

Introduction To Linear Algebra

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L1

$\begin{cases} n \text{ linear equations} \\ n \text{ unknown} \end{cases}$

Row picture

* Column Picture

Matrix form

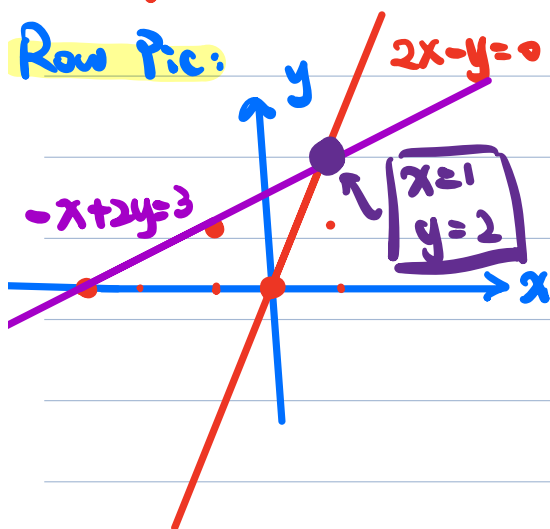
e.g.

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 3 \end{cases}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$A \quad X = "b"$

Row Pic:



Column Pic:

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Linear combination of

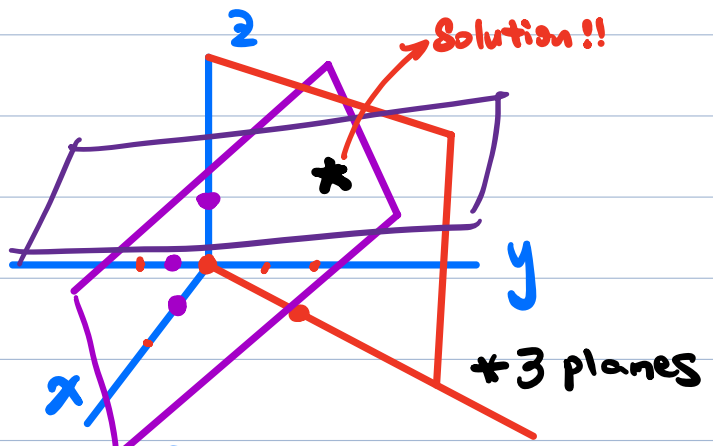
e.g. 2

$$\begin{cases} 2x - y = 0 \\ -x + 2y - z = -1 \\ -3y + 4z = 4 \end{cases}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}$$

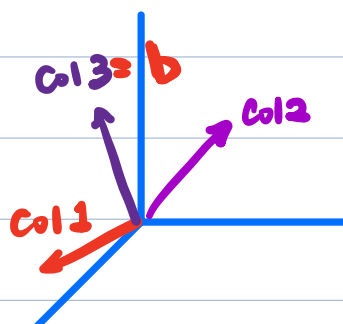
$$b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

Row P.C.:



Column P.C.:

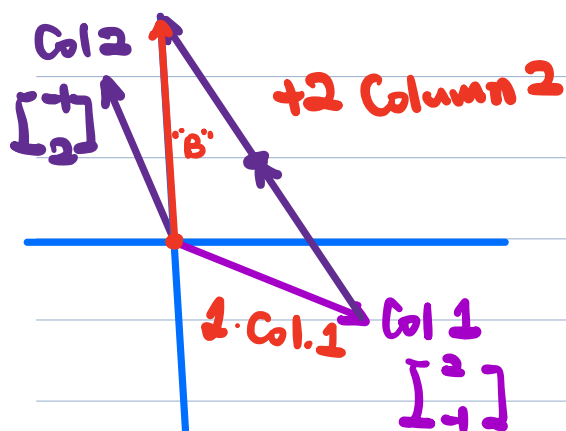
$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$



$$\begin{cases} x = y = 0 \\ z = 1 \end{cases}$$

* 3 Col.

Linear combinations of Columns



* Can I solve $AX=b$ for every b ?

$\Leftrightarrow$

Do the linear combs of the columns fill 3-D space?

For this A , answer is YES!

3个向量共面 $\Rightarrow$ No!

(因为 $A \cdot b$ 可表示)

$$Ax=b$$

Ax is a comb. of columns of A

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

Step 2: Subtract $2 \times$ row 2 from

row 3

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{(Elimination)} \quad E(3,2)$$

*

注意: 无交换律
 $A \cdot B \neq B \cdot A$

$$\underline{\underline{E_{32}(E_{21}A) = U}}$$

$$\rightarrow \frac{?}{\Downarrow} A = U \quad \text{One-in-for-All soln.}$$

$$(E_{32} \cdot E_{21}) A = U$$

"Associative Law"

Permutation Matrix
Exchange Row 1 & 2

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} b & a \end{bmatrix}$$

L2

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[\text{knock out (2,1)}]{\text{1st Pivot}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[\text{knock out (3,2)}]{\text{2nd Pivot}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

A b

$1 \cdot 2 \cdot 5 = 10$
* determinant

Pivot cannot be "0"

U (Upper trig)

Success

Elimination

Failure

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 1 & 2 & 1 \\ & 2 & \\ & & 0 \end{bmatrix}$$

Failure!
Can't Avoid

Back-Substitution

$$\begin{aligned} x + 2y + z &= 2 & x=2 \\ 2y - 2z &= 6 & y=1 \\ 5z &= -10 & z=-2 \end{aligned}$$

Matrix Multiplication

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \times \text{Col 1} \\ 4 \times \text{Col 2} \\ 5 \times \text{Col 3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 1 \cdot \text{row 1} \\ 2 \cdot \text{row 2} \\ 7 \cdot \text{row 3} \end{bmatrix}$$

Matrices

Step 1:
Subtract 3 x row 1
from row 2

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E "(2,1)"

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

P 行操作 矩阵在左

Exchange Col. 1 & 2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

on the Right!! 列操作在右

Inverses (逆矩阵)

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E⁻¹

I
(identity)

L3

Matrix multiplication (4 ways!)

Inverse of A AB A^T

Gauss-Jordan / find A⁻¹

M1:

$$\begin{bmatrix} \text{---} \end{bmatrix} \begin{bmatrix} \text{---} \end{bmatrix} = \begin{bmatrix} C_{ij} \end{bmatrix}$$

A m x n B n x p C = AB m x p

$$C_{ij} = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$$

$$= a_{i1} b_{j1} + a_{i2} b_{j2} + \dots$$

$$= \sum_{k=1}^n a_{ik} b_{kj}$$

M2:

$$\begin{bmatrix} | & | & | & | \end{bmatrix} \begin{bmatrix} | & | & | & \dots \end{bmatrix} = \begin{bmatrix} | & | & | & \dots \end{bmatrix}$$

$A \quad B \quad C = AB$
 $m \times n \quad n \times p \quad m \times p$

Columns of C are comb. of columns of A

M3:

$$\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ \vdots \end{bmatrix} \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} = \begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \\ \vdots \end{bmatrix}$$

$A \quad B \quad C = AB$
 $m \times n \quad n \times p \quad m \times p$

Rows of C are comb. of Rows of B

M4:

Column of A $\times$ Row of B

$m \times 1 \quad 1 \times p$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 12 \\ 3 & 18 \\ 4 & 24 \end{bmatrix} \quad m \times p$$

$AB = \text{sum of (Col of A)} \times \text{(Row of B)}$

e.g.

$$\begin{bmatrix} 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 6 \end{bmatrix} + \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Block

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} | & | \\ | & | \end{bmatrix}$$

$A \quad B$

$A_1 B_1 + A_2 B_3$

Inverses (square matrices)

$$A^{-1} \cdot A = I = A \cdot A^{-1}$$

if this exists same hard to prove

invertible
not singular

No inverse

Explanation 1:

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & 18 \\ 10 & 18 \end{bmatrix}$$

线性组合
不可能

Ex 2:

I can find a vector x with $Ax = 0$ (non 0)

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \times$$

e.g.

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A \quad A^{-1} \quad I$

Gauss-Jordan (solve 2 equations at once)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad 24$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = A^{-1}$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array} \right]$$

A I

↓ Elimination

- Subtract

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array} \right] \quad \text{row 1}$$

Subtract

row 2

$$\left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array} \right] \quad 2$$

I

A^{-1}

Inverse of AB . A^{-1}

$$(AB)(B^{-1}A^{-1}) = I$$

$$B^{-1}A^{-1}AB =$$

$$AA^T = I \quad * T: \text{transpose}$$

$$(A^{-1})^T A^T = I$$

↑ this is the inverse of A^T

Product of Elimination matrices

$$A = LU \quad (\text{no row exchanges})$$

$$E_{21} A = U \quad \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad (\text{Upper Trig})$$

$$A = L U \quad \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad (\text{Lower Trig})$$

$$= \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

$$E_{32} E_{31} E_{21} A = U \quad (\text{no row exchanges})$$

$$A = \frac{E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}}{\text{"L" 逆之積}} U$$

E_{32} E_{21}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 10 & -5 & 1 \end{bmatrix} = E \quad (EA=U)$$

(left of A)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

L

$$A = LU$$

$$(A = LU)$$

If no row exchanges

multipliers go directly into L

How many operations on
About $n \times n$ matrix A ?

$$\text{Count } n^2 + (n-1)^2 + \dots + 1^2$$

$$\approx \frac{1}{3}n^3 \text{ on } A$$

Cost of $b \rightarrow n^2$

Permutations 3×3

$$P^{-1} = P^T$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{e.g.}} P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6 in total (including Identity Matrix)
 $(A^3) = n!$

 $L5$

Permutation P : execute row exchanges

$$A = LU = \begin{bmatrix} \vdots & 0 \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} 1 & \vdots & \vdots \\ 0 & \ddots & \vdots \\ 0 & \vdots & 1 \end{bmatrix}$$

With Row Exchanges:

$$PA = LU$$

P = identity matrix with reordered rows

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

$\uparrow A^T$ $\uparrow R$

Transpose $(A^T)_{ij} = A_{ji}$

Symmetric Matrices

$$A^T = A \quad \text{e.g. } \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$$

$R^T \cdot R$ is always symmetric

$$\begin{bmatrix} 10 & 11 & 7 \\ 11 & - & - \\ 7 & - & - \end{bmatrix}$$

WHY? TAKE TRANSPOSE

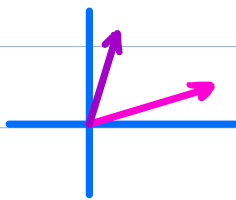
$$(R^T R)^T =$$

$$R^T R^{TT} = R^T R$$

Chp. 3 Vector spaces

Examples:

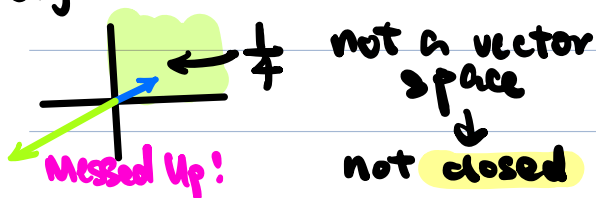
$\mathbb{R}^2 =$ all 2-D. real vectors
 = "x-y plane"
 $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dots$



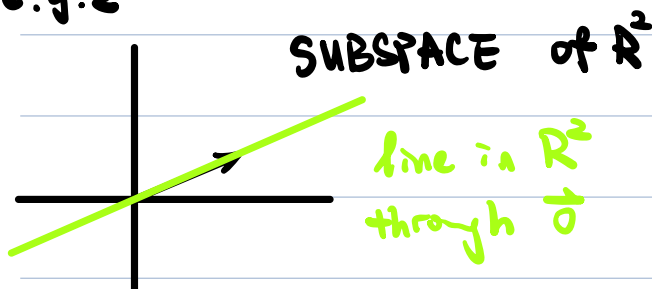
$\mathbb{R}^3 =$ all vectors with 3 components
 $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\mathbb{R}^n =$ all vectors with n components
 (red) real (column)

e.g. 1



e.g. 2



SUBSPACES of $\mathbb{R}^2$:

- ① all of $\mathbb{R}^2$
- ② any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ L
- ③ $\vec{0}$ Z

$\mathbb{R}^3$: ① $\mathbb{R}^3$ ② plane thru 0 ③ line thru 0 ④ Z

$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 4 & 1 \end{bmatrix}$ Columns in $\mathbb{R}^3$
 all their linear comb. form a subspace called column space $C(A)$

L6

Vector spaces & Subspaces

column space of A : Solving
 Nullspace of A $Ax = b$

Vector space Requirements:

$v + w$ & cv are in the space
 all combs $cv + dw$ are in the space

2 subspaces: P and L

$P \cup L =$ all vectors in P or L or both

This (is) (is not) a subspace

$P \cap L =$ all vectors in both P & L



$P \cap L$ is a subspace

Column space of A is subspace of $\mathbb{R}^3$

$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ 4×3 = all linear comb. of columns $C(A)$

Does $Ax = b$ have a solution for every b ?

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \text{ No!}$$

Which b 's allow the equation to be solved?

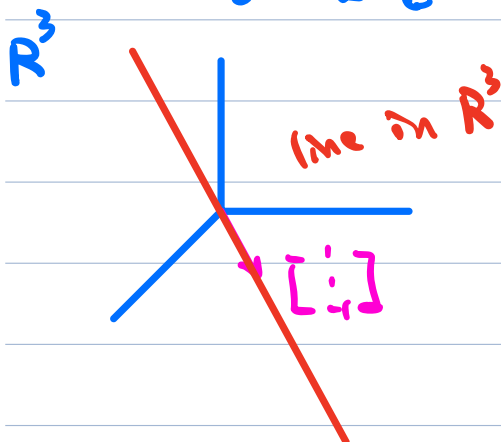
b is the column space of A

Not independent
Col. 1 + Col. 2 = Col. 3
 $\Rightarrow$ Column space is 2-D inside $\mathbb{R}^4$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

Nullspace of A = all solutions of $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $AX=0$

$N(A)$ contains $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} c \\ c \\ -c \end{bmatrix}$



Check That Solutions to $AX=0$ always give a subspace

If $Av=0$ and $Aw=0$

then $A(v+w)=0$

$$= Av + Aw = 0$$

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Computing the nullspace ($AX=0$)

Pivot variables - free variables

Special Solutions - $\text{rref}(A) = R$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{echelon}$$

2 pivot columns
free variables

rank of A

= number of pivots = 2

e.g. special solution

$$x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{\text{back substitution}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{\text{back substitution}} \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$x = c \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$n-r$ free variables

$m \times n$ r : rank

是 $n-r$ 个特解的线性组合
Nullspace

R = reduced row echelon form

$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ zeros above + below pivots

$\rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\text{rref}(A) = R = \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

notice $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ in pivot rows/cols

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ pivot cols. $\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} \leftarrow F$ free cols.

rref form

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ r pivot rows
 r pivot cols $m-r$ free cols.

Nullspace matrix

(columns = special sol.)

$RN=0$ $N = \begin{bmatrix} -F \\ I \end{bmatrix}$

Another Example:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
 $r=2$ again! pivot free col. free col.

$x = C \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = N$ $R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
 $-F \leftarrow$ $I \leftarrow$

LG

Complete Solution of $Ax=b$

Rank r

$x = x_p + x_n$

$r=m$: Solution exists

$r=n$: Solution is unique

$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

b_3 must equal $b_1 + b_2$
To have a solution

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array} \right]$$

$$\downarrow$$

$$\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array} \right]$$

$$0 = b_3 - b_2 - b_1$$

Solvability Condition on b
 $Ax=b$ solvable when b
 is $m \in (A) \iff$ If a comb.
 of rows of A gives zero row.
 the same comb. of the b s $\Rightarrow$ zero

To find complete sol'n to
 $Ax=b$

① $x_{\text{particular}}$: var. to 0
 then solve $Ax=b$ for pivot var.

$$\begin{bmatrix} 3b_1 - b_2 \\ 0 \\ \frac{b_2}{2} - b_1 \\ 0 \end{bmatrix}$$

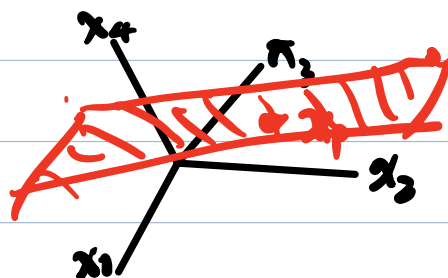
② $x_{\text{nullspace}}$

$$x = x_p + x_n$$

e.g. $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$

$$x_{\text{complete}} = \begin{bmatrix} -1 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

plot all solutions x
 in $\mathbb{R}^4$



$m \times n$ matrix A of r

(know $r \leq m, r \leq n$)

Full Column Rank,
 $r = n$: no free variables
 $N(A) = \{ \text{zero vector} \}$

Solution to $Ax=b$:
 $x = \{ x_p \}$ (unique solution if it exists)

Full row rank means $r = m$

Can solve $Ax=b$ for every b Exists

Left with $n-r$ free var.
 $n-m$

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 1 & 3 & 6/5 \\ 0 & 1 & 3 & 9/5 \end{bmatrix}$$

$r=m=n$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \text{ invertible BASIS \& dimension}$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

1 solution to
 $AX=b$

Conclusion:

$r=n < m$

$$R = \begin{bmatrix} I \\ 0 \end{bmatrix} \text{ (0 or 1) Solution}$$

$r=m < n$

$$R = \begin{bmatrix} I & F \end{bmatrix}$$

$r < m, r < n$

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \text{ (0 or } \infty \text{ solutions)}$$

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Linear independence

Spanning a space

Background:

$\star: m \times n \quad m < n$

(more unknown than equ.)

Then there are nonzero solutions to $AX=0$

Reason: There will be free variables!!

Independence

Vectors $x_1, x_2, \dots, x_n$ are independent if no comb. gives zero vector (except the zero comb.)

$$c_1 x_1 + c_2 x_2 + \dots + c_n x_n = 0$$

e.g.

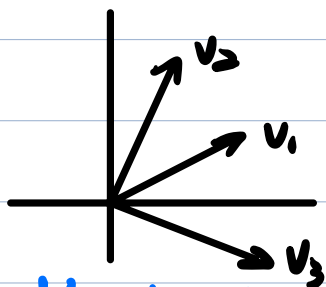
$$0v_1 + 6v_2 = 0 \quad c_1 = \dots = c_n = 0$$

$\Rightarrow v_1, v_2 \text{ NOT}$

1. ∞ solutions

2. independent

e.g.2



$$A = \begin{bmatrix} v_1 & v_2 & v_3 \\ 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ (according to background)}$$

Repeat: when $v_1, \dots, v_n$

are columns of A $\text{rank} = n$ $N(A) = \{0\}$

They are independent if no free var.
nullspace of A is $\{ \text{zero vector} \}$

They are dependent if

$$Ac = 0 \text{ for some nonzero } c \neq 0$$

$\Rightarrow \text{rank} < n$ Yes free var.

Vectors $v_1, \dots, v_d$ span a space means: The space consists of all combs of those vectors

Basis for a space is a sequence of vectors

$v_1, v_2, \dots, v_d$ with 2 properties

1. They are independent

2. They span the space

n vectors give basis if the $n \times n$ matrix with those cols.

is invertible

Every basis for the space has the same number of vectors

Def. $\uparrow$ Dimension of the space

$$\text{rank}(A) = \# \text{ pivot}$$

col. = dimension of $C(A)$

$$\dim C(A) = r$$

$$\dim N(A) = \# \text{ free var.} = n - r$$

Four Fundamental subspaces (for matrix A)

Column space $C(A)$

Null space $N(A)$

Row space = all combs of rows

$$\text{Nullspace of } A^T = N(A^T) \quad C(A^T)$$

left nullspace of A

A is $m \times n$

$C(A)$ in $\mathbb{R}^m$

$N(A)$ in $\mathbb{R}^n$

$C(A^T)$ in $\mathbb{R}^n$

$N(A^T)$ in $\mathbb{R}^m$

Q:
basis?
dimension?

$\mathbb{R}^n$	$\mathbb{R}^m$
$C(A^T)$	$C(A)$
$N(A)$	$N(A^T)$

$C(A)$ basis: pivot cols.
dimension: r

$C(A^T)$ basis: pivot cols.
dimension: r

$N(A)$ basis: special solutions
dimension: $n-r$

$N(A^T)$ basis: special solutions
dimension: $m-r$

$C(A) \neq C(A^T)$ different col spaces
Same row spaces

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow R \quad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for row space is first r rows of R

$N(A^T)$ 4th space

$$A^T \cdot y = 0$$

$$\begin{bmatrix} \end{bmatrix} \begin{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$y^T A = 0$$

$$\begin{bmatrix} y^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$E \begin{bmatrix} A_{m \times n} & I_{m \times m} \end{bmatrix}$$

↓

$$\begin{bmatrix} R_{m \times n} & E_{m \times m} \end{bmatrix}$$

$$EA = R$$

In chap 2 R was I

Then E was A^{-1}

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \leftarrow \dots \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

E A R

Notice! $\dim N(A) = 1$
 $N(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

new vector space M

All 3×3 matrices !!

Subspaces of M // upper trig // symmetric matrices

diagonal matrices

$\uparrow$
 $\dim D = 3$

L11

$\rightarrow M = \text{all } 3 \times 3 \text{ matrices}$

Bases of new vector spaces

Rank one matrices

Small world graphs

upper trig / symmetric

U 3×3

3×3

S

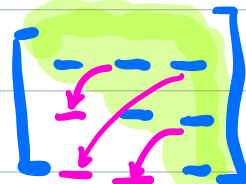
Basis for $M = \text{all } 3 \times 3$ s

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \dots \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dim M = 9$$

$$\dim S = 6$$

$$\dim U = 6$$



Intersection

$S \cap U = \text{diagonal } 3 \times 3$'s

$$\dim(S \cap U) = 3$$

Sum:

$S + U = \text{any ele. of } S +$

($S \cup U$ isn't a subspace) any ele. of U

$= \text{all } 3 \times 3$'s

$$\dim(S + U) = 9$$

$$\dim S + \dim U = \dim(S \cap U) + \dim(S + U)$$

$$\frac{d^2 y}{dx^2} + y = 0$$

$$y = \cos x \cdot \sin x$$

complete solution:

$$y = C_1 \cos x + C_2 \sin x$$

$$y = \cos x \cdot \sin x$$

BASIS

$$\dim(\text{sol. space}) = 2$$

Rank one matrices:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} \quad (r=1)$$

$$\dim C(A) = r = \dim C(A^T)$$

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$$

Rank 1 Matrix:

$$A = UV^T$$

$M =$ all 5×7 matrices

Subset of rank 4 matrices

not a subspace

$S =$ all $v \in \mathbb{R}^4$ with $v_1 + v_2 + v_3 + v_4 = 0$

In $\mathbb{R}^4$ $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ is a subspace

$$\dim(S) = 3$$

$$S = \text{nullspace of } A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}^T$$

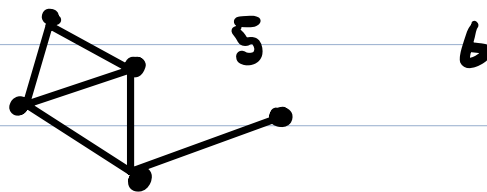
($Av = 0$)

Basis for S rank $k = 1$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad C(A) = R^1$$

$\dim N(A) = n - r$
 $N(A^T) = \{0\}$

Graph = {nodes, edges}

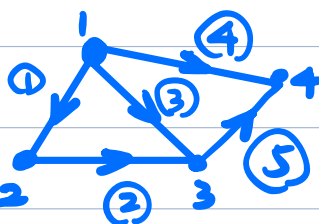


L12

Graph & Networks

Incidence Matrices

Kirchhoff's Laws



$n = 4$ nodes

$m = 5$ edges

Incidence Matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{matrix} \text{edge} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$$

node 1 2 3 4

Notice edge 1 + 2 = 3 } loop

notice edge 1 + 2 = 3
 loop = dependence

$$A \cdot x = 0$$

x_1, x_2, x_3, x_4 potentials at nodes

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$x_2 - x_1$, etc. potential differences

$$A \cdot x = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = C \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \dim N(A) = 1$$

$$r = 3$$

$$A^T \cdot y = 0$$

$$\begin{matrix} n \times m \\ 4 \times 5 \end{matrix} \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$y \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\dim N(A^T) = m - r = 2$$

x_1, x_2, x_3, x_4
potentials at nodes

$x_2 - x_1$, etc.
potential differences

C OHM'S LAW

$y = Cx$
currents $y_1, y_2, \dots, y_5$
on edges

$\downarrow A^T$

$A^T y = f$ Kirchhoff's

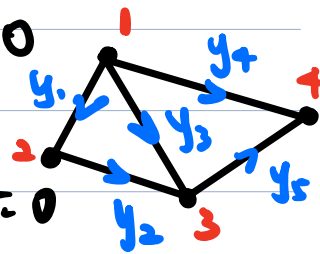
current law

$$-y_1 - y_3 - y_4 = 0$$

$$y_1 - y_2 = 0$$

$$y_2 + y_3 - y_5 = 0$$

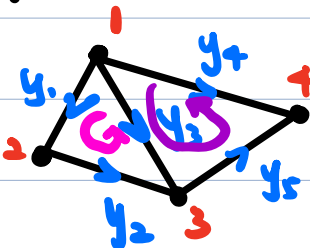
$$y_4 + y_5 = 0$$



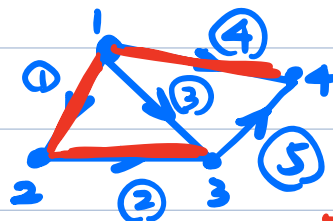
Basis for $N(A^T)$:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

G G



TREE: no loops



$$\dim N(A^T) = m - r$$

$$\# \text{ loops} = \# \text{ edges} - (\# \text{ nodes} - 1)$$

$\Leftrightarrow$

$$\# \text{ nodes} - \# \text{ edges} + \# \text{ loops} = 1$$

Euler's formula

$$A^T C A x = f$$

2.13 Revision

$$\begin{array}{c} u, v, w \text{ in } \mathbb{R}^T \\ \hline 5 \times 3 \quad u \quad \text{Pivots } r=3 \end{array}$$

$$(1) N(u) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$(2) B = \begin{bmatrix} u \\ 2u \end{bmatrix} \text{ column form}$$

$$C = \begin{bmatrix} u & u \\ u & 0 \end{bmatrix} \quad \begin{bmatrix} u \\ 0 \end{bmatrix} \quad r=3 \quad \text{TF} \quad B^2=0 \Rightarrow B=0? \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\downarrow \quad \begin{bmatrix} u & u \\ 0 & -u \end{bmatrix} \rightarrow \begin{bmatrix} u & 0 \\ 0 & -u \end{bmatrix}$$

$$\text{rank?} \quad 6 \quad \rightarrow \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$$

$$\dim N(C^T) = 4$$

$$Ax = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{if } c \text{ multiple}$$

$$A \quad 3 \times 3 \quad r=1$$

$$\dim N(A) = 2$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$Ax=b$ can be solved if b has the form $b = c \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
Don't forget other cases
 $r=m, r=n$

TF $n \times n$ indep. cols.
 $Ax=b$ always solv.

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

basis for $N(B) \subseteq \mathbb{R}^T$

$$N(CD) = N(C)$$

basis for $N(A)$ $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$Bx = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
complete solution

$$x_p + x_h = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

TF

If A, B same $\times$ subspaces, then $A=0$

example: A, B any 6×6 invertible

$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ can't be in

nullspace & be a row of A Pythagoras

Why not?

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{Contor dict!}$$

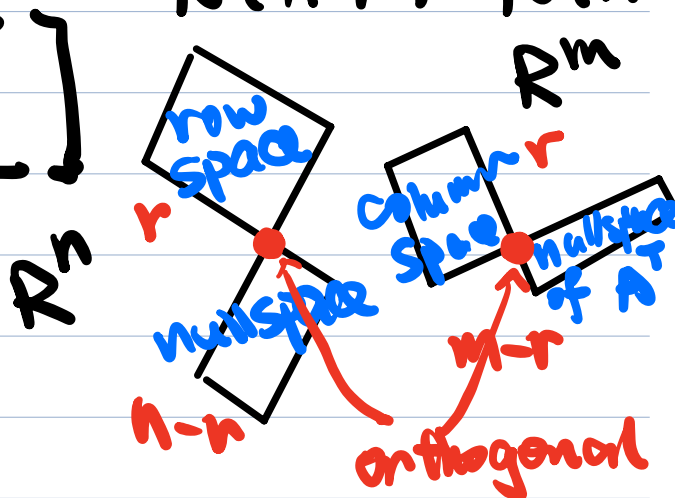
$$\begin{bmatrix} 1 & 2 & 3 \\ - & - & - \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ - & - \end{bmatrix}$$

L14

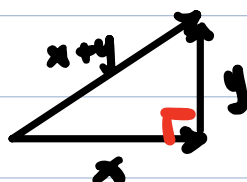
Orthogonal vectors & subspaces

nullspace $\perp$ row space

$$N(A^T A) = N(A^T)$$



Orthogonal vectors



$$x^T y = 0$$

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$

$$x^T x + y^T y = (x+y)^T (x+y)$$

$$\Leftrightarrow x^T x + y^T y =$$

$$x^T x + y^T y$$

$$+ x^T y + y^T x$$

$$\Rightarrow 2x^T y = 0$$

Check:

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$x+y = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

$$1+5=19$$

Subspace S is orthogonal to Subspace T

means: every vector in S is orthogonal to every vector in T

row space is orthogonal to the nullspace

Why?

$$Ax=0 \begin{bmatrix} \text{row 1 of } A \\ \text{row 2 of } A \\ \vdots \\ \text{row } m \text{ of } A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$c_1 (\text{row } i)^T x = 0$$

$$+ c_2 (\text{row } j)^T x$$

Nullspace & row space are orthogonal complements in $\mathbb{R}^n$

Nullspace contains all vectors $\perp$ row space

Coning: *

Solve " $Ax=b$ " when there is no solution
($m > n$)

$$A^T A$$

$n \times m \quad m \times n$

$n \times n$

symmetric

$$(A^T A)^T = A^T A^T = A^T A$$

$$Ax=b \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 30 \end{bmatrix}$$

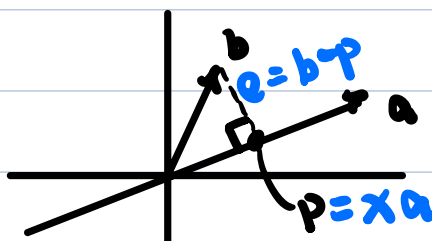
$$N(A^T A) = N(A)$$

rank of $A^T A$ = rank of A

$A^T A$ is invertible exactly if A has indep. column

L15

Projections
Least squares
PROJECTION MATRIX



$$A^T (b - xa) = 0$$

$$XA^T a = a^T b$$

$$x = \frac{A^T b}{A^T a} \quad P = ax$$

$$P = a \frac{A^T b}{A^T a} = P \cdot b$$

MATRIX

$$P = \frac{a a^T}{A^T a} \quad C(P) = \text{line thru } a$$

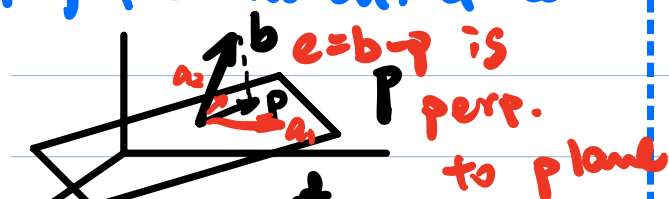
$$\text{rank}(P) = 1$$

$$* P^T = P \quad P^2 = P$$

Why project?

Because $Ax=b$ may have no solution

Solve $A\hat{x} = p$ instead
proj of b onto col. space



plane of a_1, a_2
= col. space
of $A \begin{bmatrix} a_1 & a_2 \end{bmatrix}$

$$p = A\hat{x} \quad \text{Find } \hat{x}$$

Key: $b - A\hat{x}$ is perp. to plane

$$a_1^T (b - A\hat{x}) = 0$$

$$a_2^T (b - A\hat{x}) = 0$$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T (b - A\hat{x}) = 0$$

$$e \in N(A^T) \Rightarrow e \perp C(A) \quad \text{YES!}$$

$$A^T A \hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$P = A\hat{x} = A(A^T A)^{-1} A^T b$$

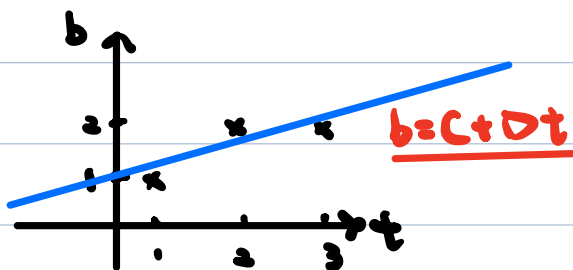
ID

matrix $P = A(A^T A)^{-1} A^T$

$$P^T = P : P^2 = P$$

Least squares

Fitting by a line



(1,1) (2,2) (3,2)

$$C + D = 1$$

$$C + 2D = 2$$

$$C + 3D = 2$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$x = \begin{bmatrix} C \\ D \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$= \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}^{-1} A^T b$$

L16

Projections

Least squares & best straight line

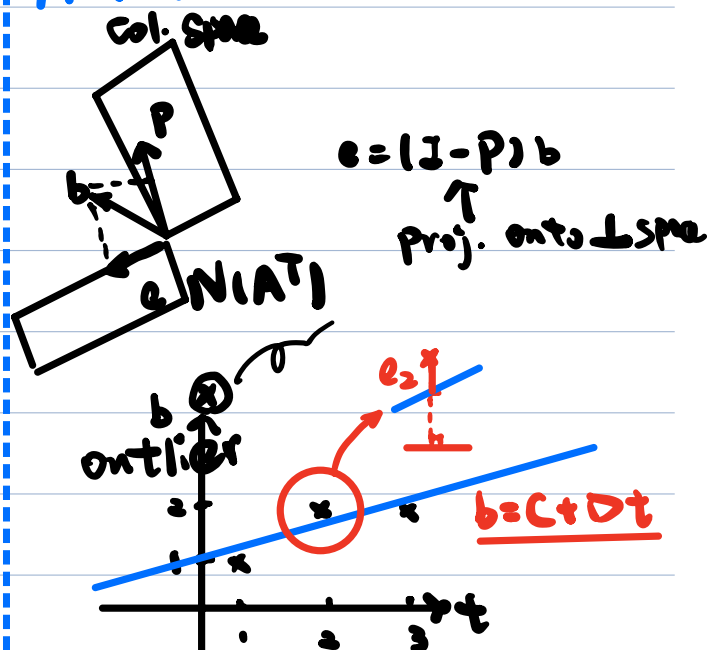
$$P = A(A^T A)^{-1} A^T$$

If b in column space $Pb = b$

If $b \perp$ column space $Pb = 0$

$$b = Ax$$

$$A(A^T A)^{-1} A^T A x = Ax = b$$

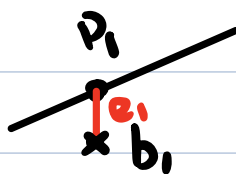


(1,1) (2,2) (3,2)

$$\text{Minimize } \|Ax - b\|_2^2$$

$$= e_1^2 + e_2^2 + e_3^2$$

(least squares)



Q:

$$\text{Find } \hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$$

$$A^T A \hat{x} = A^T b$$

P

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 5 \\ 6 & 14 & 11 \end{bmatrix}$$

$$\Rightarrow 3C + 6D = 5$$

$$6C + 14D = 11$$

$$D = 1/2 \quad C = 2/3$$

Prove:

$\Leftarrow x$ must be 0

IDBA:

$$x^T A^T A x = 0 = (Ax)^T (Ax)$$

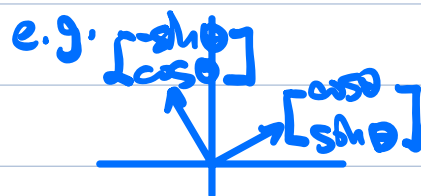
$$\Rightarrow Ax = 0$$

A has
indep.
cols. $A \neq 0$

$$x = 0$$

Columns definitely independent

If they are perp. write
vector
orthonormal
vectors



Calculus's Way

$$e_1^2 + e_2^2 + e_3^2$$

$$= (C+D-1)^2 + (C+2D-2)^2 + (C+3D-3)^2$$

求极值 $\rightarrow$ Solve!

Best line: $\frac{2}{3} + \frac{1}{2}t$

$$b = p + e$$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix} \begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}$$

L17

Orthogonal basis: $q_1 \dots q_n$

Orthogonal matrix Q: square

Gram-Schmidt $A \rightarrow Q$

orthonormal vectors

$$q_i q_j^T = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

If A has indep. columns

then $A^T A$ is invertible

Suppose $A^T A x = 0$

$$Q = \begin{bmatrix} | & & | \\ q_1 & \dots & q_n \\ | & & | \end{bmatrix}$$

$$Q^T Q = I$$

If Q is square:
 then $Q^T Q = I$ tells us
 $Q^T = Q^{-1}$

examples:

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$Q = \frac{1}{3} \begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

Project onto Q 's column space

$$P = Q(Q^T Q)^{-1} Q^T$$

$= Q Q^T$ if Q is square

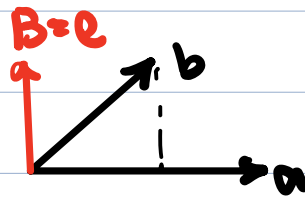
$$A^T A \hat{x} = A^T b$$

$$\Rightarrow Q^T Q \hat{x} = Q^T b$$

$$\Leftrightarrow \hat{x} = Q^T b$$

Gram-Schmidt:

independent vectors a, b



orthogonal
 $A \perp B$

orthonormal

$$q_1 = \frac{A}{\|A\|} \quad q_2 = \frac{B}{\|B\|}$$

$$B = b - \frac{A^T b}{A^T A} A$$

$$q_3 = \frac{C}{\|C\|}$$

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

e.g.

$$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$B = b - \frac{a^T b}{a^T a} a$$

$$= b - a = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \quad A = QR$$

$$A \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_1^T & a_2^T \\ 0 & a_3^T \end{bmatrix}$$

④ 2 equal rows $\rightarrow \det = 0$

(prove:

exchange those rows: same matrix

$$\det A = -\det A'$$

$$\det A = \det A'$$

$$\rightarrow \det = 0$$

⑤ Subtract $1 \times \text{row } i$ from row k

DET doesn't change

L18

Determinants $\det A$

Properties 1, 2, 3 4-13

$\pm$ signs

(prove:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ -c & -d \end{vmatrix}$$

$$\textcircled{1} \det I = 1 \quad \det P = -1$$

② Exchange rows, reverse

the sign of $\det$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

③ Row of zeros $\rightarrow \det = 0$

(prove:

$$5 \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 5 \cdot 0 & 5 \cdot 0 \\ c & d \end{vmatrix} = A$$

$$\det A = 5 \det A \rightarrow \det A = 0$$

$$\textcircled{3a} \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

⑦

$$\det U = \begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & \dots & 0 & d_n \end{bmatrix} = d_1 \cdot d_2 \dots d_n$$

product of pivots
(if no row exchange)

③b

$$\det \begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \det \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \det \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

⑧ $\det A = 0$

when A is singular

$\det A \neq 0$ when A is invertible

$A \text{ invertible} \rightarrow U \rightarrow D \rightarrow d_1 \dots d_n$

BIG FORMULA!!

⑨ $\det AB = \det A \cdot \det B$

$\det A^{-1} = \frac{1}{\det A}$

$\det A^2 = (\det A)^2$

$\det 2A = 2^n \det A$

⑩ $\det A^T = \det A$

$\det A = \sum \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \dots a_{n\omega}$
 $n! \text{ terms}$ $(\alpha, \beta, \gamma, \dots, \omega)$
 $= \text{perm. of } (1, 2, \dots, n)$

Example

$A \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

(4, 3, 2, 1)

$\rightarrow +1$

(3, 2, 1, 4)

$\rightarrow -1$

$\det A = 0 \rightarrow \text{singular}$

row 3 + row 1 - row 2

= row 1

prove!

L19

Formula for $\det A$ ($n!$ terms)

Cofactor formula

Tridiagonal matrices

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$

Method * $\begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$
 $\leftarrow + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = ad - bc$

Cofactors 3×3

$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32})$
 $+ a_{12}(a_{21}a_{33} - a_{23}a_{31})$
 $+ a_{13}(a_{22}a_{31} - a_{21}a_{32})$

$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{33} \\ 0 & a_{32} & 0 \end{vmatrix}$

$+ \begin{vmatrix} 0 & a_{22} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \\ a_{31} & 0 & 0 \end{vmatrix}$

$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$
 $- a_{13}a_{22}a_{31}$

NEED A GENERAL FORMULA!!

$\begin{vmatrix} a_{11} & \circ & \circ \\ \circ & a_{22} & a_{23} \\ \circ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} \circ & a_{12} & \circ \\ a_{21} & \circ & a_{23} \\ a_{31} & \circ & a_{33} \end{vmatrix}$
 $+ \begin{vmatrix} \circ & \circ & a_{13} \\ a_{21} & a_{22} & \circ \\ a_{31} & a_{32} & \circ \end{vmatrix}$