

$$= \begin{bmatrix} L_{n-1}^T D_{n-1}^{-1} L_{n-1}^T & a \\ a^T & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} L_{n-1}^T & 0 \\ a^T L_{n-1}^{-1} D_{n-1}^{-1} & * \end{bmatrix} \begin{bmatrix} D_{n-1} & 0 \\ 0 & \textcircled{a_{nn} - a^T L_{n-1}^{-1} D_{n-1}^{-1} L_{n-1}^T a} \end{bmatrix} \begin{bmatrix} L_{n-1}^{-1} & x^T \\ 1 & 1 \end{bmatrix}$$

$$\Leftrightarrow a_{nn} - a^T L_{n-1}^{-1} D_{n-1}^{-1} L_{n-1}^T a > 0$$

$$\therefore (15) \det(A) > 0 \Rightarrow > 0 \quad \#$$

(6) $\Rightarrow$ (11): $\checkmark$

命题: 1 > A 半正定

2 > A 的特征值都非负

3 > 存在可逆矩阵 T s.t. $A = TT^T$

4 > A 有 LDL^T 分解, D 的对角元非负

证: (HW) ~~5 > A 的顺序主子式非负~~ $\times$

~~6 > A 的顺序主子式半正定~~ $\times$

A 正定 $x^T A x = 0 \Leftrightarrow x = 0$

A 半正定 $x^T A x = 0 \Leftrightarrow Ax = 0$

正 $\checkmark$ $\sim$
半正 $\checkmark$ $\sim$

命题: 若 $A = A^T \in \mathbb{R}^{n \times n}$, 若 $x, y \neq 0$, $x^T A x > 0$
 $y^T A y < 0$ 则存在 $z \neq 0$ s.t. $z^T A z = 0$ ($A \neq 0$)

Proof: 假设 $z = x + ky$ $z \neq 0$

$$0 = z^T A z = (x + ky)^T A (x + ky)$$

$$= x^T A x + 2k x^T A y + k^2 y^T A y$$

$$\Delta = (2x^T A y)^2 - 4(x^T A x)(y^T A y) > 0$$

$\therefore \exists k_0$ s.t. $z \neq 0$

#

$$A = Q \Lambda Q^T = L D L^T = T \Sigma T^T$$

$$A \mapsto X B X^T$$

定义: 对 A, 若存在可逆矩阵 X s.t. $B = X^T A X$

则称 A 合同于 B

命题: 合同等价关系

合同标准形:

$$\text{实对称 } A \quad A = Q \Lambda Q^T \quad A = L J L^T$$

$$(A \rightarrow \Lambda \rightarrow J)$$

$$\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}^T = \begin{bmatrix} d_1^2 \lambda_1 & & \\ & \ddots & \\ & & d_n^2 \lambda_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} I_p & & \\ & -I_{n-p} & \\ & & 0 \end{bmatrix}$$

命题: (Sylvester 惯性定律)

实对称矩阵合同标准形唯一

$$\text{Proof: } A = X J X^T = \tilde{X} \tilde{J} \tilde{X}^T$$

$$J = \begin{pmatrix} I_p & & \\ & -I_{n-p} & \\ & & 0 \end{pmatrix} \quad \tilde{J} = \begin{pmatrix} I_1 & & \\ & I_{r-1} & \\ & & 0 \end{pmatrix}$$

$$Y J Y^T = \tilde{J}$$

$$\exists M = R \left(X^T \begin{bmatrix} I_p \\ 0 \end{bmatrix} \right) \quad \dim M = p$$

$$N = R \left(X^T \begin{bmatrix} 0 \\ I_{n-p} \end{bmatrix} \right) \quad \dim N = n - p$$

$$\forall x \in M, x \neq 0 \quad x^T A x = X^T J X x = \tilde{x}^T \tilde{J} \tilde{x} = \tilde{x}^T \begin{bmatrix} I_p \\ 0 \end{bmatrix} \tilde{x} = \tilde{x}^T \tilde{x} > 0$$

$$\forall x \in N \quad x^T A x = \tilde{x}^T \tilde{J} \tilde{x} = \tilde{x}^T \begin{bmatrix} 0 \\ I_{n-p} \end{bmatrix} \tilde{x} = -\tilde{x}^T \tilde{x} < 0$$

$$x \in M \cap N \Rightarrow x \in M \Rightarrow \begin{cases} x = 0 \\ x \neq 0, x^T A x > 0 \end{cases}$$

$$\Rightarrow x \in N \Rightarrow x^T A x \leq 0$$

$$\Rightarrow \dim(M \cap N) = 0$$

$$p + n - q \leq n \Rightarrow p \leq q \Rightarrow p = q$$

SVD: (不唯一)

$$A = (U_r \ U_0) \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_r^T \\ V_0^T \end{pmatrix}$$

V_0 是 $N(A)$ 的一组标准正交基

$$V_r = N(A^T) = R(A^T) \quad \dots$$

$$U_0 \dots N(A^T) \quad \dots$$

$$U_r \dots R(A) \quad \dots$$

$$A = U_r \Sigma_r V_r^T \quad (\text{简化奇异值分解}) \quad (\text{简化SVD})$$

$$\text{例: } A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \\ 0 & 0 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 3 & 4 & 0 \\ 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 4 & 5 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix}$$

谱分解:

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 45 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$\Rightarrow$

$$U_r = A V_r \Sigma_r^{-1} = \begin{pmatrix} 3 & 0 \\ 4 & 5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 3 & 3 \\ 4 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & -1/\sqrt{10} \\ 0 & 0 \end{pmatrix}$$

必考!

$$A = \begin{pmatrix} 1/\sqrt{10} & 3/\sqrt{10} & 0 \\ 3/\sqrt{10} & -1/\sqrt{10} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$P_{R(A)} = U_r U_r^T = A V_r \Sigma_r^{-1} U_r^T = A A^+$$

(像 A^+ 一样)

唯一 $\leftarrow$ "广义逆" (Moore-Penrose)

$$AX=b$$

最小二乘解: $P_{R(A)} b = A \cdot x$

$$A A^+ b = A \cdot x$$

$$A \cdot (x - A^+ b) = 0$$

$$x = A^+ b + \tilde{x} \in N(A) \quad \|A^+ b + \tilde{x}\| \geq \|A^+ b\|$$

$$A^+ b = V_r \Sigma_r^{-1} U_r^T b \in R(W) = N(A)^\perp$$

$$\therefore \begin{cases} \|b - Ax\| = \min \\ \|x\| = \min \end{cases} \text{ 的解是 } A^+ b \quad !!$$

$$J = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \quad \lambda: n \neq 0 \quad \tau: n-1 \neq 1, 1 \neq 0$$

$$\tilde{J} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \quad \lambda: 10^t e^n \quad p=0 \dots n-1 \quad \tau: n-1 \neq 1, 1 \neq 10^n$$

$$\text{dist}(J, \tilde{J}) = \frac{\max \|Jx - \tilde{J}x\|}{\|x\|}$$

$$\|J - \tilde{J}\| = 10^n \quad |\lambda - \tilde{\lambda}| = 10^t \quad |\tau - \tilde{\tau}| \leq 10^n$$

定义: 对 $A \in \mathbb{R}^{m \times n}$, 称 $\max \frac{\|Ax\|}{\|x\|}$ 为 A 的谱范数

记为 $\|A\|$

$$\forall x, \frac{\|Ax\|}{\|x\|} \leq \max = \|A\|$$

$$\Rightarrow \|Ax\| \leq \|A\| \cdot \|x\|$$

$$\forall A: |\tau - \tilde{\tau}| \leq \|A - \tilde{A}\| \quad |\lambda - \tilde{\lambda}| \leq c_n \|A - \tilde{A}\|^{1/n}$$

$$\text{命题: } ① \|A\| \geq 0 \quad \|A\|=0 \Leftrightarrow A=0$$

$$② \|kA\| = |k| \cdot \|A\|$$

$$③ \|A+B\| \leq \|A\| + \|B\|$$

$$④ \|AB\| \leq \|A\| \cdot \|B\|$$

$$⑤ \|A^T\| = \|A\|$$

$$⑥ \|uAv^T\| = \|A\| \text{ 若 } u, v \text{ 正交}$$

$$\text{命题: } \|A\| = \sqrt{\lambda_{\max}(A^T A)}$$

$$\text{Proof: } \|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sqrt{\max_{x \neq 0} \frac{x^T A^T A x}{x^T x}}$$

$$= \sqrt{\lambda_{\max}(A^T A)} = \sqrt{\lambda_{\max}(A A^T)} = \sqrt{\lambda_{\max}(A^T A)}$$

命题: A 的奇异值 $\sigma_1 \geq \dots \geq \sigma_n$
 $u_1 \dots u_n$

则 $\sigma_1 = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$
 $\vdots$
 $\sigma_i = \max_{\substack{x \neq 0 \\ x \perp \text{span}\{u_1, \dots, u_{i-1}\}}} \frac{\|Ax\|}{\|x\|}$

$A = U \Sigma V^T = U_r \Sigma_r W^T$
 $(m+n+1) \quad (m+r+n = r(m+n+1)) \uparrow$

SVD 可能省空间

$\approx U_k \Sigma_k V_k^T$ $k(m+n+1)$
 $k < r$

"低秩逼近"

$\|A - U_k \Sigma_k V_k^T\| = ?$

$= \|U_r \Sigma_r V_r^T - U_k \Sigma_k V_k^T\|$
 $e.g. \begin{cases} U_r = (U_k \ U_{r-k}) \\ \Sigma_r = \begin{pmatrix} \Sigma_k & \\ & \Sigma_{r-k} \end{pmatrix} \\ V_r^T = \begin{pmatrix} V_k^T & V_{r-k}^T \end{pmatrix} \end{cases} \Rightarrow \|U_{r-k} \Sigma_{r-k} V_{r-k}^T\| = \sigma_{k+1}$
 $(= \min)$

HW: 1 (15), 2 ~ 10.

Review:

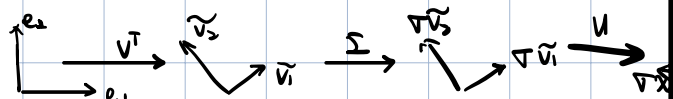
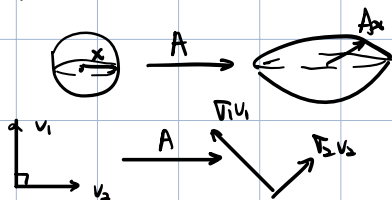
SVD $A = U \Sigma V^T$ $\Sigma = \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix}$ $\Sigma_r = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}$ $\sigma_1 \geq \dots \geq \sigma_r > 0$

$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sigma_1(A)$

$\|Ax\| \leq \|A\| \|x\|$

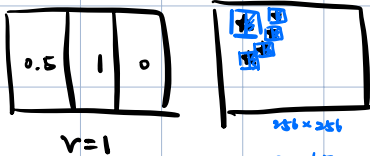
$AV = U \Sigma$

$A = U \Sigma V^T$



低秩逼近

$A \approx A_r$ 秩很小



$A = U_r \Sigma_r V_r^T$
 $= \sigma_1 u_1 v_1^T + \dots + \sigma_r u_r v_r^T$
 $A_k = \sigma_1 u_1 v_1^T + \dots + \sigma_k u_k v_k^T$ ($k < r$) **截断SVD**

定理: $\min_{r(B) \leq k} \|A - B\| = \|A - A_k\| = \sigma_{k+1}$

A_k 是 A 的最佳秩 k 逼近

Proof: 只需证 $\min_{r(B) \leq k} \|A - B\| = \sigma_{k+1}$

$\Leftrightarrow \forall B, r(B) \leq k \Rightarrow \|A - B\| \geq \sigma_{k+1}$

$\|A - B\| = \max_{\|x\|=1} \|(A - B)x\|$

$= \|(A - B)x_0\|$ ($\|x_0\| = 1$)

$Bx_0 = 0 \quad \|(A - B)x_0\| = \|Ax_0\| \geq \sigma_{k+1}$

$\uparrow \exists x_0 \quad x_0 \perp \text{span}\{v_1, \dots, v_k\}$

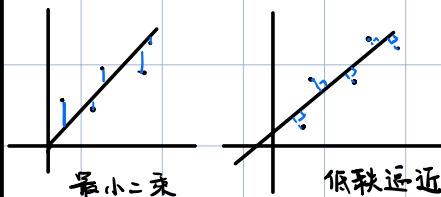
$\Leftrightarrow x_0 \in N(B) \cap \text{span}\{v_1, \dots, v_k\}$

$\dim N(B) \geq n - k$ $\dim(\text{span}\{v_1, \dots, v_k\}) = k + 1$

$\dim N(B) \cap \text{span}\{v_1, \dots, v_k\} \geq n - k + k + 1 - n = 1 \quad \checkmark$

$a_1 \dots a_n \in \mathbb{R}^d$

$\hat{a}_1 \dots \hat{a}_n \in M$ $\dim M = r$



主成分分析 (PCA)

$\min_Q \sum_{i=1}^n \|I - Q Q^T\| a_i = ?$ $A = (a_1 \dots a_n)_{d \times n}$

$= \sum a_i^T (I - Q Q^T) a_i$

$= \sum a_i^T a_i - \sum a_i^T Q Q^T a_i$

$= \text{tr}(A^T A) - \text{tr}(A^T Q Q^T A)$

$= \text{tr}(A A^T) - \text{tr}(Q^T A A^T Q)$

$\max_{Q^T Q = I_r} \text{tr}(Q^T A A^T Q) = ?$

$= \text{tr}(Q^T U \Sigma V^T V \Sigma^T U^T Q)$

$= \text{tr}(\tilde{Q}^T \Sigma \tilde{Q})$

$= \text{tr}(\tilde{Q}^T (\tilde{v}_1^2 \dots \tilde{v}_r^2) \tilde{Q})$

$= \text{tr}((\tilde{Q} \ Q_n)^T (\tilde{v}_1^2 \dots \tilde{v}_r^2) (\tilde{Q} \ Q_n))$

$= \text{tr}(Q_n^T Q_n)$

命题: $\max_{1 \leq k \leq n} \text{tr}(Q^T A A^T Q) = \text{tr}(U_k^T A A^T U_k)$

$k=1$ 时 $\max_{q^T q=1} q^T A A^T q = \max \|A^T q\|^2 = \sigma_1^2 \quad \checkmark$

设 $k=i$ $\checkmark$ 考虑 k

$\text{tr}((Q_i \quad 1)^T A A^T (Q_i \quad 1))$

$= \text{tr}(Q_i^T A A^T Q_i) + q_{i+1}^T A A^T q_{i+1}$

$\leq \text{tr}(U_i^T A A^T U_i) \quad \text{又需证} \leq \text{tr}(U_k^T A A^T U_k) = \sigma_k^2$

$\|A^T q_{i+1}\|^2$

$\begin{cases} q_{i+1} \perp \text{span}\{q_1, \dots, q_m\} \\ q_{i+1} \perp \text{span}\{u_1, \dots, u_m\} \end{cases}$ 同时满足

$\Leftrightarrow q_{i+1} \in N(W_i)$

维数不等式 $\Rightarrow q_{i+1} \quad \checkmark$
 $k+m+k+1-1$

定义: 一个矩阵每个元素的平方和 $\sqrt{\text{tr}(A A^T)}$

$= \sqrt{\sum_{i,j} |a_{ij}|^2}$ 为 A 的 **Frobenius 范数**, 记为 $\|A\|_F$

性质: $\|A\|_F \geq 0, \|A\|_F = 0 \Leftrightarrow \|A\| = 0$

$\|kA\|_F = k \|A\|_F$

$\|A+B\|_F \leq \|A\|_F + \|B\|_F$

$\|A^T\|_F = \|A\|_F$

$\|AB\|_F \leq \frac{\|A\|_F \|B\|_F}{\|A\|_F \cdot \|B\|_F} = \|A\|_F \|B\|_F$

$\|A\|_F = \sqrt{\sum \sigma_i^2}$

定理: $\|A - B\|_F = \|A - A_k\|_F$

PA_k 的 Frobenius 范数意义下的最佳秩 k 逼近
令 QQ^T 是 B 上的正交投影

$\|A - B\|_F^2 = \|A - QQ^T A + QQ^T A - QQ^T B + QQ^T B - B\|_F^2$

$= \|A - QQ^T A\|_F^2 + \|QQ^T A - QQ^T B\|_F^2$
 $+ 2 \text{tr}((A - QQ^T A)^T (QQ^T A - QQ^T B)) = 0$

$= \|I - QQ^T\|_F^2 + \|QQ^T(A - B)\|_F^2$

$\geq \|(I - QQ^T)A\|_F^2 = \text{tr}(A^T(I - QQ^T)A)$

$QQ^T = I - \tilde{U} \tilde{U}^T \quad \tilde{U} \leq k$

$= \|A\|_F^2 - \|\tilde{U}^T A\|_F^2 = \|A\|_F^2 - \text{tr}(\tilde{U}^T A A^T \tilde{U})$

$\therefore \|A - QQ^T A\|_F^2 \geq \sum_{j=1}^k \sigma_j^2 - \sum_{j=1}^k \sigma_j^2 = \sum_{j=1}^k \sigma_j^2$
 $= \sum_{j=1}^k \sigma_j^2 \geq \sum_{j=1}^k \sigma_j^2 = \|A - A_k\|_F^2 \quad \#$

HW: 11-13

12, 20 Chp. 7

$\mathbb{R}^n, \mathbb{C}^n, \mathbb{M}_n(\mathbb{R}), \mathbb{M}_n(\mathbb{C})$

$\mathbb{R}^{m \times n}, \mathbb{C}^{m \times n}$

定义: 给定 $\mathbb{C}$ 的子集 F , 若 F 包含一个非零复数, 且对四则运算封闭 ($\forall a, b \in F, a+b, a-b, a \cdot b, a/b \in F, b \neq 0$)

则称 F 是一个 **数域**

$Ax=b$ 例: $\mathbb{Q}(i) = \{a+bi \mid a, b \in \mathbb{Q}\} \quad \checkmark$

$\mathbb{Q}(i) = \{a+bi \mid a, b \in \mathbb{Q}\} \quad \checkmark$

定义: 给定数域 F 和非空集合 V , 若 V 上有两种运算:

① 加法: $\forall a, b \in V$, 有唯一元素与之对应, 记为 $a+b$

② 数乘: $\forall a \in V, k \in F$, 有唯一元素与之对应, 记为 ka

且两种运算满足八条运算法则:

- ① 加法结合律: $\forall (a, b, c) \in V, (a+b)+c = a+(b+c)$
- ② 加法交换律: $\forall (a, b) \in V, a+b = b+a$
- ③ 有零元素: $\exists 0 \in V, \text{s.t. } \forall a \in V, a+0 = a$
- ④ 有负元素: $\forall a \in V, \exists b \in V, \text{s.t. } a+b = 0$
- ⑤ 有单位数: $\mu(1, a) = a \quad \forall a \in V$
- ⑥ 数乘结合律: $\mu(k, \mu(l, a)) = \mu(kl, a)$
- ⑦ 数乘对加法的分配律: $\mu(k+l, a) = \mu(k, a) + \mu(l, a)$
- ⑧ 数乘对数域的分配律: $\mu(k, a+b) = \mu(k, a) + \mu(k, b)$

则称 V 是数域 F 上的一个 **线性空间**

其中的元素可以称为 **向量**

例: $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\} = V$

$\forall (a, b) \in V, a+b = ab$

$\mu(k, a) = a^k$

V 非空 $\checkmark \quad F = \mathbb{R}$

运算封闭 $\checkmark \quad$ 运算法则 $\checkmark$

$V = \{a^k \mid k \in \mathbb{R}\}$

$\forall (a, b) \in V, a+b = ab$

$\mu(k, a) = a^k$

$$\sum_{j=k+1}^n \dots$$

$\mathbb{R}$ 是 $\mathbb{R}$ 上的线性空间

$\mathbb{R}$ 是 $\mathbb{C}$ 上的线性空间

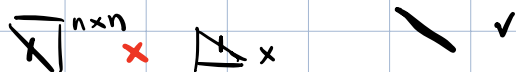
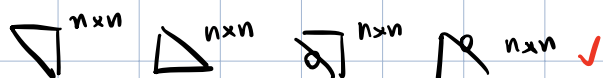
$\mathbb{C}$ 是 $\mathbb{R}$ 上的

$\mathbb{C}$ 是 $\mathbb{C}$ 上的

坐标空间: $\mathbb{F}^n, (\mathbb{R}^n, \mathbb{C}^n)$ 是 $\mathbb{F}$ 上的

$N(A), Q(A), N(A^T), R(A^T)$

矩阵空间: $\mathbb{F}^{n \times n}$ 是 $\mathbb{F}$ 上的



$$\mathbb{F}_S^{n \times n} = \{A \in \mathbb{F}^{n \times n} \mid A = A^T\}$$

$$\mathbb{F}_N^{n \times n} = \{A \in \mathbb{F}^{n \times n} \mid A = -A^T\}$$

$$\mathbb{F}_n^{n \times n} = \{\text{可逆矩阵}\} \quad \times$$

$$\{\text{正交矩阵}\} \quad \times$$

函数空间: $\text{Func}(D) \supset C(D) \supset C^1(D) \supset C^2(D) \dots$

$C^\infty(D) \supset C^w(D) \supset \mathbb{R}[x] \supset \mathbb{R}[x]_n$

光滑函数空间 解析 ~ 多项式 ~ ($\deg < n$) 多项式

$$\{a_0 + a_1x + \dots + a_nx^n \mid a_n \neq 0\} \quad \times$$

$$(+) + (-) = 0$$

定义: 数域 $\mathbb{F}$ 上线性空间 V 的非空子集 M 若关于 V 的线性运算也构成 $\mathbb{F}$ 的线性空间, 则称 M 是 V 的 (线性) 子空间

$M \neq \emptyset, M \subseteq V \rightarrow$ 线性空间:

命题: M 是 V 的子空间 $\Leftrightarrow M$ 关于 V 的加法和数乘封闭

乘封闭

Proof: $\Rightarrow \checkmark$

$\Leftarrow$ (1) $\checkmark$ (2) $\checkmark$ (3) $\checkmark$ (4) $\checkmark$ (5) $\checkmark$ (6) $\checkmark$

if

$\rightarrow \checkmark$

$$(3): M \neq \emptyset \Rightarrow \exists a \in M \Rightarrow a + (-a) = 0 \in M \checkmark$$

$$0 \cdot a = 0 \checkmark$$

$$(4): \forall a \in M, (-1) \cdot a \in M, a + (-1)a = 0 \cdot a = 0 \checkmark$$

命题: 1) 零元素唯一

2) 负

$$3) a+b=a+c \Rightarrow b=c$$

$$(4) a+b=c \Rightarrow a=c-b$$

$$(5) 0 \cdot a = 0_V; k \cdot 0_V = 0_V; (-1)a = -a$$

$$(6) ka=b \Rightarrow a=\frac{1}{k}b$$

Proof:

(1) $0, \tilde{0}$ 是零元素.

$$0 = 0 + \tilde{0} = \tilde{0} + 0 = \tilde{0}; 0 = \tilde{0}$$

(2) b, c 都是 a 的负元素

$$b = b + 0 = b + a + c = 0 + c = c$$

(3) $b \cdot c$

$$(a+b) + (-a) = (a+c) + (-a)$$

$$\Rightarrow b+0 = c+0 \Rightarrow b=c$$

(4)

$$a = a+b+(-b) = c+(-b) = c-b$$

(5)

$$(0+0) \cdot a = 0a + 0a$$

$$= 0a + 0a$$

$$\therefore 0a = 0_V$$

$$k(a_V + b_V) = k a_V + k b_V$$

$$= k a_V + 0_V \therefore k a_V = 0_V$$

...

子空间的运算:

$$M_1 \cap M_2 = \{a \in V \mid a \in M_1, a \in M_2\} \text{ 交}$$

$$M_1 + M_2 = \{m_1 + m_2 \mid m_1 \in M_1, m_2 \in M_2\} \text{ 和}$$

$$M_1 \cup M_2 \quad \times$$



$$\Delta^{n \times n} + \Delta^{n \times n} = \mathbb{F}^{n \times n} \text{ 不是直和}$$

$$\forall a \in \mathbb{F}^{n \times n} \quad a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ 不唯一}$$

$$\mathbb{F}_S^{n \times n} \cap \mathbb{F}_N^{n \times n} = \{0\}$$

$$\mathbb{F}_S^{n \times n} + \mathbb{F}_N^{n \times n} = \mathbb{F}^{n \times n}$$

$$A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$$

交集为 0

命题: $M = M_1 + M_2$

$a = a_1 + a_2$ 表示法唯一

$\Leftrightarrow 0 = 0 + 0$ 分解法唯一

$\Leftrightarrow M_1 \cap M_2 = \{0\}$ 此时称 $M_1 + M_2$ 为直和,

Proof: $1) \Rightarrow 2) \checkmark$ 记为 $M_1 \oplus M_2$

$2) \Rightarrow 3) \checkmark$ 取 $a \in M_1 \cap M_2$

$$0 = 0 + 0 = a + (-a) \Rightarrow a = 0$$

有限

$3) \Rightarrow 1) \checkmark$ $a = a_1 + a_2 = b_1 + b_2$

$$(a_1, b_1) = (a_2, b_2)$$

$$\in M_1 \quad \in M_2$$

$$\therefore M_1 \cap M_2 = \{0\} \quad \therefore \begin{cases} a_1 = b_1 \\ a_2 = b_2 \end{cases}$$

线性组合: $k_1 a_1 + \dots + k_n a_n \quad a_1, \dots, a_n \in V$
 $k_1, \dots, k_n \in \mathbb{F}$

线性表示: $b = k_1 a_1 + \dots + k_n a_n$ b 被 $(a_1, \dots, a_n)$ 线性表示

线性生成: $\text{span}\{a_1, \dots, a_n\}$ 生成 自白子空间

线性无关: $k_1 a_1 + \dots + k_n a_n = 0 \Rightarrow k_1 = \dots = k_n = 0$
(相关)

极大线性无关部分组: $\sim$

(同 chp. 2)

筛选法

线性等价: 不变量: 向量组的秩

基和维数:

HW: 1~5, 7~11, 13

1. $\mathbb{R}^+$ $\therefore$ 非空 $\checkmark$

加法, 乘法封闭:

$$\forall a, b \in \mathbb{R}^+$$

$$a \oplus b = ab \in \mathbb{R}^+$$

$$k \odot a = a^k \in \mathbb{R}^+ \quad \checkmark$$

12.25

线性空间: $\mathbb{F}, V$ 2+8

子空间 2 交, 和, 直和

线性组合 $k_1 a_1 + k_2 a_2 + \dots + k_n a_n$

线性表示 $\text{span}\{a_1, \dots, a_n\} = \{k_1 a_1 + \dots + k_n a_n \mid k_i \in \mathbb{F}\}$
线性生成
线性无关/相关 $k_1 a_1 + \dots + k_n a_n = 0 \Rightarrow k_1 = \dots = k_n = 0$

极大线性无关部分组

筛选法

线性等价 向量组的秩

定义 (基和维数):

对数域 $\mathbb{F}$ 上的线性空间 V , 若存在 $a_1, \dots, a_n \in V$ 线性无关且生成 V , 则称 $a_1, \dots, a_n$ 是 V 的一组基, 又称 V 是 n 维线性空间, 又称 V 的维数是 n , 记作 $\dim V = n$

① $\dim \{0\} = 0$. 无基

② 若 V 中存在任意多个线性无关的向量, 则称 V 是无限维线性空间, 记作 $\dim V = \infty$

例: $e^{\lambda_1 x}, e^{\lambda_2 x}, e^{\lambda_3 x} \in C^\infty(\mathbb{R})$ 线性无关
($\lambda_1, \lambda_2, \lambda_3$ 互不相同)

Proof: $k_1 e^{\lambda_1 x} + k_2 e^{\lambda_2 x} + k_3 e^{\lambda_3 x} = 0 \quad \forall x \in \mathbb{R} \Rightarrow k_1 = k_2 = k_3 = 0$

$$x = 0, 1, 2$$

$$\begin{cases} k_1 + k_2 + k_3 = 0 \\ k_1 e^{\lambda_1} + k_2 e^{\lambda_2} + k_3 e^{\lambda_3} = 0 \\ k_1 e^{2\lambda_1} + k_2 e^{2\lambda_2} + k_3 e^{2\lambda_3} = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ e^{\lambda_1} & e^{\lambda_2} & e^{\lambda_3} \\ e^{2\lambda_1} & e^{2\lambda_2} & e^{2\lambda_3} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0$$

$$\det(-) = (-1)^1 (e^{\lambda_1} - e^{\lambda_2})(e^{\lambda_1} - e^{\lambda_3})(e^{\lambda_2} - e^{\lambda_3}) \neq 0$$

$$k_1 = k_2 = k_3 = 0 \quad \#$$

Proof 2: $f(x) = f'(x) = f''(x) = 0$

$$\begin{pmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} & e^{\lambda_3 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} & \lambda_3 e^{\lambda_3 x} \\ \lambda_1^2 e^{\lambda_1 x} & \lambda_2^2 e^{\lambda_2 x} & \lambda_3^2 e^{\lambda_3 x} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0$$

$$\det(-) = e^{\lambda_1 x + \lambda_2 x + \lambda_3 x} \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} \neq 0 \quad \forall x$$

$$k_1 = k_2 = k_3 = 0 \quad \#$$

$\forall n, e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ 线性无关

$\lambda_1, \dots, \lambda_n$ 互不相同

$$\dim C^\infty(\mathbb{R}) = \infty$$

$$\therefore \dim C(\mathbb{R}) = \dim \text{Func}(\mathbb{R}) = \infty$$

$$\dim C^w(\mathbb{R}) = \infty$$

$$\dim \mathbb{R}[x] = ? \rightarrow \begin{cases} \dim \mathbb{R}[x] = \infty \\ \dim \mathbb{R}[x]_n = n \end{cases}$$

$$\mathbb{R}[X]_n = \{a_0 + a_1 X + \dots + a_{n+1} X^{n+1} \mid a_i \in \mathbb{R}\}$$

$$= \text{span}(1, X, \dots, X^{n+1})$$

$$k_1 X + k_2 X + \dots + k_n X^{n+1} = 0 + 0X + 0X^2 + \dots + 0X^{n+1}$$

解: 取 $x_1, \dots, x_n$ 互不相同

$$\textcircled{1} \begin{pmatrix} x_1 & \dots & x_n \\ x_2 & \dots & x_n \\ \vdots & \ddots & \vdots \\ x_n & \dots & x_n \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = 0$$

$\det \neq 0$

$$\textcircled{2} \begin{pmatrix} 1 & x & \dots & x^{n-1} \\ 2x & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \vdots \\ (n-1)x^{n-2} & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = 0$$

③ 代数基本定理: 至多有 $n+1$ 个根, 取 $\mathbb{R} \setminus \{x_1, \dots, x_{n+1}\}$ 中元素 x_0 , 若 $\exists k_i \neq 0$, 则 $f(x_0) \neq 0 \Rightarrow k_1 = \dots = k_n = 0$
有限维空间: 基存在, 基扩充定理

命题(维数公式)

(若 M_1, M_2 是有限维)

$$\dim(M_1 \cap M_2) + \dim(M_1 + M_2) = \dim(M_1) + \dim(M_2)$$

取 $M_1 \cap M_2$ 的基 $a_1, \dots, a_p$

将其扩充成 M_1 的基 $a_1, \dots, a_p, b_1, \dots, b_{q-p}$

$\dots \dots M_2 \quad a_1, \dots, a_p, c_1, \dots, c_{r-p}$

下证: $a_1, \dots, a_p, b_1, \dots, b_{q-p}, c_1, \dots, c_{r-p}$ 是 $M_1 + M_2$ 的基

$$\forall x \in M_1 + M_2 \quad x = \underbrace{x_1}_{M_1} + \underbrace{x_2}_{M_2} \quad \text{表之} \checkmark$$

无关:

$$k_1 a_1 + \dots + k_p a_p + l_1 b_1 + \dots + l_{q-p} b_{q-p} + m_1 c_1 + \dots + m_{r-p} c_{r-p} = 0$$

$x_1 = x_2$
 $\in M_1 \cap M_2$
无关 $\checkmark$ $\#$

命题: M_1, M_2 是直和 $\Leftrightarrow \dim(M_1 + M_2) = \dim M_1 + \dim M_2$

$\dim \mathbb{R}^{m \times n}$

$$A = \sum_{i,j} a_{ij} e_j e_i^T$$

$$\therefore \mathbb{R}^{m \times n} = \text{span} \{ e_j e_i^T \mid i=1, \dots, m, j=1, \dots, n \}$$

$$k_{ij} e_j e_i^T = 0 \quad k_{ij} = 0$$

$$\dim \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}^{n \times n} = \frac{n(n+1)}{2}$$

$$\dim \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}^{n \times n} = \frac{n(n+1)}{2}$$

$$\dim \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}^{n \times n} = n$$

$$\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}^{n \times n} \cap \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}^{n \times n} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}^{n \times n}$$

$$\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}^{n \times n} + \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}^{n \times n} = \mathbb{R}^{n \times n}$$

$$\dim \mathbb{R}_s^{n \times n} = \frac{n(n+1)}{2}$$

$$\dim \mathbb{R}^{n \times n} = \frac{n(n-1)}{2}$$

定义: 对 F 上两个线性空间 U, V 映射 $f: U \rightarrow V$

$$\text{满足 } f(a+b) = f(a) + f(b) \quad \forall a, b \in U$$

$$f(ka) = kf(a) \quad \forall k \in F, a \in U$$

则 f 是 U 到 V 的线性映射

U 到 V 线性映射全体, 记作 $\text{Hom}(U, V)$

$$f \in \text{Hom}(U, V)$$

线性映射的运算:

$$f+g: U \rightarrow V$$

$$x \mapsto f(x) + g(x)$$

$$(f+g)(x) = f(x) + g(x)$$

$$(kf)(x) = kf(x)$$

$$f: U \rightarrow V$$

$$gf: U \rightarrow W$$

$$g: V \rightarrow W$$

$$x \mapsto g(f(x))$$

$$(gf)(x) = g(f(x))$$

命题: $\text{Hom}(U, V)$ 关于 “+”, “.” 构成 F 上的线性空间

$$\text{核 } N(f) = \{x \mid f(x) = 0\} \subseteq U$$

$$\text{像集 } R(f) = \{f(x) \mid x \in U\} \subseteq V \quad \text{是子空间}$$

$$(\dim N(f) + \dim R(f) = \dim U)$$

定义: 给 F 上的线性空间 U 上的线性变换 f ,

若存在 $\lambda \in F, x \neq 0 \in U$, 使得 $f(x) = \lambda x$, 则称 λ 是 f 的一个特征值, x 是 f 的一个属于 λ 的特征向量

$$S: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$$

$$X \mapsto \frac{X+X^T}{2}$$

$$N(S) = \{X \mid S(X) = 0\} = \{X \mid \frac{X+X^T}{2} = 0\} = \mathbb{R}_n^{n \times n}$$

$$R(S) = \{S(X) \mid X\} = \{\frac{X+X^T}{2} \mid X\} = \mathbb{R}_S^{n \times n}$$

$$\text{特征值: } S(X) = \lambda X \quad \frac{X+X^T}{2} = \lambda X$$

$$\begin{aligned} S^2 &= S & S(X) &= 0 & N(0I - S) &= N(S) \\ \lambda^2 &= \lambda & S(X) &= X & N(I - S) &= R(S) \\ \lambda &= 0, 1 \end{aligned}$$

$$L_A: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$$

$$X \mapsto AX$$

$$N(L_A) = \{X \mid AX = 0\} = \left\{ \begin{matrix} BC \\ n \times (n-r) \end{matrix} \mid \begin{matrix} B \text{ 是 } N(A) \text{ 的基} \\ C \text{ 任意} \end{matrix} \right\}$$

$$R(L_A) = \{AX \mid X\}$$

$$AX = \lambda X = DE \mid D \text{ 是 } |A| \text{ 的基, } E \text{ 任意}$$

$$N(\lambda I - L_A) = \{FG \mid F \text{ 是 } N(\lambda I - A) \text{ 的基}, G \text{ 任意}\}$$

$$D: C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$$

$$f \mapsto f'$$

$$N(D) = \mathbb{C}$$

$$R(D)$$

$$Df = \lambda f \quad f' = \lambda f, \quad f \neq 0 \quad \lambda \in \mathbb{R}$$

$$\forall \lambda \in \mathbb{R}, \lambda \text{ 是 } D \text{ 的特征值}$$

$$k e^{\lambda x} \in \text{---}$$

$$HW: 7.2 \quad 1-4, 5(\text{奇}), 6, (1.4), 12, 13$$

$$7.3 \quad 1-7, 9-10, 12-13$$

12.27

定义: 对 $\mathbb{F}$ 上线性空间 U, V , U 到 V 的线性映射 f 是双射, 则称 U 同构于 V , 又称 f 是 U 到 V 的同构 (映射), U 到 U 的同构映射称为 U 上的自同构 ($\sim$)

命题: f 是 U 到 V 的同构, g 是 V 到 W 的同构

则 $g \circ f$ 是 U 到 W 的同构

2) f^{-1} 是 V 到 U 的同构

命题: 线性空间的同构关系是等价关系

Proof: 1) $\checkmark$

2) f^{-1} 双射 $\checkmark$

f^{-1} 是线性? $\Leftrightarrow f^{-1}(k_1 x_1 + k_2 x_2) = k_1 f^{-1}(x_1) + k_2 f^{-1}(x_2)$

$$\Leftrightarrow f \circ f^{-1}(k_1 x_1 + k_2 x_2) = f(k_1 f^{-1}(x_1) + k_2 f^{-1}(x_2))$$

$$k_1 x_1 + k_2 x_2 = k_1 x_1 + k_2 x_2 \quad \checkmark$$

$$\text{例: } T: a+bi \mapsto \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

$$\mathbb{C} \rightarrow \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \subseteq \mathbb{R}^{2 \times 2}$$

是同构

$$T: F^n \rightarrow F^n$$

$$x \mapsto Tx \quad \text{是自同构}$$

不变量? 维数 (dim)

标准形? F^n

$\mathbb{F}$ 上的 n 维线性空间 U

基 $e_1, \dots, e_n$

$$\forall x \in U, x = x_1 e_1 + \dots + x_n e_n \stackrel{\text{记号}}{=} (e_1, \dots, e_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\text{唯一: } x = x_1 e_1 + \dots + x_n e_n$$

$$= \tilde{x}_1 e_1 + \dots + \tilde{x}_n e_n$$

$$\Rightarrow x_1 = \tilde{x}_1, \dots, x_n = \tilde{x}_n \quad \#$$

$$T_{e_1, \dots, e_n}: U \rightarrow \mathbb{F}^n$$

$$x = x_1 e_1 + \dots + x_n e_n \mapsto \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \stackrel{\text{是映射}}{=}$$

称 $x_1, \dots, x_n$ 为 x 在基 $e_1, \dots, e_n$ 下的坐标

$T: e_1, \dots, e_n$ 是线性映射, 是双射
故 $T: e_1, \dots, e_n$ 是 U 到 $\mathbb{F}^n$ 的同构

命题: $\mathbb{F}$ 上 n 维线性空间 U , $x_1, \dots, x_n \in U$

$$x_i = (e_1, \dots, e_n) \hat{x}_i$$

$x_1, \dots, x_n$ 的线性关系与 $\hat{x}_1, \dots, \hat{x}_n$ 的线性关系相同

EP: 1) $x_{i_1}, \dots, x_{i_r}$ 线性无关 $\Leftrightarrow \hat{x}_{i_1}, \dots, \hat{x}_{i_r}$ 线性无关 e.g.

$$2) x_{i_{r+1}} = k_1 x_{i_1} + \dots + k_r x_{i_r} \Leftrightarrow \hat{x}_{i_{r+1}} = k_1 \hat{x}_{i_1} + \dots + k_r \hat{x}_{i_r} \quad (E_1, E_2, E_3, E_4) = (E_1, E_2, E_3, E_4) \begin{pmatrix} 1 & 1 & 1 \\ & & \end{pmatrix}$$

例: $\mathbb{R}^{2 \times 2}$ 有一组基: $E_{11}, E_{12}, E_{21}, E_{22}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} E_{11} + a_{12} E_{12} + a_{21} E_{21} + a_{22} E_{22}$$

$$T_{E_{11}, E_{12}, E_{21}, E_{22}}: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^4$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}$$

$$T_{E_{11}, E_{12}, E_{21}, E_{22}}: A \mapsto \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$$

$\mathbb{F}$ 上 n 维线性空间 U

基: $e_1, \dots, e_n$

$t_1, \dots, t_n$

$$t_1 = t_{11} e_1 + t_{12} e_2 + \dots + t_{1n} e_n$$

$$t_2 = t_{21} e_1 + t_{22} e_2 + \dots + t_{2n} e_n$$

$$(t_1, \dots, t_n) = (e_1, \dots, e_n) \underbrace{\begin{pmatrix} t_{11} & t_{12} & \dots \\ t_{21} & t_{22} & \dots \\ \vdots & \vdots & \ddots \\ t_{n1} & t_{n2} & \dots \end{pmatrix}}_{=T}$$

称 T 为

从基 $e_1, \dots, e_n$ 到基 $t_1, \dots, t_n$ 的过渡矩阵

命题: $U: (t_1, \dots, t_n) = (e_1, \dots, e_n) \cdot T$

则 $t_1, \dots, t_n$ 是 U 的一组基 $\Leftrightarrow T$ 可逆

$$x = (e_1, \dots, e_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (t_1, \dots, t_n) \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= (e_1, \dots, e_n) \cdot T \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= (t_{11} e_1 + \dots + t_{1n} e_n, \dots) \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= y_1 (t_{11} e_1 + \dots + t_{1n} e_n) + \dots$$

$$= (y_1 t_{11} + y_2 t_{12} + \dots + y_n t_{1n}) e_1 + \dots$$

$$= [T^* \cdot y] e_1 \dots e_n = (e_1, \dots, e_n) (T \cdot y)$$

$$\Rightarrow x = (e_1, \dots, e_n) \hat{x} = (t_1, \dots, t_n) \hat{y}$$

$$\hat{x} = T \hat{y} \quad ((e_1, \dots, e_n) T) \hat{y} = (e_1, \dots, e_n) (T \hat{y})$$

基变换公式: $(t_1, \dots, t_n) = (e_1, \dots, e_n) T$

坐标变换公式: $x = (e_1, \dots, e_n) \hat{x} = (t_1, \dots, t_n) \hat{y}$
 $\hat{x} = T \hat{y}$

$$T_{E_{11}, E_{12}, E_{21}, E_{22}}(A) = T \cdot T_{E_{11}, E_{12}, E_{21}, E_{22}}(A)$$

$$A = QR$$

$$(a_1, \dots, a_r) = (q_1, \dots, q_r) \nabla$$

$$x = A \hat{x} = Q \hat{y} \quad x = A \hat{y}$$

$$(A \ B) \xrightarrow{\text{Gauss-Jordan}} \left(\begin{array}{c|c} I & T \\ \hline 0 & 0 \end{array} \right)$$

$$B = A(A^T B)$$

$$(e_1, \dots, e_n) (k \hat{x} + l \hat{y}) = k(e_1, \dots, e_n) \hat{x} + l(e_1, \dots, e_n) \hat{y}$$

$$U \text{ (dim} = n) \xrightarrow{T: e_1, \dots, e_n} \mathbb{F}^n$$

$$x \longrightarrow \hat{x}$$

$$(t_1, \dots, t_n) \longrightarrow (\hat{t}_1, \dots, \hat{t}_n) = T$$

$$A \longrightarrow ?$$

$f: U \rightarrow V$

U 基: $e_1, \dots, e_n$

V 基: $i_1, \dots, i_m$

命题: $f, g \in \text{Hom}(U, V)$ 若 $f(e_i) = g(e_i), i=1, \dots, n$,

则 $f = g$

② $\forall a_1, \dots, a_n \in V \quad \exists f \in \text{Hom}(U, V) \text{ s.t.}$

$$f(e_i) = a_i$$

③ $\forall a_1, \dots, a_n \in V$ 存在唯一 $f \in \text{Hom}(U, V)$ s.t.

$$f(e_i) = a_i$$

Proof: ① $f(x) = f(x_1)e_1 + \dots + f(x_n)e_n$
 $= x_1 f(e_1) + \dots + x_n f(e_n)$

$$x_1 g(e_1) + \dots + x_n g(e_n) = g(x)$$

$$\therefore f(x) = g(x)$$

② $f: U \rightarrow V$

$$(e_1, \dots, e_n) \hat{x} \mapsto (a_1, \dots, a_n) \hat{x}$$

是映射
线性 $\checkmark$ #

$$f(e_1) = f_{11}e_1 + f_{12}e_2 + \dots + f_{1m}e_m$$

i

$$f(e_i) = f_{i1}e_1 + f_{i2}e_2 + \dots + f_{im}e_m$$

$$(f(e_1), \dots, f(e_n)) = (e_1, \dots, e_m) \cdot \begin{bmatrix} f_{11} & \dots & f_{1m} \\ \vdots & & \vdots \\ f_{n1} & \dots & f_{nm} \end{bmatrix}$$

$$= f(e_1, \dots, e_m) \text{ 所以 } F \text{ 是 } f \text{ 在基 } e_1, \dots, e_n; \in \mathbb{F}^{m \times n}$$

$$\forall e_1, \dots, e_n, i_1, \dots, i_m:$$

$$\text{Hom}(U, V) \rightarrow \mathbb{F}^{m \times n}$$

$i_1, \dots, i_m$ 下的表示矩阵

$$f \mapsto F \text{ (} f \text{ 的表示矩阵)}$$

双射. 线性 $\Rightarrow$ 同构

$$\text{HW: } 7.3 \rightarrow 18, 19$$

$$7.4 \rightarrow 1, 2(1), 3-4, 7-8, 10-11$$

1.3

$$\begin{matrix} V \\ \dim = n \end{matrix} \quad e_1, e_2, \dots, e_n$$

$$\forall \{e_i\}:$$

$$x_1 e_1 + \dots + x_n e_n \mapsto \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

取坐标映射

$$\forall \{e_i\} = (e_1, \dots, e_n) \hat{x}$$

$$= (e_1, \dots, e_n) (k\hat{x} + l\hat{y}) = k \forall \{e_i\} \hat{x} + l \forall \{e_i\} \hat{y}$$

$$= k(e_1, \dots, e_n) \hat{x} + l(e_1, \dots, e_n) \hat{y}$$

$$(e_1, \dots, e_n) (A\hat{x}) = ((e_1, \dots, e_n)A) \hat{x}$$

$$(t_1, \dots, t_n) = (e_1, \dots, e_n) T$$

$$x = (e_1, \dots, e_n) \hat{x} = (t_1, \dots, t_n) \hat{y}$$

$$\Rightarrow \hat{x} = T\hat{y}$$

$$U: i_1, \dots, i_m$$

$$\dim = m \quad \forall \{e_i\}, \{f_i\} \quad \text{Hom}(V, U) \rightarrow \mathbb{F}^{m \times n}$$

$$f \mapsto F$$

$$f(e_1, \dots, e_n) = (f(e_1), \dots, f(e_n)) = (e_1, \dots, e_m) \begin{bmatrix} f_{11} & \dots & f_{1n} \\ \vdots & & \vdots \\ f_{m1} & \dots & f_{mn} \end{bmatrix}$$

$$(kf + lg)(e_1, \dots, e_n) = (e_1, \dots, e_m) (k \forall \{f_i\} + l \forall \{g_i\})$$

$$\forall \{e_i\}, \{f_i\} \quad \text{Hom}(V, U) \rightarrow \mathbb{F}^{m \times n} \quad f: V \rightarrow U$$

$$\forall \{f_i\}, \{g_i\} : \text{Hom}(U, W) \rightarrow \mathbb{F}^{l \times m} \quad g: U \rightarrow W$$

$$\forall \{e_i\}, \{g_i\} : \text{Hom}(V, W) \rightarrow \mathbb{F}^{l \times n} \quad gf: V \rightarrow W$$

$$\forall \{e_i\}, \{g_i\} (gf) = \forall \{f_i\} \{g_i\} (g) \cdot \forall \{e_i\} \{f_i\} (f)$$

$$f_2 f_1 (e_1, \dots, e_n) A_1 A_2 A_3 \hat{x}$$

线性映射 向量组 坐标矩阵

$$N(f) \rightarrow N(F)$$

$$R(f) \rightarrow R(F)$$

$$f \text{ 为恒等} \rightarrow F \text{ 为恒等}$$

$$f \text{ 奇异三元组} \rightarrow F \dots$$

$$f(E_{11}, E_{12}, E_{21}, E_{22}) = (f(E_{11}), f(E_{12}), f(E_{21}), f(E_{22}))$$

$$= (E_{11}, \frac{1}{2}E_{11} + \frac{1}{2}E_{12}, \frac{1}{2}E_{12} + \frac{1}{2}E_{21}, E_{22})$$

$$= (E_{11}, E_{12}, E_{21}, E_{22}) \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(P_6) \mathbb{F}^{4 \times 4}: E_{11}, E_{22}, \frac{E_{11}+E_{12}}{2}, \frac{E_{21}-E_{22}}{2}$$

$$f(E_{11}, E_{12}, E_{21}, E_{22}) = (E_{11}, E_{22}, \frac{E_{11}+E_{12}}{2}, \frac{E_{21}-E_{22}}{2})$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$N(f) = \{A \mid f(A) = 0\}$$

$$= \mathbb{F}_N^{n \times n} = \text{span} \left(\frac{E_{12}-E_{21}}{2} \right)$$

$$\begin{array}{ccc} V & \xrightarrow{T} & \mathbb{F}^{2 \times 2} \\ \downarrow & & \downarrow \\ N(f) & \xleftarrow{T^{-1}} & N(F) \end{array}$$

$$T^{-1}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = (E_{11} E_{12} E_{21} E_{22}) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{span}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \mathbb{F}^{2 \times 2}$$

$$\begin{aligned} f(A) &= \lambda A & f(A^2) &= \lambda^2 A & \lambda^2 &= \lambda \\ & & &= f(A) = \lambda A & \Rightarrow \lambda &= 0, 1 \end{aligned}$$

$$F x = \lambda x$$

$$\det(\lambda I - F) = (\lambda - 1)^2 \begin{vmatrix} \lambda - \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - \frac{1}{2} \end{vmatrix} = (\lambda - 1)^3 \lambda$$

$$\lambda = 0: N(0I - F) = \text{span}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\lambda = 1: N(I - F) = \text{span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$$

$$T^{-1}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad T^{-1} \downarrow$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$f: V \rightarrow U$$

$$V: e_1 \dots e_n \xrightarrow{T} t_1 \dots t_n$$

$$f(e_1 \dots e_n) = (f_1 \dots f_n) F$$

$$U: u_1 \dots u_m \xrightarrow{S} s_1 \dots s_m$$

$$f(t_1 \dots t_n) = (s_1 \dots s_m) \tilde{F}$$

$$f(e_1 \dots e_n) = (f_1 \dots f_n) F$$

$$f(e_1 \dots e_n) T = (f_1 \dots f_n) S \tilde{F}$$

$$f(e_1 \dots e_n) T = (f_1 \dots f_n) S \tilde{F}$$

$$\underline{S \tilde{F} T^{-1} = F}$$

$$\tilde{F} = S^{-1} F T \quad \text{相似}$$

$$f: V \rightarrow V$$

$$V: e_1 \dots e_n \xrightarrow{T} t_1 \dots t_n$$

基变换公式:

$$f(e_1 \dots e_n) = (e_1 \dots e_n) F$$

$$(t_1 \dots t_n) = (e_1 \dots e_n) T$$

$$f(t_1 \dots t_n) = (t_1 \dots t_n) \tilde{F}$$

坐标变换:

$$T_{f|e_1} = T T_{f|t_1}$$

$$\tilde{F} = T^{-1} F T$$

$$\text{矩阵变换: } T_{f|t_1}(f) = T^{-1} T_{f|e_1}(f) T$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ f & & f \end{array}$$

$$D = T^{-1} A T$$

$$\begin{array}{c} \{t_i\} \\ \downarrow \\ A \end{array} \begin{array}{c} \{e_i\} \\ \downarrow \\ A \end{array}$$

$$A t_i = d_i t_i$$

$$A \text{ span}(t_i) \subseteq \text{span}(t_i) \quad \text{不变子空间}$$

$$V = \text{span}(t_1) \oplus \text{span}(t_2) \oplus \dots \oplus \text{span}(t_n)$$

$$\text{例: } \frac{d}{dt}(x(t)) = A(x(t))$$

$$x(t) = e^{At} \cdot c$$

$$A = T J T^{-1}$$

$$\frac{d}{dt} \tilde{x}(t) = J \tilde{x}(t)$$

$$3 \quad QR \quad \text{求小=求}$$

$$4 \quad \det$$

$$5. \quad \text{谱分解. 对称相似. 相似}$$

$$6. \quad \text{正交相似. 正定合同}$$

$$SVD$$

$$7. \quad \text{基和表示}$$

计算:

Golub & Van Loan Matrix Computation

矩阵论:

Horn & Johnson Matrix Analysis

Bernstein Matrix Mathematics

线性代数:

吴易 线性代数五讲