

Cofactor of $a_{ij} = C_{ij}$

$= \pm \det (n-1 \text{ matrix with row } i \text{ \& col } j \text{ erased})$

$i+j$
even

odd

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

Ex:

$$A_4 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$|A_1| = 1 \quad |A_2| = 0$$

$$|A_3| = -1$$

$$|A_4| = 1 \cdot |A_3| - 1 \cdot 0 = -1$$

$$\Rightarrow |A_n| = |A_{n-1}| - |A_{n-2}|$$

$$|A_5| = 0 \quad |A_6| = 1 \quad |A_7| = 1$$

$$(1, 0, -1, -1, 0, 1, 1, \dots)$$

L20

Formula for A^{-1}

Cramer's Rule for $x = A^{-1}b$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} C^T$$

← cofactor

↑ products of n entries

products $n-1$ entries

$$\text{check: } AC^T = (\det A) I$$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix} =$$

$$\begin{bmatrix} \det A & 0 & \dots & 0 \\ 0 & \det A & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \det A \end{bmatrix}$$

why 0? :

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} \dots & C_{ni} \\ \vdots & \vdots \\ \dots & C_{nn} \end{bmatrix}$$

Identical row 1 & n $\Rightarrow \det = 0$

$$Ax = b$$

$$x = A^{-1}b = \frac{1}{\det A} C^T b$$

CRAMER'S RULE

$$x_i = \frac{\det B_i}{\det A} \quad x = \begin{bmatrix} x_1 \\ \vdots \end{bmatrix}$$

$|\det A| = \text{volume of box}$

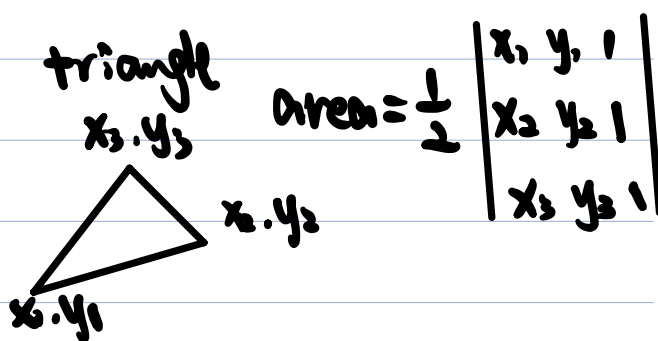
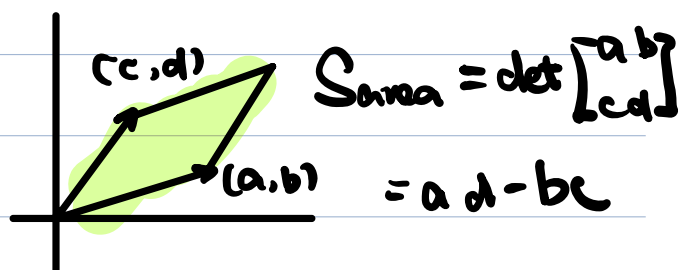
A with col. 1 replaced by $b = B_1 = \begin{bmatrix} | & n-1 \\ b & \text{col. of} \\ & A \end{bmatrix}$

A with col. j replaced by $b = B_j$

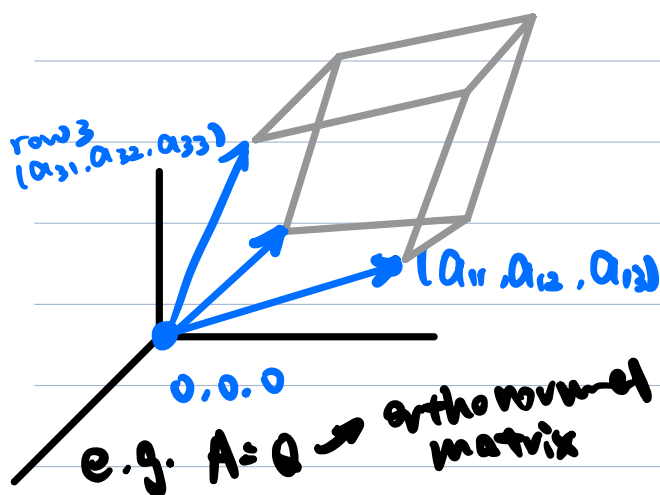
$$x_j = \frac{\det B_j}{\det A}$$

$$x_2 = \frac{\det B_2}{\det A}$$

$[x_n]$



3×3
 $|\det A| = \text{volume of box}$



$$Q^T Q = I \rightarrow \text{orthonormal}$$

$$\det(Q^T) \cdot \det Q = \det I$$

$\downarrow$ (1)

$$(\det Q)^2 = \det I$$

$$|\det Q| = 1$$

Eigenvalues - Eigenvectors

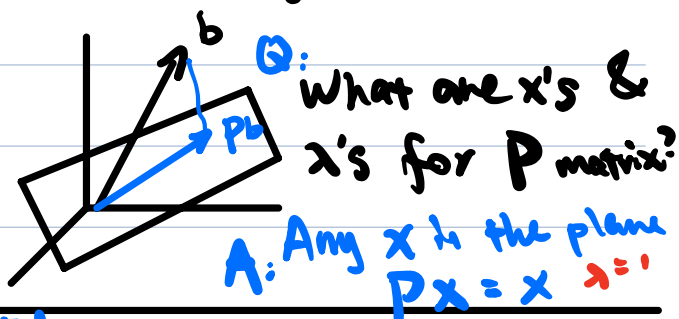
$$\det[A - \lambda I] = 0$$

$$\text{TRACE} = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

Eigenvectors: x
 Ax parallel to x

$$\Leftrightarrow Ax = \lambda x$$

if A is singular, $\lambda = 0$ is ok



Any $X \perp$ plane: $PX=0$
 $\lambda=0$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \lambda = 1$$

$$x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad Ax = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \lambda = -1$$

Fact:

Sum of λ 's = $a_{11} + a_{22} + \dots + a_{nn}$ trace = $0 + 0 = \lambda_1 + \lambda_2$

How to solve $Ax = \lambda x$

$$(A - \lambda I)x = 0$$

SINGULAR

$$\det(A - \lambda I) = 0$$

e.g. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)^2 - 1$$

$$= \lambda^2 - 6\lambda + 8 = 0$$

$$\lambda = 2 \text{ or } 4$$

$$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{A}x = \lambda x \quad \lambda \neq 0$$

then $(A + 3I)x = (\lambda + 3)x$

NOT SO GREAT

* $Ax = \lambda x$, B has eigenvalues α_i

wrong: $Bx = \alpha x$
 $(A+B)x = (\lambda + \alpha)x$

Example $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
 90° rotation

$$\det = 1 = \lambda_1 \cdot \lambda_2$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1$$

$$\lambda_1, \lambda_2 = \pm i$$

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{U-trig}$$

$$\det \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0$$

$\lambda = \lambda = 3$

$$(A - \lambda I)x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

No 2nd indep x
 $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $x_2 = \text{none}$
 (2 dependent x 's)

Degenerate Matrix

2.2.2

Diagonalising a matrix $S^{-1}AS = \Lambda$

Powers of A | equation $u_{k+1} = Au_k$

Suppose n indep. eigenvectors of A

Put them in columns of S

invertible

$$AS = A \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \dots & \lambda_n x_n \\ \vdots & & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & \dots & x_n \\ \vdots & & \vdots \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} = S\Lambda$$

$$AS = S\Lambda$$

diagonal
eigenvalue
matrix Λ

$$\Rightarrow S^{-1}AS = \Lambda$$

$$\Leftrightarrow A = S\Lambda S^{-1}$$

$$\text{if } Ax = \lambda x$$

$$A^2 x = A\lambda x = \lambda Ax = \lambda^2 x$$

$$A^k = S \Lambda^k S^{-1}$$

Theorem

$A^k \rightarrow 0$ as $k \rightarrow \infty$
if all $|\lambda_i| < 1$

A is sure to have n independent vectors
(and be diagonalizable)
if all the λ 's are different

(No repeated λ 's)

eig(rand(10, 10))

Repeated possibility // may
or may not have n indep. vectors

e.g.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\lambda_1 = \lambda_2 = 2$$

$$\det(A - \lambda I)$$

$$= (2 - \lambda)^2 = 0$$

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Equation $u_{k+1} = Au_k$

start with given vector u_0

$$u_{100} = \begin{bmatrix} F_{100} \\ F_{100} \end{bmatrix} = \Lambda^{100} S c$$

$$= \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \end{bmatrix}^{100}$$

$$u_k = A^k \cdot u_0 = S \Lambda^k S^{-1} u_0$$

To really solve: write

$$u_0 = C_1 x_1 + C_2 x_2 + \dots + C_n x_n$$

$$A^{100} u_0 = C_1 \lambda_1^{100} x_1 + \dots + C_n \lambda_n^{100} x_n$$

$$= S \Lambda^{100} C \quad \text{pure power}$$

Fibonacci example:

0, 1, 1, 2, 3, 5, 8, 13, ...

F₁₀₀ = ?

$$\begin{cases} F_{k+2} = F_{k+1} + F_k \\ F_{k+1} = F_k + F_{k-1} \end{cases}$$

Trick:

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \text{ unknown}$$

$$u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$$

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$A^{100} u_0 = \Lambda^{100} S C$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = 1.618 \quad x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} = -0.618 \quad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad L^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 - \lambda_2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^{100} & 0 \\ 0 & \lambda_2^{100} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1^{101} & \lambda_2^{101} \\ \lambda_1^{100} & \lambda_2^{100} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1^{101} - \lambda_2^{101} \\ \lambda_1^{100} - \lambda_2^{100} \end{bmatrix}$$

$$\therefore F_{100} = \frac{\lambda_1^{100} - \lambda_2^{100}}{\lambda_1 - \lambda_2}$$

2.3

Differential Eqns $\frac{du}{dt} = Au$
Exponential e^{At} of a matrix

Example:

$$\frac{du_1}{dt} = -u_1 + 2u_2$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$\lambda = 0, -3$$

$$\lambda_1 = 0 \quad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -3 \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

singular

Solution: pure exponential

$$u(t) = C_1 e^{\lambda_1 t} x_1 + C_2 e^{\lambda_2 t} x_2$$

check:

Plug in $e^{\lambda_1 t} x_1$

$$\frac{du}{dt} = Au \quad \lambda_1 e^{\lambda_1 t} x_1 = A e^{\lambda_1 t} x_1$$

$$= C_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Use $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

At $t=0$

$$C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution: $\begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$

$$\Rightarrow \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Steady state:

$$u(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

① Stability $u(t) \rightarrow 0$

need

Real Part of

$$e^{\lambda t} \rightarrow 0$$

$\lambda < 0$

e.g. $|e^{(\lambda_1 + i\omega)t}| = e^{-\lambda_1 t}$
 $|e^{i\omega t}| = 1$

② Steady State

$$\lambda_1 = 0 \text{ \& other } \operatorname{Re} \lambda < 0$$

③ Blow up if any $\operatorname{Re} \lambda > 0$

2x2 stability $\operatorname{Re} \lambda_1 < 0$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \operatorname{Re} \lambda_2 < 0$$

trace $a+d$

$$a+d = \lambda_1 + \lambda_2 < 0$$

$$\det > 0 \quad |\lambda_1 \lambda_2|$$

$\frac{du}{dt} = Au$ Set $u = S v$
 e^{λ} vectors

$$\frac{dv}{dt} = S^{-1} A S v = \Lambda v$$

$$\Rightarrow v_i = e^{\lambda_i t} v_i(0)$$

$$u(t) = S e^{\Lambda t} S^{-1} u(0) = e^{At} u(0)$$

Matrix exponential

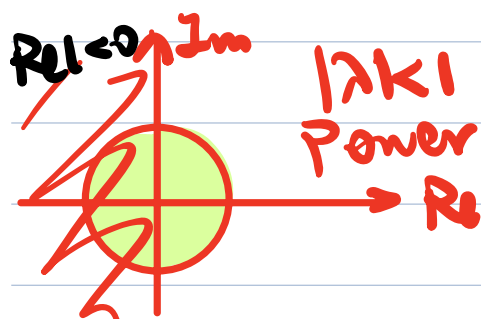
$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \dots + \frac{(At)^n}{n!}$$

$$|I - At| = I + At + (At)^2 + (At)^3 + \dots$$

$$|\lambda(At)| < 1$$

$$CA^k = S e^{\lambda k} S^{-1}$$

$$\lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda_n \end{bmatrix} \quad \lambda^k = \begin{bmatrix} e^{\lambda k} & 0 \\ 0 & e^{\lambda_n k} \end{bmatrix}$$



EXP

$$y'' + by' + ky = 0$$

$$u = \begin{bmatrix} y' \\ y \end{bmatrix} \quad u' = \begin{bmatrix} y'' \\ y' \end{bmatrix} = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$$

L24

Markov matrices

Steady State

Fourier Series of Projections

$$A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.11 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$$

① All entries ≥ 0

② All columns add to 1

1. $\lambda = 1$ is an eigenvalue

2. All other $|\lambda_i|$ are < 1

$$u_k = A^k u_0 = c_1 \lambda^k x_1 + \dots$$

$c_1 x_1$ steady state
 x_1 part of u_0

3. eigenvector $x_1 \geq 0$

→ prove:

$$A - I = \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix}$$

$A - I$ is singular
($N(A^T) \neq 0$)

eigenvalues of A & A^T

eigenvalues of A^T are the same

$$\begin{cases} \det(A - \lambda I) = 0 \\ \det A^T = \det A \end{cases} \Rightarrow \det(A^T - \lambda I) = 0$$

$$= \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.33 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$U_{k+1} = A U_k \quad A \text{ matrix}$$

E.g.

$$\begin{bmatrix} U_{\text{cal}} \\ U_{\text{mass}} \end{bmatrix}_{t=k+1} = \begin{bmatrix} .9 & .2 \\ .1 & .8 \end{bmatrix} \begin{bmatrix} U_{\text{cal}} \\ U_{\text{mass}} \end{bmatrix}_k$$

$$U_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} \rightarrow U_1 = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$$

$$U_{100} = S \lambda^{100} C$$

$$\det \begin{bmatrix} .9 - \lambda & .2 \\ .1 & .8 - \lambda \end{bmatrix} = 0$$

$$\Rightarrow \lambda = 0.7 / 1$$

$$\lambda_1 = 0.7 \quad X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 1 \quad X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$U_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix} = -\frac{2000}{3} X_1 + \frac{1000}{3} X_2$$

$$U_{100} = \frac{2000}{3} \cdot (0.7)^{100} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \approx \begin{bmatrix} \frac{2000}{3} \\ \frac{1000}{3} \end{bmatrix}$$

$$\frac{1099}{3} \cdot (1)^{100} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

projections with orthonormal basis $q_1, \dots, q_n$

Any

$$V = x_1 q_1 + x_2 q_2 + \dots + x_n q_n$$

$$\rightarrow [q_1 \dots q_n] [x] = V$$

$$q_i^T V = x_1 q_i^T q_1 + 0 \dots + 0$$

$$x = Q^T V = Q^T V$$

$$x_1 = q_1^T V$$

orthonormal

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x \dots$$

FOURIER SERIES:

functions: $f(x) = f(x + 2\pi)$ PERIODIC FUN.

vectors:

$$V^T W = v_1 w_1 + \dots + v_n w_n$$

functions:

$$f^T g = \int_0^{2\pi} f(x) g(x) dx$$

$$\int_0^{2\pi} \sin x \cos x dx = \frac{1}{2} (\sin^2 x) \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} f(x) \cos x dx$$

$$= \int_0^{2\pi} (\cos x)^2 dx = \pi$$

$$\Rightarrow a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x dx$$

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Review for QUIZ 2

$$Q = [q_1 \dots q_m]$$

Projections - Least Squares

Gram-Schmidt

(2) $\det A$ Properties 1-3

Big formula ($n!$ terms, $\pm$)

Cofactors / A^T

(3) Eigenvalues

$$Ax = \lambda x$$

$$\det(A - \lambda I) = 0$$

$$\text{Diagonalize } S^{-1}AS = \Lambda$$

Powers A^k

$$Q^T Q = I$$

$$a = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad P?$$

$$P = A(A^T A)^{-1} A^T$$

$$= \frac{a a^T}{a^T a} = \frac{1}{9} \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

$$\lambda = 0, 0, 1$$

evector
 $a = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \lambda = 1$

$$u_{k+1} = P u_k$$

$$u_0 = \begin{bmatrix} 9 \\ 9 \\ 0 \end{bmatrix}$$

$$u_1 = P u_0 = a \frac{a^T u_0}{a^T a} = \frac{27}{9} a$$

$$u_k = u_1 = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} = 3a$$

$$t=1 \quad y=4$$

$$=2 \quad =5$$

$$=3 \quad =8$$

$$y = Dt$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} D = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix}$$

$A \times b$

$$\hat{D} = (A^T A)^{-1} A^T b$$

$$= \frac{1}{14} 38 = \frac{19}{7}$$

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

plane sol. sp of A

$$B = \begin{bmatrix} 1-\lambda & & \\ & 1-2\lambda & \\ & & 1-3\lambda \end{bmatrix}$$

$$6-14\lambda = 0 \quad \lambda = \frac{6}{14} = \frac{3}{7} \quad \begin{bmatrix} \frac{4}{7} \\ \frac{1}{7} \\ -\frac{2}{7} \end{bmatrix}$$

③ 4×4 $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ $|\lambda| = 1$

(a) invertible? $\Leftrightarrow$ no zero $L \neq 0$
 evaluates

(b) $\det A^T = \frac{1}{\lambda_1 \cdots \lambda_4}$

(c) Trace of $A+I$

$$A+I = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 4$$

$$A_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$D_n = \det A$$

use cofactor: $D_n = -D_{n-1} - D_{n-2}$

$$D_n = 1 \cdot D_{n-1} - 1 \cdot D_{n-2}$$

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda + 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

Symmetric Matrices

Eigenvalues / Eigenvectors

Start: Positive Definite Matrices

$$A = A^T$$

① The eigenvalues are REAL

② the eigenvectors $\perp$
 can be chosen

Usual case $A = S \Lambda S^{-1}$

Symmetric case: $A = Q \Lambda Q^{-1}$

Why real eigenvalues?

$$Ax = \lambda x \xrightarrow{\text{always}} \bar{A} \bar{x} = \bar{\lambda} \bar{x}$$

$$\bar{x}^T A x = \bar{x}^T \lambda x$$

$$\bar{x}^T A^T = \bar{x}^T \bar{\lambda} \Rightarrow A \bar{x} = \bar{\lambda} \bar{x}$$

$$\bar{x}^T A^T x = \bar{x}^T \bar{\lambda} x$$

$$A^T = A \Rightarrow \bar{\lambda} = \lambda$$

if $\bar{x}^T x \neq 0 \Rightarrow \lambda$ is real

$$[\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \bar{x}_1 x_1 + \dots$$

$$\lambda X^T X > 0$$

$\therefore \lambda$ is real

$$\begin{aligned} &\downarrow \\ &(a-ib) \\ &(a+ib) \\ &= a^2 + b^2 \end{aligned}$$

Good Matrices

Real λ 's
Perpendicular x 's

$$A = \bar{A}^T$$

$$A = Q \Lambda Q^{-1}$$

$$= \begin{bmatrix} | & | & \dots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \bar{q}_1^T \\ \bar{q}_2^T \\ \vdots \\ \bar{q}_n^T \end{bmatrix}$$

$$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \dots$$

Projection Matrix

Every Symm matrix is
a comb of perp projection
matrices

Signs of pivots same as
signs of λ 's

pos pivots = # pos. λ 's

Positive definite matrix
symmetric

all eigenvalues are positive
all pivots are positive

$$\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{pivots } 5, \frac{11}{5}$$

$$\lambda^2 - 8\lambda + 11 = 0$$

$$\lambda = 4 \pm \sqrt{5}$$

all sub-
determinants
are positive

L2

complex vectors matrices

inner products

Fourier matrix F_n

discrete Fourier

FAST Transform = FFT

n^2 mults

$\downarrow$
 $n \log_2 n$ mults

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

length
in C_n

length? $z^T z$ No Good!

$\bar{z}^T z$ is good
 $\rightarrow$ length²

$z^H z$ Hermitian

inner product?

$$\bar{y}^T x = y^H x$$

symmetric: $A^T = A$

no good if A is complex

Hermitian

$$A^H = A$$

$$\bar{A}^T = A = \begin{bmatrix} 2 & 2+i \\ 3-i & 5 \end{bmatrix} \quad (W_{64})^2 = W_{32}$$

Perpendicular

$q_1, q_2, \dots, q_n$

$$Q = \begin{bmatrix} | & & | \\ q_1 & \dots & q_n \\ | & & | \end{bmatrix}$$

$$\bar{q}_i q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad Q^T Q = I$$

orthogonal $\Downarrow$
unitary $Q^H Q = I$

$$F_n = \begin{bmatrix} 1 & w & w^2 & \dots & w^{n-1} \\ 1 & w^2 & w^4 & \dots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \dots & w^{(n-1)^2} \end{bmatrix}$$

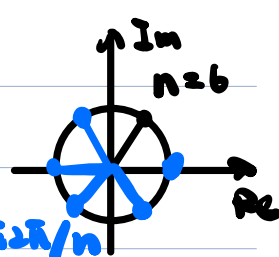
$$(F_n)_{ij} = w^{ij}$$

$$i, j = 0 \dots n-1$$

$$w^n = 1$$

$$w = e^{i2\pi/n}$$

$$= \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$$



$$n=4 \quad w^4 = 1$$

$$w = e^{2\pi i/4} = i$$

$$i, i^2 = -1, i^3 = -i, i^4 = 1$$

$$F_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \quad \text{cols. orthonormal} \quad F_4^H F_4 = I$$

$$[F_{64}] =$$

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$2(32^2) + 32$$

$$D = \begin{bmatrix} 1 & & & \\ & w & & \\ & & w^2 & \\ & & & \ddots \\ & & & & w^{31} \end{bmatrix}$$

$$\begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \rightarrow \begin{bmatrix} F_{16} & F_{16} \\ F_{16} & F_{16} \end{bmatrix} \rightarrow \begin{bmatrix} P & P \\ P & P \end{bmatrix}$$

L28

Positive Definite Matrix (Tests)

Tests for Minimum ($x^T A x > 0$)

Ellipsoids in $\mathbb{R}^n$

① $\lambda_1 > 0 \quad \lambda_2 > 0 \quad A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

② $a > 0 \quad ac - b^2 > 0$

③ pivots $a > 0 \quad \frac{ac-b^2}{a} > 0$

④ $x^T A x > 0$

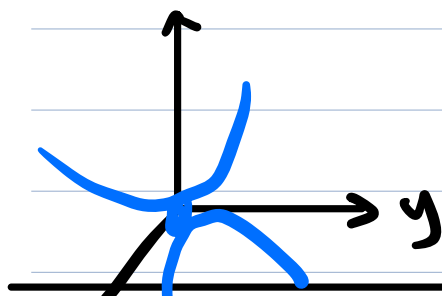
e.g. $\begin{bmatrix} 2 & 6 \\ 6 & 19 \end{bmatrix}$

$\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$

pos semidefinite. $\lambda = 0, 20$
pivots 2

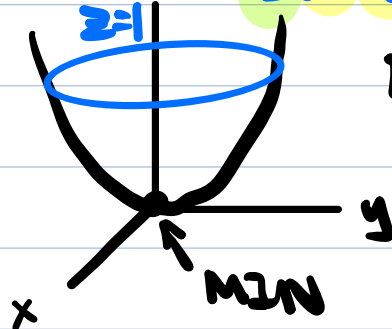
$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 6x_2 \\ 6x_1 + 18x_2 \end{bmatrix}$

$= 2x_1^2 + 12x_1x_2 + 18x_2^2$
quadratic form
 $\uparrow \quad \uparrow \quad \uparrow$
 $a x^2 \quad 2bxy \quad c y^2 \quad ? > 0$



$\frac{1}{2} n \log n$

$f(x,y) = 2x^2 + 12xy + 20y^2$
 $= 2(x+3y)^2 + 2y^2$



1st derivative

calculus:
 $\min \sim \frac{d^2y}{dx^2}$
1st deriv = 0

18.06 MIN: matrix of 2nd Deriv is pos def
 $f(x_1, x_2, \dots, x_n)$

$1 = 2x^2 + 12xy + 20y^2$

椭圆 (Ellipse)

$A = LU$

$\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

$\frac{d^2 f_x}{dx^2} = f_{xx}$

x^T

3x3 example

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

dets 2, 3, 4

pivots 2, $\frac{3}{2}$, $\frac{9}{2}$

eigenvalues 2, $2-i$, $2+i$

$$x^T A x = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 > 0$$

$$= (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_1^2 + x_3^2$$

$$A = Q \Lambda Q^T$$

L29

$A^T A$ is positive definite!

Similar matrices A, B

$$B = M^{-1} A M \text{ / JORDAN FORM}$$

positive definite means

$$x^T A x > 0 \text{ (except for } x=0)$$

If A, B are pos def

$$x^T (A+B) x > 0 \Leftrightarrow \begin{matrix} x^T A x > 0 \\ x^T B x > 0 \end{matrix}$$

so is $A+B$:

A man

$A^T A$: square, symmetric
pos def.

prove:

$$x^T A^T A x = (Ax)^T (Ax)$$

$$\text{length}^2 \geq 0$$

when not 0?

rank n A

$n \times n$ $n \times n$

A & B are similar

means: for some M

$$B = M^{-1} A M$$

e.g. $S^{-1} A S = \Lambda$ A is similar to Λ

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \quad \lambda = 3, 1$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -15 \\ 1 & 6 \end{bmatrix}$$

$M^{-1} \quad A \quad M \quad B$

Similar matrices have

same eigenvalues!

prove:

$$A x = \lambda x \quad (B = M^{-1} A M)$$

$$M^{-1} A M M^{-1} x = \lambda M^{-1} x$$

$$BM^{-1}x = \lambda M^{-1}x$$

Eigenvector of B is M^{-1}
(eigenvector of A)

Bad case:

e.g. $\lambda_1 = \lambda_2 = 4$

one family has $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ (one matrix)

big family include $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ ← Jordan Form

more members of family:

$$\begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \text{ trace} = 8 \text{ det} = 16$$

same # of eigenvectors

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \lambda = 0, 0, 0, 0$$

2e vector
 $\dim N(A) = 2$

not similar to

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Jordan block
1 vector

$$J_i = \begin{bmatrix} \lambda_i & & 0 \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}$$

Every square A is similar to a

Jordan matrix J # blocks =

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix} \text{ # eigenvectors}$$

Good case: J is A

L30

Singular Value Decomposition
= SVD

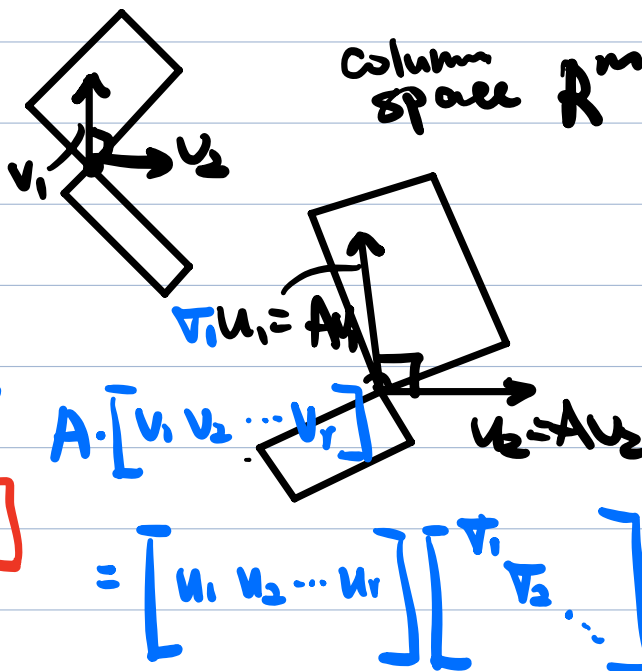
$$A = U \Sigma V^T \quad // \quad \Sigma \text{ diagonal} \\ U, V \text{ orthogonal}$$

symm pos def

$$A = Q \Lambda Q^T$$

~~$$A = S \Lambda S^{-1}$$~~

$\mathbb{R}^n$ row space



$$AV = U\Sigma \quad \text{e.g. } A = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \quad \text{find } u_1, u_2$$

v_1, v_2 orthonormal in row space $\mathbb{R}^2$

u_1, u_2 in col. space $\mathbb{R}^2$

$$\sigma_1 > 0 \quad \sigma_2 > 0$$

$$Av_1 = \sigma_1 u_1 \Rightarrow A = U\Sigma V^T$$

$$Av_2 = \sigma_2 u_2 = U\Sigma V^T$$

$$A^T A = V \Sigma^T U^T U \Sigma V^T = V \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} V^T$$

$$A^T A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

$$\begin{bmatrix} 1/32 \\ 1/32 \end{bmatrix} = 32 \begin{bmatrix} 1/128 \\ 1/128 \end{bmatrix}$$

$$\begin{bmatrix} 1/18 \\ -1/18 \end{bmatrix} = 18 \begin{bmatrix} 1/324 \\ -1/324 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/32 & 0 \\ 0 & 1/18 \end{bmatrix} \begin{bmatrix} 1/16 & 1/18 \\ 1/18 & -1/16 \end{bmatrix}$$

$$AA^T = U \Sigma \cancel{V^T V} \Sigma^T U^T = U \Sigma \Sigma^T U^T$$

$$AA^T = \begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 32 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 18 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

e.g. 2

$$v_1 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

row space

$$\begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$n(A)$

column space

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$n(A^T)$

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

$U \quad \Sigma \quad V^T$

$$A^T A = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

$v_1 \dots v_r$	orthonormal basis	row space
$u_1 \dots u_r$	"	column space
$v_{r+1} \dots v_n$	"	nullspace
$u_{r+1} \dots u_m$	"	$n(A^T)$

NON-EXAMPLE

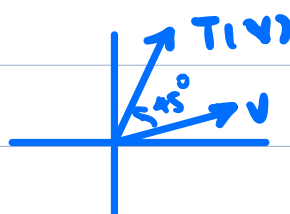
$$T(v) = \|v\|$$

$$\mathbb{R}^3 \rightarrow \mathbb{R}^1$$

$$Av_i = T_i u_i$$

e.g. 2 Rotation by 45°

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$



e.g. 3 Matrix A

$$T(w) = Aw$$

$$A(v+w) = Av + Aw$$

$$A(cv) = cAv$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



$$\text{start: } T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

e.g. $T(v) = A \begin{matrix} \swarrow \\ v \end{matrix}$ $\leftarrow$ 2×3 matrix
 $\swarrow$ $\nwarrow$
 output in $\mathbb{R}^2$ input in $\mathbb{R}^3$

Information needed to know $T(w)$
 for all inputs

23)

Linear transformations T

without coordinates : no matrix

with coordinates $\rightarrow$ MATRIX

e.g. 1 Projection

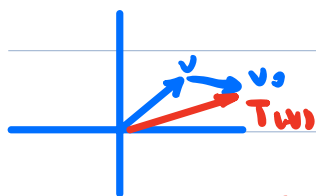
$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$T(v+w) = T(v) + T(w)$$

$$T(cv) = cT(v)$$

$$\Rightarrow T(cv+dw) = cT(v) + dT(w)$$

e.g. 2 Shift whole plane by v



$$T(2v) \neq 2T(v)$$

NOT linear

$T(0) = 0$ | linear case
 which is not the case
 here

$T(v_1), T(v_2), \dots, T(v_n)$ for any basis $v_1, \dots, v_n$

Choose $v = c_1 v_1 + \dots + c_n v_n$

So $T(v) = c_1 T(v_1) + \dots + c_n T(v_n)$

construct matrix A that represents

$T: \underset{\text{basis}}{R^n} \rightarrow R^m$ $1 \times n \text{ tr } T$

Choose $v_1, \dots, v_n$ for inputs R^n

$w_1, \dots, w_m$ for outputs R^m

eigenvector basis leads to diagonal matrix Λ

Proj onto 45° line

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = w_1, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = w_2$

$P = \frac{aa^T}{a^T a} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Rule to find A Given bases

1st col of A : $\begin{matrix} v_1, \dots, v_n \\ w_1, \dots, w_m \end{matrix}$

Write $T(v_1) = a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m$

2nd col of A :

$T(v_2) = a_{12} w_1 + \dots + a_{m2} w_m$

$T = \frac{d}{dx}$ Input: $c_1 + c_2 x + c_3 x^2$ basis: $1, x, x^2$
 Output: $c_2 + 2c_3 x$ basis: $1, x$
 (linear! deriv)

$A \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \end{bmatrix}$

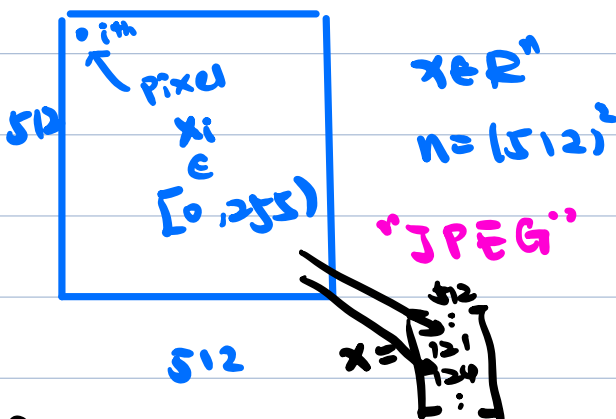
$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

L31

Change of Basis

Compression of Images

Transformation $\leftrightarrow$ Matrix



Standard Basis:

$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \dots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

Better basis

$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Fourier basis 8×8



Signal x 512

lossless $\downarrow$ change basis

coeffs c

lossy $\downarrow$ compression

$\hat{c}$ (many zeros)

threshold: throw away coeff. with too small $\hat{c}$'s

$$\hat{x} = \sum \hat{c}_i v_i$$

videos: sequence of photos: correlated with respect to $w_1, \dots, w_8$

Wavelets

$$\begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} \dots$$

standard basis

$$\begin{bmatrix} p_1 \\ \vdots \\ p_8 \end{bmatrix} = c_1 w_1 + \dots + c_8 w_8$$

$$P = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_8 \end{bmatrix} = WC$$

$$C = W^T P$$

Good BASIS:

① FAST FFT, FWT

② Good compression

! Few is enough!

Change of basis:

column

of W = new basis vectors

$$[x]_{\text{old basis}} \rightarrow [c]_{\text{new basis}}$$

T

$$x = Wc$$

with respect to $v_1, \dots, v_8$

it has matrix A

It has matrix B

relation: Similar

$$B = M^{-1} A M$$

What is A ? Using basis $v_1, \dots, v_8$

Know T completely from $T(v_1)$

$T(v_2), \dots, T(v_8)$

$$\text{Cuz every } x = c_1 v_1 + c_2 v_2 + \dots + c_8 v_8$$

$$\text{Then } T(x) = c_1 T(v_1) + \dots + c_8 T(v_8)$$

Write $T(v_1) = a_{11}v_1 + \dots + a_{81}v_8$
 $T(v_2) = a_{12}v_1 + \dots + a_{82}v_8$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \dots \\ \vdots & \vdots & \ddots \\ a_{81} & a_{82} & \dots \end{bmatrix}$$

Eigenvector basis

$$T(v_i) = \lambda_i v_i$$

What is A?

$$A = \begin{bmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \\ 0 & 0 & \dots \end{bmatrix}$$

L33

Quiz 3 Review

Linear transformations

On final, not on quiz

6.1-2 λ & x

6.3 $du/dt = Au$ & e^{At}

6.4 $A = A^T$ 6.5 positive definite

6.6 Similar $P = M^{-1}AM$ 6.7 $A = U\Sigma V^T$
 $= Q\Lambda Q^T$ SVD

$$\frac{dy}{dt} = Au = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} u$$

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 + c_3 e^{\lambda_3 t} x_3$$

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda^3 - 2\lambda = 0$$

$$\lambda(\lambda^2 + 2) = 0$$

$$\Rightarrow u(t) = c_1 x_1 + c_2 e^{\sqrt{2}it} + c_3 e^{-\sqrt{2}it}$$

Periodic

$$E_i \cdot T = 2\pi i \quad e^0 = 1$$

$$T = R \cdot \sqrt{2}$$

Orthogonal vectors

$$AA^T = A^T A$$

sym / anti-sym / orth.

$$u(t) = e^{At} u(0)$$

$$e^{At} = S e^{\Lambda t} S^{-1} \rightarrow \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

(if $A = S\Lambda S^{-1}$)

$$\lambda_1 = 0 \quad \lambda_2 = i \quad \lambda_3 = -i$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

(a) diagonalizable? all c

(b) symmetric?

all real c

(c) positive definite? No

Semi - $c > 0$

(d) Markov? No

(e) one half of A = proj. matrix: $\text{medi. ad. } 1/0$ or 1 sol. to

$$P^2 = P \text{ so } \lambda^2 = \lambda \text{ so } \lambda = 0/1$$

Need $c = 0/2$

SVD:
every

$$A = U \Sigma V^T \quad \begin{matrix} \text{(orthog.)} \\ \text{(orthog.) (diag.)} \end{matrix}$$

$$A^T A = (V \Sigma^T U^T)(U \Sigma V^T) = V(\Sigma^T \Sigma) V^T$$

$$v_1 \dots v_r \\ v = \text{e vector} \\ \text{for } A^T A \quad v_i^2 = \lambda_i(A^T A)$$

$$A A^T = V^T (\Sigma \Sigma^T) V$$

L34

4 subspaces

Left-Inverses Pseudo-

Right-Inverses inverses

2-sided inverse

$m=n$ full rank

$$A A^T = I = A^T A$$

left inverse

full col. rank $r = n$

nullspace = $\{0\}$

medi. ad. $1/0$ or 1 sol. to $Ax = b$

$$\underbrace{(A^T A)^T A^T}_{A_{\text{left}}^T} \cdot A = I \quad \begin{matrix} m \times n & n \times n \end{matrix}$$

right-inverse

full row rank $r = m < n$

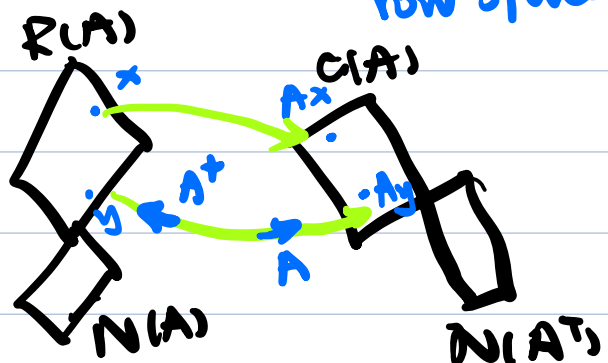
$n(A^T) = \{0\}$ ind. rows

∞ sol. to $Ax = b$

$$A \underbrace{A^T (A A^T)^T}_{A_{\text{right}}^T} = I$$

$A (A^T A)^T A^T$ Proj. Mat.

$A^T (A A^T)^T A$ Proj. onto row space



Find the pseudo inverse
 A^+

① Start from SVD:

$$A = U \Sigma V^T$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Sigma^+ = \begin{bmatrix} 1/\sigma_1 & 0 & 0 \\ 0 & 1/\sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

rank r
m rows
n columns

$$\Sigma \Sigma^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

m x m

$$\Sigma^+ \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

n x n

Ex 35 Given $Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ no soln. $\rightarrow r < m$

$Ax = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ 1 soln. $\rightarrow r = n$

$$m=3 > n = r$$

$$N(A) = \{0\}$$

e.g. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ / $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$...

T/F:

1. $\det A^T A = \det A A^T$ (X)

2. $A^T A$ is invertible (V)

3. $A A^T$ is pos. def. (X)

$$3 \times n \quad n \times 3$$

$$A A^T = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$A^T y = c \quad \text{at least 1 sol'n}$$

$$\downarrow$$

n x m

In fact every c

∞ solns for $\underline{c}$

because $n < r$

$$\dim N(A^T) = m - r > 0$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

a) solve $Ax = v_1 - v_2 + v_3$

$$x = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

b) suppose $v_1 - v_2 + v_3 = 0$

then this x is in $N(A)$

so sol. never unique

c) if v_1, v_2, v_3 are orthogonal

then $\frac{0}{\|v_1\|} v_1 + \frac{0}{\|v_2\|} v_2$ is closest to v_3

$$A = \begin{bmatrix} .2 & .4 & .3 \\ .4 & .2 & .3 \\ .4 & .4 & .4 \end{bmatrix}$$

$$\text{col. 1} + \text{col. 2} = 2 \neq \text{col. 3}$$

$$\lambda = 0, 1, -0.2$$

$$A^T A = \begin{bmatrix} .2 & .4 & .3 \\ .4 & .2 & .3 \\ .4 & .4 & .4 \end{bmatrix}$$

$$u(1) = H \cdot u(0) \rightarrow L: \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \leftarrow b$$

$$= c_1 \lambda_1^k \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \lambda_2^k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \lambda_3^k \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

0 1 -0.2

$$\Rightarrow u(100) = c_2 x_2$$

$$c_2 = 1 \quad x_2 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

2x2 A

① Proj onto $a = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

$$P = \frac{aa^T}{a^T a} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

evaluate & vectors

② $0, \begin{bmatrix} 1 \\ 2 \end{bmatrix}; 3, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ given

$$A = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_S \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}}_\Lambda \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{S^{-1}}$$

③ $A \neq B^T B$ any B

2x2

→ orthog vectors: sym

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \text{ orthog.}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \end{bmatrix} = \begin{bmatrix} 11/3 \\ -1 \end{bmatrix}$$

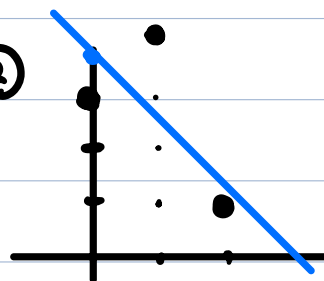
① Proj of b onto $C(A)$

is

?

$11/3$ col. 1
-1 col. 2

②



$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \leftarrow b$$

$$\Downarrow$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2000 MIT 18.06

DONE:

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