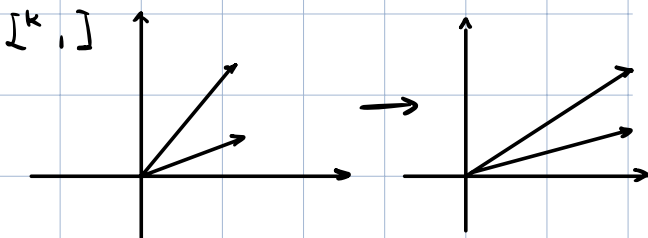
几何含义:

$$Ax \parallel x$$



e_1 是特征向量

$$x = k_1 e_1 + k_2 e_2 \text{ 不是 } \sim (k_1 \neq 0)$$



$e_1, e_2 \checkmark$ 其余方向 x

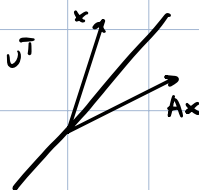
$$\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} = I - 2v v^T$$

$$Av = v$$

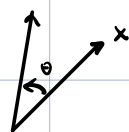
$$A v_2 = -v_2$$

$$v_2 \perp v$$

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



Ax



$\theta \neq k\pi$

没有 $\sim$ (在 $\mathbb{R}$ 上)

$$Ax = \lambda x$$

$$\Leftrightarrow (\lambda I - A)x = 0$$

$$\Leftrightarrow x \in N(\lambda I - A)$$

$$\Leftrightarrow \lambda I - A \text{ 不可逆}$$

$$\Leftrightarrow \det(\lambda I - A) = 0$$

算法: 给定方阵 A

① 计算 $\det(\lambda I - A)$

② 计算所有的根 $\lambda_1, \dots, \lambda_s$ 作为特征值

③ 对 $i=1, \dots, s: N(\lambda_i I - A)$ 的一组基 $x_{i,1}, \dots, x_{i,r_i}$

$\det(\lambda I - A)$ 是一个首项系数为 1 的 n 次多项式
称为 A 的特征多项式, 记为 $P_A(\lambda)$

$$e.g. \begin{bmatrix} 7 & -5 & 5 \\ -2 & 3 & -2 \\ -10 & 10 & 8 \end{bmatrix}$$

解:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-7 & 5 & -5 \\ 2 & \lambda-3 & 2 \\ 10 & -10 & \lambda-8 \end{vmatrix}$$

$$= (\lambda-7)(\lambda-3)(\lambda-8) + 10 \times 5 \times 2 + 2 \cdot (-10) \cdot (-5) - 10 \cdot (-5) \cdot (\lambda-3) - 2 \cdot (-10) \cdot (\lambda-7) - 2 \cdot 5 \cdot (\lambda-8)$$

$$= \dots = (\lambda-2)(\lambda^2+1)$$

2, i, -i 是 A 的特征值

$$\text{当 } \lambda_1 = 2 \text{ 时 } \lambda_1 I - A = \begin{bmatrix} -5 & 5 & -5 \\ 2 & -1 & 2 \\ 10 & -10 & 6 \end{bmatrix}$$

$$\rightarrow \begin{pmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ 则 } N(\lambda_1 I - A) \text{ 的一组基为 } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ 则属于 } \lambda_1 = 2 \text{ 的特征向量全体为 } \left\{ k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \mid k \neq 0, k \in \mathbb{C} \right\}$$

当 $\lambda_2 = i$ 时

$$\lambda_2 I - A = \begin{bmatrix} i-7 & 5 & -5 \\ 2 & i-3 & 2 \\ 10 & -10 & i+8 \end{bmatrix}$$

$$\Rightarrow x_2 = \begin{pmatrix} 1-2i \\ -1+i \\ -2+4i \end{pmatrix}$$

当 $\lambda_3 = -i$ 时

$$\Rightarrow x_3 = \begin{pmatrix} 1+2i \\ -1-i \\ -2-4i \end{pmatrix} \quad \begin{cases} Ax = ix & \text{共轭} \\ A\bar{x} = \bar{A}\bar{x} = \bar{A}\bar{x} \\ A\bar{x} = -i\bar{x} \end{cases}$$

求 A 在 R 上的特征值和特征向量

A 在 R 上特征值为 2

特征向量全体 $N(\lambda I - A)$ 是 $\left\{ k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \mid k \neq 0, k \in \mathbb{R} \right\}$

命题: 上(下)三角矩阵的特征值是所有对角元素

正交矩阵的特征值是 0, 1

$$\det(\lambda I - QR^T) = \lambda^{n-r} \det(\lambda I - R^T Q R) = \lambda^{n-r} \det(\lambda I)$$

$$P_A(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \dots (\lambda - \lambda_s)^{n_s}$$

n_i 称为特征值 λ_i 的(代数)重数

λ_i 是 A 的 n_i 重特征值

= $n-1$ 次项系数相反数

= 常数项的 $(-1)^n$ 倍

$$n_1 \lambda_1 + n_2 \lambda_2 + \dots + n_s \lambda_s = n-1 \text{ 次项系数相反数} = \text{trace}(A)$$

$$\lambda_1^{n_1} \lambda_2^{n_2} \dots \lambda_s^{n_s} = \text{常数项的 } (-1)^n \text{ 倍} = \det(A)$$

命题:

$$P_A(\lambda) = \lambda^n - \text{trace}(A) \lambda^{n-1} + \dots + (-1)^n \det(A)$$

证: ① $\det(\lambda I - A) = P_A(\lambda)$

常数项 $P_A(0) = \det(-A)$

$$\text{② } P_A(\lambda) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1n} \\ \vdots & \lambda - a_{22} & & \vdots \\ -a_{n1} & \dots & \dots & \lambda - a_{nn} \end{vmatrix} = \prod_{i=1}^n \text{sgn}(T_i) a_{11} \dots a_{nn} = \prod_{i=1}^n (\lambda - a_{ii}) \quad (\text{行列式展开})$$

$N(\lambda I - A)$ 称为属于 λ 的特征子空间

命题: (Gershgorin 圆盘定理)

$$\text{定义圆盘 } D = \{ z \mid |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \}$$

所有特征值都 $\in \bigcup_{i=1}^n D_i$

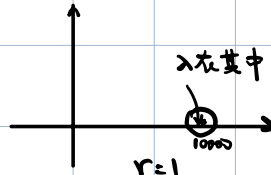
Proof: $|\lambda - a_{ii}| > \sum_{j \neq i} |a_{ij}|$ 的 λ

$|\lambda I - A|$ 对角占优 $\Rightarrow \lambda I - A$ 可逆

应用:

$$\begin{pmatrix} 10000 & 1 \\ 1 & 10000 \end{pmatrix}$$

$$(\lambda - 10000)^2 \leq 1$$



HW: 1. 8. 2. 4. 6-18. 22-23

Review:

特征值和特征向量

$$Ax = \lambda x \quad x \neq 0$$

算法: 1) 特征多项式 $P_A(\lambda) = \det(\lambda I - A)$

2) 特征值 $\lambda_1 \dots \lambda_s$: $P_A(\lambda)$ 的根

3) 特征向量 $x_{1,1} \dots x_{1,n_1}, x_{2,1} \dots x_{2,n_2}, \dots$

$$P_A(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_s)^{n_s}$$

$n_1, n_2, \dots, n_s$ 代数重数; $n_1 + \dots + n_s = n$

$m_1, m_2, \dots, m_s$ 几何重数; $m_i \leq n_i$

$$A^k x_0 = A^k (k_1 x_1 + k_2 x_2) = k_1 A^k x_1 + k_2 A^k x_2 = k_1 \lambda_1^k x_1 + k_2 \lambda_2^k x_2$$

$$A^k x_0 = k_1 \lambda_1^k x_1 + k_2 \lambda_2^k x_2 = k_1 x_1 + k_2 \frac{1}{2^k} x_2$$

$$A \cdot X = A \cdot [x_1, x_2] = [Ax_1, Ax_2] = [\lambda_1 x_1, \lambda_2 x_2]$$

$$= [x_1, x_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = X \Lambda$$

$$\therefore A^k x_0 = A^k X \tilde{x} = X \Lambda^k \tilde{x} \quad I = X \cdot \tilde{I}$$

$$\Rightarrow A^k \cdot I = X A^k \tilde{I} \Rightarrow \underline{A = X \Lambda X^{-1}}$$

$$\textcircled{1} x_{k+1} = A x_k$$

$$\textcircled{2} \frac{dx}{dt} = A \cdot x \Rightarrow \frac{dx}{dt} = X \Lambda X^{-1} x$$

$$\frac{d(X^{-1}x)}{dt} = \Lambda(X^{-1}x) \therefore \frac{dy}{dt} = \Lambda y$$

$$\frac{dy_i}{dt} = \lambda_i y_i$$

定义: 对于一个方阵A, 若存在可逆矩阵X, 使得

$X^{-1} A X = \Lambda$ 是对角矩阵, 则称:

① A 是可对角化的

② X 对角化 A

若 A 可对角化, 则称 $A = X \Lambda X^{-1}$ 为 A 的谱分解

命题: A 可对角化 $\Leftrightarrow$ A 有 n 个线性无关的特征向量;

Proof: $\Leftarrow$: $A x_1 = \lambda_1 x_1 \cdots A x_n = \lambda_n x_n$

$$A (x_1 \cdots x_n) = (x_1 \cdots x_n) \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\therefore A = (x_1 \cdots x_n) \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} (x_1 \cdots x_n)^{-1}$$

代数重数是 1 的特征值称为单特征值

$$\Rightarrow: A X = X \Lambda$$

$$A x e_i = X \Lambda e_i = X \lambda_i e_i = \lambda_i (X e_i)$$

$\Rightarrow$ X 的每一列都是 A 的特征向量, 相互线性无关

定义: 对 A 的特征值 λ , 称 $\dim N(\lambda I - A)$ 为 λ (作为 A 特征值的) 几何重数

命题: $1 \leq$ 几何重数 $\leq$ 代数重数

$$\text{Proof: } A x_1 = \lambda_0 x_1 \cdots A x_m = \lambda_0 x_m$$

$$A (x_1 \cdots x_m) = (x_1 \cdots x_m) \begin{bmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_0 \end{bmatrix}$$

$$A (x_1 \cdots x_m | x_{m+1} \cdots x_n) = \begin{bmatrix} \lambda_0 x_1 & \cdots & \lambda_0 x_m & x_{m+1} & \cdots & x_n \end{bmatrix}$$

$$= X \begin{bmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_0 & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_{s-1} \end{bmatrix}$$

$$\det(\lambda I - A) = \det(\lambda I - X \Lambda X^{-1})$$

$$= \det(\lambda I - \begin{bmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_0 & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_{s-1} \end{bmatrix})$$

$$= \det(\lambda I - \begin{bmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_0 & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_{s-1} \end{bmatrix}) = (\lambda - \lambda_0)^m \cdot \det(\lambda I - \Lambda_{s-1})$$

$\therefore \lambda_0$ 的代数重数 $\geq m =$ 几何重数

命题: 矩阵的不同特征值对应的特征向量总是线性无关

Proof: $\lambda_1, \dots, \lambda_s$ 是 A 的特征值, 两两不同

$x_1, \dots, x_s$ 是 A 的特征向量

若 $\exists k_i$ s.t. $k_1 x_1 + \dots + k_s x_s = 0$

$$k_1 \lambda_1 x_1 + \dots + k_s \lambda_s x_s = A(k_1 x_1 + \dots + k_s x_s) = 0$$

$$k_1 \lambda_1^2 x_1 + \dots + k_s \lambda_s^2 x_s = A^2(k_1 x_1 + \dots + k_s x_s) = 0$$

$\vdots$ 可逆 (Vandermonde 矩阵)

$$(k_1 x_1 \cdots k_s x_s) \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{s-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_s & \lambda_s^2 & \cdots & \lambda_s^{s-1} \end{bmatrix} = (0)$$

$$\therefore (k_1 x_1 \cdots k_s x_s) = (0) \Rightarrow k_1 = \dots = k_s = 0$$

定义: 矩阵的代数重数与几何重数相等的特征值, 称为半单特征值 (单特征值 $\leq$ 半单特征值)

命题: A 可对角化 $\Leftrightarrow$ A 的特征值都半单

证: $\Rightarrow: AX = X\Lambda$

$$AX_i = \lambda_i x_i$$

$$\det(\lambda I - A) = \det(\lambda I - \Lambda)$$

$$\Leftarrow: A = (x_1 \dots x_s \ x_{s+1} \dots x_{m_1} \dots x_{m_s}) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_s \end{pmatrix} \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}^{-1}$$

在 $\mathbb{R}$ 上可对角化 $\Leftrightarrow A = X\Lambda X^{-1}$
 $A, X, \Lambda \in \mathbb{R}^{n \times n}$

例: 实对称矩阵可对角化 $A = I \Lambda I^{-1}$

2个特征值互不相同的可对角化

$$\textcircled{1} A^2 = A \Leftrightarrow A(I-A) = 0 \Rightarrow R(I-A) = N(A)$$

$$\Rightarrow \dim N(I-A) + \dim N(0 \cdot I - A) = n$$

$$A^2 = I \Leftrightarrow (I-A)(I+A) = 0 \Rightarrow \dots \quad \checkmark$$

④ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}_{n \times n}$ 代数重数是 n
几何重数是 1
Jordan 块

HW: 1-4, 6(奇), 7(奇), 8-17, 20-21

Review:

对角化、谱分解

A

$$\textcircled{1} p_n(\lambda) = 0$$

$$\textcircled{2} \lambda_1 \dots \lambda_s \text{ (代数重数 } n_1 \dots n_s)$$

$$\textcircled{3} N(\lambda_i I - A) \text{ 的基 } x_{i,1} \dots x_{i,m_i}$$

若 $m_i = n_i, i = 1 \dots s$ 则 A 可对角化

$$A \cdot [x_{1,1} \dots x_{1,m_1} \dots x_{s,1} \dots x_{s,m_s}] = X \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_s \end{bmatrix} X^{-1}$$

$$A = X \Lambda X^{-1}$$

A 可对角化 $\Leftrightarrow A$ 有 n 个线性无关的特征向量

$\Leftrightarrow A$ 的所有特征值半单

$\Leftrightarrow$ 存在多项式 $p(\lambda)$ 无重根, s.t. $p(A) = 0$

命题: $\begin{bmatrix} A_1 & \dots & A_s \end{bmatrix}$ 可对角化 $\Leftrightarrow A_i$ 有对角化 $i = 0, 1 \dots s$

Proof:

$$\Leftarrow A_i = x_i \Lambda_i x_i^{-1}$$

$$\begin{bmatrix} A_1 & \dots & A_s \end{bmatrix} = \begin{bmatrix} x_1 & \dots & x_s \\ & & \end{bmatrix} \begin{bmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_s \end{bmatrix} X^{-1}$$

$\Rightarrow$: 不妨设 $s = 2$

$$\det(\lambda I - \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}) = \det(\lambda I - A_1) \det(\lambda I - A_2)$$

$$\lambda_i \text{ 代数重数 } n_{i,1} + n_{i,2} \quad n_{i,1} \quad n_{i,2}$$

$$\text{考虑 } N \left(\begin{bmatrix} \lambda_i I - A_1 & \\ & \lambda_i I - A_2 \end{bmatrix} \right) \subseteq N(\lambda_i I - A_1) \cup N(\lambda_i I - A_2)$$

$$\begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Leftrightarrow A_1 x_1 = 0 \quad A_2 x_2 = 0$$

$$A_1 y_1 = A_1 y_2 = \dots = A_1 y_t = 0$$

$$A_2 z_1 = A_2 z_2 = \dots = A_2 z_s = 0$$

$$\begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} y_1 \dots y_t & 0 \\ 0 & z_1 \dots z_s \end{bmatrix} = 0$$

$$m_{i,1} + m_{i,2} = n_{i,1} + n_{i,2}$$

$$m_{i,1} \leq n_{i,1} \Rightarrow m_{i,1} = n_{i,1} \quad m_{i,2} = n_{i,2}$$

$$m_{i,2} \leq n_{i,2}$$

$$\Rightarrow A_i \text{ 可对角化 } i = 1, 2 \dots n$$

$$A^n = X \Lambda^n X^{-1}$$

$$\frac{dx}{dt} = A x \quad \frac{d(X^{-1}x)}{dt} = \Lambda(X^{-1}x) \quad \frac{dy}{dt} = \Lambda y$$

$$\frac{dy_i}{dt} = \lambda_i y_i \quad y_i = e^{\lambda_i t} c_i$$

$$\Rightarrow y = e^{\Lambda t} c \quad x = X e^{\Lambda t} X^{-1} c$$

定义: 对方阵 A, 若存在可逆 X, s.t.

$$A = X B X^{-1}, \text{ 则称 } A \text{ 相似于 } B$$

命题: 相似是等价关系

注: 相似是特殊的相似

相似变换: $A \mapsto X^{-1}AX$

命题: 不变量 ① 秩 ② 特征多项式 $\left\{ \begin{array}{l} \text{代数重数} \\ \text{迹} \\ \text{行列式} \end{array} \right.$

③ 几何重数

Proof: $AX_{ij} = \lambda_i X_{ij} \quad j=1, 2, \dots, m_i$

$$(X^{-1}AX)X^{-1}X_{ij} = \lambda_i (X^{-1}X_{ij})$$

$X^{-1}X_{i1}, \dots, X^{-1}X_{i:m_i}$ 是 $X^{-1}AX$ 的特征向量

线性无关

$$m_i(X^{-1}AX) \geq m_i \geq m_i(X^{-1}AX)$$

$$= =$$

几何重数不变 #

标准形:

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_s \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_s \end{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}^{-1}$$

命题: 对方阵 A , 存在可逆 X s.t. $X^{-1}AX = T$

是上三角矩阵

Proof: 取 A 的一个特征对 (λ_1, x_1) $AX_1 = \lambda_1 X_1$

把 x_1 扩充成可逆矩阵 $X_1 = [x_1, x_{12}]$

$$AX_1 = A[x_1, x_{12}] = [\lambda_1 x_1, x_{12} \tilde{X}_{12}]$$

$$= X_1 \begin{bmatrix} \lambda_1 & 0 & \tilde{x}_{12} \\ 0 & \tilde{\lambda}_2 & \tilde{x}_{12} \end{bmatrix}$$

$\tilde{x}_{12}$ $n-1$ 阶, 递归

$$= X_1 \begin{bmatrix} \lambda_1 & \tilde{x}_{12} \\ 0 & \tilde{\lambda}_2 \tilde{X}_{12} \end{bmatrix}$$

$$= X_1 \underbrace{\begin{bmatrix} 1 & \\ & \tilde{X}_{12} \end{bmatrix}}_X \underbrace{\begin{bmatrix} \lambda_1 & * \\ 0 & \tilde{T}_2 \end{bmatrix}}_T \underbrace{\begin{bmatrix} 1 & \\ & \tilde{X}_{12}^{-1} \end{bmatrix}}_{\#}$$

注:

1> T 的元素是 A 的特征值

2> 通过选择 X , 可使 T 的对角元任意排列

定理 (Hamilton-Cayley): $P_A(A) = 0$

Proof: $P_A(\lambda) = \det(\lambda I - A) = \det(\lambda I - T) = P_T(\lambda)$

$$P_T(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$$

$$P_T(A) = (A - \lambda_1 I) \dots (A - \lambda_n I)$$

$$= (XTX^{-1} - \lambda_1 I) \dots (XTX^{-1} - \lambda_n I)$$

$$= (T - \lambda_1 I) \dots (T - \lambda_n I) = P_T(T)$$

$$P_T(T) = \begin{pmatrix} \lambda_1 & * & \dots & * \\ & \lambda_2 & & * \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \dots \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

$$\begin{bmatrix} k & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & a \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} k & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \lambda_1 & ka \\ & \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & k \\ & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & a \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & -k \\ & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & a + k\lambda_2 - k\lambda_1 \\ & \lambda_2 \end{bmatrix}$$

定理 (Jordan 分解) 对方阵 A , 存在可逆 X s.t.

$$X^{-1}AX = J = \begin{bmatrix} J_1(\lambda_1) & & \\ & \ddots & \\ & & J_s(\lambda_s) \end{bmatrix}$$

其中 $J_i(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & 0 & \\ & \lambda_i & 1 & \\ & & \ddots & \\ 0 & & & \lambda_i \end{bmatrix}_{n_i \times n_i}$ 是 n_i 阶 Jordan 块

注:

① Jordan 块在不计次序时唯一

② $\lambda_1, \dots, \lambda_s$ 可能相同

③ 若 $\lambda_i \in \mathbb{R}$, $A \in \mathbb{R}^{n \times n}$ 则 $J, X \in \mathbb{R}^{n \times n}$

④ $A = XJX^{-1}$ 称为 A 的 Jordan 分解

J ———— 标准形

证: 略 (过困难)

同时可对角化:

$$\left. \begin{array}{l} X^{-1}AX = \Lambda_A \\ X^{-1}BX = \Lambda_B \end{array} \right\} ?$$

命题: 若 A, B 都可对角化, 则 A, B 可同时对角化 $\Leftrightarrow AB = BA$

Proof: $\Rightarrow AB = X \Lambda X^{-1} X \Lambda_B X^{-1} = X \Lambda A_B X^{-1}$

$BA = X \Lambda_B \Lambda X^{-1} \neq$

$\Leftarrow \exists x_0 \text{ s.t. } x_0 A x_0^{-1} = \Lambda A$

$x_0 B x_0^{-1} = \tilde{B}$

$\Lambda \tilde{B} = \tilde{B} \Lambda$

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_s \end{bmatrix} \begin{bmatrix} \tilde{B}_{11} & \dots & \tilde{B}_{1s} \\ \vdots & \ddots & \vdots \\ \tilde{B}_{s1} & \dots & \tilde{B}_{ss} \end{bmatrix} = \begin{bmatrix} \tilde{B}_{11} & \dots & \tilde{B}_{1s} \\ \vdots & \ddots & \vdots \\ \tilde{B}_{s1} & \dots & \tilde{B}_{ss} \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_s \end{bmatrix}$$

$\lambda_i \tilde{B}_{ij} = \lambda_j \tilde{B}_{ij}$

$\lambda_i \neq \lambda_j, \tilde{B}_{ij} = 0 \quad \tilde{B} = \begin{bmatrix} \tilde{B}_{11} & & \\ & \ddots & \\ & & \tilde{B}_{ss} \end{bmatrix}$

HW: 1-3, 5-6, 7 (1-3)

12, 6

Review:

$$X^{-1} A X = J = \begin{bmatrix} J_{n_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{n_s}(\lambda_s) \end{bmatrix}$$

$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}_{n_i \times n_i}$

e.g.

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_1 & 1 & \\ & & \ddots & \ddots \\ & & & \lambda_1 & 1 \\ & & & & \ddots & \ddots \\ & & & & & \lambda_s & 1 \\ & & & & & & \ddots & \ddots \\ & & & & & & & \lambda_s & 1 \\ & & & & & & & & \lambda_s \end{bmatrix}$$

对角线为 λ 的
块数 $\sim$ λ 的
代数重数
相同 $\lambda \sim$ λ 的
几何重数

倍加
倍乘
对称
正交

$A = A^T \in \mathbb{R}^{n \times n}$

$A = X \Lambda X^{-1}$

$A^T = X^{-T} \Lambda^T X^T = X^{-T} \Lambda X^T$

Q: $X^{-1} = X^T$? $X^T X = I$

Q: $A \in \mathbb{R}^{n \times n}$?

$Ax = \lambda x \quad x \neq 0 \quad x \in \mathbb{C}^n$

$\bar{A} \bar{x} = \bar{\lambda} \bar{x} \quad y^T A x = \lambda y^T x$

$Ax = \lambda x$: 若 $y^T x \neq 0$ 尝试 $y = x \quad x^T x = 0$? $\times$

$\lambda = \frac{y^T A x}{y^T x}$ 再尝试: $y = \bar{x}$

A 实对称 $\bar{A} = A$

$\Rightarrow \lambda = \frac{\bar{x}^T A x}{\bar{x}^T x}$

$\bar{A} \bar{x} = \bar{\lambda} \bar{x}$:

$x^T \bar{A} \bar{x} = \bar{\lambda} x^T \bar{x}$

$\bar{\lambda} = \frac{x^T \bar{A} \bar{x}}{x^T \bar{x}}$

$\Rightarrow \lambda$ 是实数

实对称矩阵 $A \rightarrow \Lambda \in \mathbb{R}^{n \times n}$

$\Leftrightarrow$ 命题: 实对称矩阵在 $\mathbb{C}$ 上的特征值都是实数

设

$Ax_1 = \lambda_1 x_1$

$Ax_2 = \lambda_2 x_2$

$\lambda_1 \neq \lambda_2$

$x_1^T A x_1 = \lambda_1 x_1^T x_1$

$x_1^T A x_2 = \lambda_2 x_1^T x_2$

若 $x_1^T x_1 \neq 0$

则 $\lambda_1 = \lambda_2$ 矛盾

$\therefore x_2^T x_1 = 0 \quad \checkmark$

$\Rightarrow x^T x = I$

命题: 属于不同特征值的特征向量相互正交

$\Rightarrow$ 定理: 对实对称矩阵 A , 存在实正交矩阵 Q .

s.t. $A = Q \Lambda Q^T$, 其中 Λ 是实对称矩阵

Proof: 取 A 的特征对 $(\lambda_i, x_i) \quad \lambda_i \in \mathbb{R} \quad x_i \in \mathbb{R}^n \quad \|x_i\|=1$

把 x_i 扩充成正交矩阵 $[x_1, x_2]$

$A[x_1, x_2] = [Ax_1, Ax_2] = [\lambda_1 x_1, \lambda_2 x_2]$

$= [x_1, x_2] \cdot \begin{bmatrix} \lambda_1 & * \\ 0 & * \end{bmatrix}$

$x^T A x = \begin{bmatrix} \lambda_1 & * \\ 0 & * \end{bmatrix}$ 对称 $= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix}$ $\tilde{A}_2$ 对称

归纳内法: $A_2 = Q_2 \Lambda_2 Q_2^T$

$$X^T A X = \begin{bmatrix} \lambda_1 & & \\ & Q_2 \Lambda_2 Q_2^T & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & & \\ & Q_2 & \\ & & \ddots \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \Lambda & \\ & & \ddots \end{bmatrix} \begin{bmatrix} 1 & & \\ & Q_2^T & \\ & & \ddots \end{bmatrix}$$

$$A = \underbrace{X \begin{bmatrix} 1 & & \\ & Q_2 & \\ & & \ddots \end{bmatrix}}_Q \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \Lambda & \\ & & \ddots \end{bmatrix}}_\Lambda \underbrace{\begin{bmatrix} 1 & & \\ & Q_2^T & \\ & & \ddots \end{bmatrix} X^T}_{Q^T}$$

谱分解 $A = Q \Lambda Q^T = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \dots$

考法: 计算实对称矩阵谱分解 $+ \lambda_n q_n q_n^T$

定义: 若存在实正交矩阵 Q s.t. $A = Q B Q^T$

则称 A 正交相似于 B

① 正交相似是等价关系

② 正交相似是特殊的相似

对称性是不变量

类比:

实数 $\sim$ 对称

复数 $\sim$ 正交

$$\begin{pmatrix} z = r e^{i\theta} \\ A = S Q = S e^{iS} \end{pmatrix}$$

$$\lambda = \frac{x^T A x}{x^T x} = \frac{(x^T Q) \Lambda (Q^T x)}{x^T Q Q^T x} = \frac{y^T \Lambda y}{y^T y} = \frac{\lambda_1 y_1^2 + \dots + \lambda_n y_n^2}{y_1^2 + \dots + y_n^2}$$

定义: 对 $x \neq 0$ 称 $\frac{x^T A x}{x^T x}$ 为 x 关于 A 的 Rayleigh 商

命题: $A = A^T \in \mathbb{R}^{n \times n}$

特征值 $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ $[q_1, \dots, q_n]$ 正交

$$\lambda_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x}$$

$$\lambda_i = \max_{x \perp \text{span}\{q_1, \dots, q_{i-1}\}} \frac{x^T A x}{x^T x}$$

$$\lambda_n = \min_{x \neq 0} \frac{x^T A x}{x^T x}$$

$$\lambda_i = \min_{x \perp \text{span}\{q_{i+1}, \dots, q_n\}} \frac{x^T A x}{x^T x}$$

Proof:

$$\textcircled{1} A = Q \Lambda Q^T$$

$$\frac{x^T A x}{x^T x} = \frac{y^T \Lambda y}{y^T y} = \frac{\lambda_1 y_1^2 + \dots + \lambda_n y_n^2}{y_1^2 + \dots + y_n^2}$$

$$\leq \frac{\lambda_1 y_1^2 + \lambda_1 y_2^2 + \dots + \lambda_1 y_n^2}{y_1^2 + \dots + y_n^2} = \lambda_1 \quad \checkmark$$

$$y = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ 等号成立 } \therefore \max_{x \neq 0} \frac{x^T A x}{x^T x} = \lambda_1$$

$$\textcircled{2} \max_{x \in \text{span}\{q_1, \dots, q_i\}} \frac{x^T A x}{x^T x} = \max_{y \in \text{span}\{e_1, \dots, e_i\}} \frac{y^T \Lambda y}{y^T y} = \frac{\lambda_1 y_1^2 + \dots + \lambda_i y_i^2}{y_1^2 + \dots + y_i^2} \leq \lambda_i \quad \checkmark$$

$$i \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \text{ 等号成立}$$

$$A: \lambda_1 \geq \lambda_2$$

$$A - \mu I: \lambda_1 - \mu \geq \lambda_2 - \mu > 0$$

$$\frac{x^T (A - \mu I) x}{x^T x} = \frac{x^T A x}{x^T x} - \mu$$

$$\max_{y^T y = 1} y^T \Lambda y \quad \lambda_1 y_1^2 + \lambda_2 y_2^2$$



“1” 直线”

定理: (Courant-Fischer min-max) *

$$A = A^T \in \mathbb{R}^{n \times n} \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

$$\lambda_i = \min_{\dim M = n-i+1} \max_{x \neq 0, x \in M} \frac{x^T A x}{x^T x} = \max_{\dim M = i} \min_{x \neq 0, x \in M} \frac{x^T A x}{x^T x}$$

$$i=1: \lambda_1 = \max_{\substack{\dim M = n \\ x \in M \\ x \neq 0}} \frac{x^T A x}{x^T x} \quad \checkmark$$

$$i=n: \lambda_n = \min_{\substack{\dim M = 1 \\ x \in M \\ x \neq 0}} \frac{x^T A x}{x^T x} \quad \checkmark$$

$i=2:$



Proof: 只需证 $\max_{x \neq 0} \frac{x^T A x}{x^T x} \geq \lambda_i$

$$\max_{\substack{y \neq 0 \\ y \in N^\perp}} \frac{y^T A y}{y^T y} = \frac{\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2}{y_1^2 + y_2^2 + \dots + y_n^2}$$

$y_1 = \dots = y_n = 0 \quad y \in \text{span}(e_1, \dots, e_i)$

$\dim N^\perp = i-1$

$$\max_{y \in N} \frac{y^T A y}{y^T y} = \max_{y \in N} \frac{y^T A y}{y^T y} \geq \lambda_i$$

$y \in \text{span}(e_i, \dots, e_n)$
 $y \in N \cap \text{span}(e_i, \dots, e_n) \dim > 1$

Cauchy 分离不等式:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ 对称}$$

$A: \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

$A_{22}: \mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1}$

$\Rightarrow \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \dots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$

HW: 1-10, 12-14, 16-17, 19

12.11

$A = A^T \in \mathbb{R}^{n \times n}$

$A = Q \Lambda Q^T$

$\lambda_{\max}(A) \leq \frac{x^T A x}{x^T x} \leq \lambda_{\max}(A)$

正交相似 $\rightarrow$ 合同 $\rightarrow$ 相似

定义: 对实对称矩阵 A, 若

- ① $\forall x \neq 0 \quad x^T A x > 0$, 则称 A 正定
- ② $x^T A x \geq 0$, 则称 A 半正定 $\sqrt{A + \epsilon I}$ 正定 $\epsilon \rightarrow 0$
- ③ $\forall x \neq 0 \quad x^T A x < 0$, 则称 A 负定
- ④ $x^T A x \leq 0$, 则称 A 半负定
- ⑤ 以上皆不满足, 则称 A 不定

$(\exists x \text{ s.t. } x^T A x > 0)$
 $(\exists y \text{ s.t. } y^T A y < 0)$

相似: "定性" 不变量

保持定性
 $A \rightarrow Q A Q^T$ 正交
 $\rightarrow X A X^T$ 可逆

命题: 若 $A = A^T \in \mathbb{R}^{n \times n}$, 则如下叙述

- 等价:
- 1> A 正定
 - 2> A 的特征值皆正数
 - 3> 存在可逆矩阵 T s.t. $A = T T^T$
 - 4> A 有 LDL^T 分解, 且 D 的对角元素皆正
 - 5> A 的顺序主子阵的行阵式皆正 (顺序主子式)
 - 6> A 的顺序主子阵都正定

Proof:

$1 \Rightarrow 2: \checkmark$

$2 \Rightarrow 3: A = Q \Lambda Q^T \quad \Lambda = D^2 \Rightarrow A = \underbrace{Q Q^T}_{I} \underbrace{(Q \Lambda Q^T)}_{D^2}$

$3 \Rightarrow 4: T^T = QR$
 $A = T T^T = R^T Q^T Q R = R^T R$
 $R = \tilde{D} \cdot L^T$
 $\tilde{D} = \text{diag}(|r_{ii}|)$

$\therefore A = L \tilde{D}^T \tilde{D} L^T$ $\tilde{D}$ diag 皆正

$4 \Rightarrow 5:$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & \\ & L_{22} \end{bmatrix} \begin{bmatrix} D_{11} & \\ & D_{22} \end{bmatrix} \begin{bmatrix} L_{11}^T & L_{12}^T \\ & L_{22}^T \end{bmatrix}$$

$$= \begin{bmatrix} L_{11} D_{11} L_{11}^T & L_{11} D_{11} L_{12}^T \\ L_{12} D_{11} L_{11}^T & * \end{bmatrix}$$

$A_i = L_i D_i L_i^T$

$5 \Rightarrow 6:$ 对 n 归纳: $n=1 \checkmark$

设 $n-1 \checkmark$

$A = \begin{bmatrix} A_{n-1} & a \\ a^T & a_{nn} \end{bmatrix}$ 只需证 A 正定