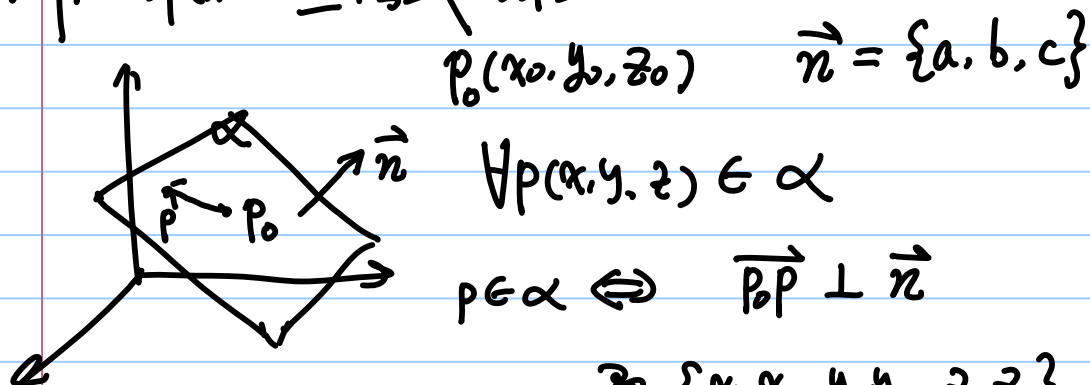


§1. 线性方程组的 Gauss 消元法.

便笺标题

2015/9/29

例1: 平面的法式方程:



$$\Rightarrow \{x-x_0, y-y_0, z-z_0\} \perp \{a, b, c\}$$

$$\Rightarrow \alpha \text{ 的方程为 } a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

例2: 讨论两个平面 $2x-y+3z=1$, $4x+2y+5z=4$.
 $x+z=3$ 是否交于一直线?

解:
$$\begin{cases} 2x-y+3z=1 & ① \\ 4x+2y+5z=4 & ② \\ x+z=3 & ③ \end{cases} \quad \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 4 & 2 & 5 & 4 \\ 1 & 0 & 1 & 3 \end{array} \right]$$

(代入消元)
(加减消元)

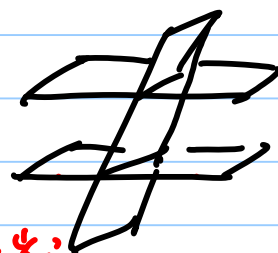
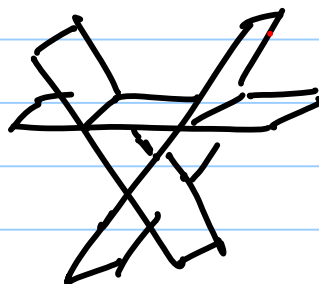
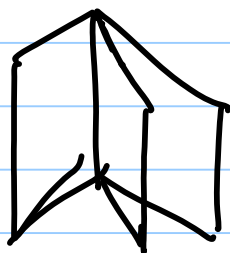
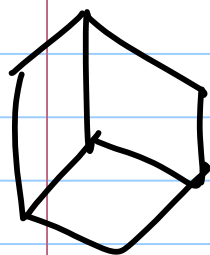
$$\begin{aligned} & ① \times (-2) + ② \rightarrow \begin{cases} 2x-y+3z=1 & ④ \\ 4y-z=2 & ⑤ \end{cases} \quad r_1 \times (-2) + r_2 \\ & ① \times (-\frac{1}{2}) + ③ \rightarrow \begin{cases} 2x-y+3z=1 & ④ \\ 4y-z=2 & ⑤ \\ \frac{1}{2}y - \frac{1}{2}z = \frac{5}{2} & ⑥ \end{cases} \quad r_1 \times (-\frac{1}{2}) + r_3 \end{aligned}$$

$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 4 & -1 & 2 \\ 0 & \frac{1}{2} & \frac{5}{2} & \frac{5}{2} \end{array} \right]$

$$\begin{aligned} & ⑥ \times 2 \rightarrow \begin{cases} 2x-y+3z=1 & ⑦ \\ 4y-z=2 & ⑧ \\ y-z=5 & ⑨ \end{cases} \rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 4 & -1 & 2 \\ 0 & 1 & -1 & 5 \end{array} \right] \\ & \text{交换 } ⑧ \text{ 和 } ⑨ \end{aligned}$$

$$\begin{aligned} & ⑧ \leftrightarrow ⑨ \rightarrow \begin{cases} 2x-y+3z=1 & ⑩ \\ y-z=5 & ⑪ \\ 4y-z=2 & ⑫ \end{cases} \rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 3 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 4 & -1 & 2 \end{array} \right] \end{aligned}$$

① $x(-4) + ②$ $\rightarrow$ $\begin{cases} 2x - y + 3z = 1 \\ y - z = 5 \\ 3z = -18 \end{cases} \rightarrow \begin{bmatrix} 2 & -1 & 3 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 3 & 18 \end{bmatrix} \rightarrow (9, -1, -6)$



Note: 1. 基本思想: 消元 $\begin{cases} \text{代入} \rightarrow \text{秩} \\ \text{加减} \rightarrow \text{Gauss 消元} \end{cases}$

2. 线性方程组 3 种变换: ① 交换两个方程

② 某个方程乘以非零常数

③ 某一方程乘以常数加到其他方程

例 3: $\begin{cases} x_1 + 3x_2 - 5x_3 = -1 & ① \\ 2x_1 + 6x_2 - 3x_3 = 5 & ② \\ 3x_1 + 9x_2 - 10x_3 = 2 & ③ \end{cases}$

① $\times (-2) + ②$ $\rightarrow$ $\begin{cases} x_1 + 3x_2 - 5x_3 = -1 \\ 7x_3 = 7 \\ 5x_3 = 5 \end{cases}$ $\rightarrow \begin{bmatrix} 1 & 3 & -5 & -1 & 1 \\ 0 & 0 & 7 & 7 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$\rightarrow \begin{cases} x_1 = 4 - 3k \\ x_2 = k \\ x_3 = 1 \end{cases}$

k 为任意实数

自由变量

• 抽象出一般结构:

$$(*) \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

1. 若 $(c_1, c_2, \dots, c_n)$ 使 $(*)$ 左右两端值相等 $\forall$ 则 $(c_1, c_2, \dots, c_n)$ 为 $(*)$ 的解

2. 基本问题:

- ① 存在性. ② 个数问题. ③ 解的公式
④ 解法问题. ⑤ 解不唯一时解之结构.

• 定义 $m \times n$ 个数构成数表称为一个 $m \times n$ 阶矩阵 (Matrix)

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \begin{matrix} A_{m \times n} \\ A = [a_{ij}]_{m \times n} \end{matrix}$$

$$A \in M_{m \times n}(\mathbb{R}) \quad A \in \mathbb{R}^{m \times n}$$

当 $m=n$ 时, A 称为方阵 $A_n \quad A \in M_n(\mathbb{R})$

$$A \in \mathbb{R}^{n \times n}$$

• 矩阵的某些初等变换

① 交换两行

② 某行要以非零数为

③ 某行要以某数加到其他行

• 在方程组 (*) 中, 称 $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ 为 (*) 系数矩阵

称 $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & | & b_m \end{bmatrix}$ 为 (*) 的增广矩阵

• 阶梯形矩阵

$$x+y=2$$

$$2x+4y=4$$

① 零行在最下方

② 元素不全为零的行 左边数起 每行第一个不为 0 的元素, 称为主元. 主元的位置 随行 标而增大而列标增大

Note: 求解方程组的过程就是将增广矩阵利用初等行变换化为阶梯形矩阵的过程

• 最简阶梯型 (简化行阶梯型)

① 是一个阶梯型矩阵

② 主元所在列 其他元素均为 0.

$$\text{12) 4. } \begin{cases} x_1 - x_2 - x_3 + 3x_5 = -1 \\ 2x_1 - 2x_2 - x_3 + 2x_4 + 4x_5 = -2 \\ 3x_1 - 3x_2 - x_3 + 4x_4 + 5x_5 = -3 \\ x_1 - x_2 + x_3 + x_4 + 8x_5 = 2. \end{cases}$$

$$\text{Aug: } \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 3 & -1 \\ 2 & -2 & -1 & 2 & 4 & -2 \\ 3 & -3 & -1 & 4 & 5 & -3 \\ 1 & -1 & 1 & 1 & 8 & 2 \end{array} \right]$$

$$\begin{array}{l} r_1 \times (-2) + r_2 \\ \hline r_1 \times (-3) + r_3 \\ r_1 \times (-1) + r_4 \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 2 & 4 & -4 & 0 \\ 0 & 0 & 2 & 1 & 5 & 3 \end{array} \right]$$

$$\begin{array}{l} r_2 \times (-2) + r_3 \\ \hline r_2 \times (-2) + r_4 \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 9 & 3 \end{array} \right]$$

$$\begin{array}{l} r_4 \times (-\frac{1}{3}) \\ \hline r_4 \leftrightarrow r_3 \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{array}{l} r_3 \times (-2) + r_2 \\ \hline \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{array}{l} r_2 + r_1 \\ \hline \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 7 & 1 \\ 0 & 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

主元
 x_1, x_3, x_4

$$\begin{cases} x_1 = 1 + k_1 - 7k_2 \\ x_2 = k_1 \\ x_3 = 2 - 4k_2 \\ x_4 = 3k_2 - 1 \\ x_5 = k_2 \end{cases}$$

例 5.
$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 - 5x_3 = 2 \\ 2x_1 + 3x_2 - 4x_3 = 5 \end{cases}$$

解:
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & -5 & 2 \\ 2 & 3 & -4 & 5 \end{array} \right] \xrightarrow[r_1 \times (-2) + r_2]{r_1 \times (-1) + r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -6 & 1 \\ 0 & 1 & -6 & 3 \end{array} \right]$$

$$\xrightarrow{r_2 \times (-1) + r_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -6 & 1 \\ 0 & 0 & 0 & 2 \end{array} \right] \quad \text{无解}$$

•
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right] \xrightarrow{\text{若 } a_{11} \neq 0} \text{经过一系列初等行变换}$$

其中 $c_{11}, c_{22}, \dots, c_{rr} \neq 0$ 主元

$$\left[\begin{array}{cccc|c} c_{11} & c_{12} & \dots & c_{1r} & d_1 \\ 0 & c_{22} & \dots & c_{2r} & d_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & c_{rr} & d_r \\ 0 & \dots & 0 & 0 & d_{r+1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{array} \right]$$

若 $d_{r+1} \neq 0$ 时, 无解

若 $d_{r+1} = 0$ 时, 有解

- $r < n$ 时 无解? 有解? 有 $(n-r)$ 个自由变量
- $r = n$ 时 唯一解

例 6: t 为任意值时

$$\begin{cases} -x_1 - 4x_2 + x_3 = 1 \\ tx_2 - 3x_3 = 3 \\ x_1 + 3x_2 + (t+1)x_3 = 0 \end{cases}$$

无解? 有唯一解? 有无解?

$$\left[\begin{array}{ccc|c} -1 & -4 & 1 & 1 \\ 0 & t & -3 & 3 \\ 1 & 3 & t+1 & 0 \end{array} \right] \xrightarrow{r_1+r_3} \left[\begin{array}{ccc|c} -1 & -4 & t+1 & 1 \\ 0 & t & -3 & 3 \\ 0 & -1 & t+2 & 1 \end{array} \right]$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} -1 & -4 & 1 & 1 \\ 0 & -1 & t+2 & 1 \\ 0 & t & -3 & 3 \end{array} \right]$$

$$\xrightarrow{r_2 \times (t)+1} \left[\begin{array}{ccc|c} -1 & -4 & 1 & 1 \\ 0 & -1 & t+2 & 1 \\ 0 & 0 & (t-1)(t+3) & t+3 \end{array} \right]$$

若 $t=1$ 时 无解

若 $t=3$ 时 无解

若 $t \neq 1$ 且 $t \neq -3$ 时 唯一解

• ④ 方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

当 $b_1 = b_2 = \dots = b_m = 0$ 时 称为齐次方程组
否则 称为非齐次方程。

• 推论 1: 齐次方程总有解

2: 齐次方程组中 当 $m < n$ 时 总有非零解

$$\left[\begin{array}{c|c} \text{---} & 0 \\ \text{---} & 0 \\ \text{---} & \vdots \\ \text{---} & 0 \end{array} \right] \xrightarrow{m \times n} \left[\begin{array}{c|c} \text{---} & 0 \\ \text{---} & \vdots \\ \text{---} & 0 \end{array} \right]$$

非零行 行数 $r \leq m < n$ 有无数解

例 P₁₄₆ 1(2)(3)

2(3)(4)

3.

4. 5.

§2 行列式の定義: Cramer's Leibniz.

Note Title

2015/10/8

$$1. \begin{cases} a_{11}x + a_{12}y = b_1 & ① \\ a_{21}x + a_{22}y = b_2 & ② \end{cases}$$

解: ①より $a_{11} \neq 0$.

$$① \times \left(-\frac{a_{21}}{a_{11}}\right) + ②$$

$$\begin{bmatrix} a_{11} & a_{12} & | & b_1 \\ a_{21} & a_{22} & | & b_2 \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & | & b_1 \\ 0 & a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}} & | & b_2 - b_1 \cdot \frac{a_{21}}{a_{11}} \end{bmatrix}$$

$$\left(a_{22} - a_{12} \cdot \frac{a_{21}}{a_{11}}\right)y = b_2 - b_1 \cdot \frac{a_{21}}{a_{11}}$$

よ $a_{11}a_{22} - a_{12}a_{21} \neq 0$ のとき $\frac{1}{a_{11}a_{22} - a_{12}a_{21}}$

$$\begin{cases} x = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \\ y = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \end{cases}$$

$a_{11}x + a_{12}y = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} + \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} \cdot a_{12}$
 $= \frac{b_2a_{11} - b_1a_{21} + b_1a_{12}a_{22} - b_2a_{12}a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$

$$\text{定義} \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{cases} x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \\ y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \end{cases}$$

$$x = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$

Cramer's 法

$|A|$

$$2. \begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 & \textcircled{1} \\ a_{21}x + a_{22}y + a_{23}z = b_2 & \textcircled{2} \\ a_{31}x + a_{32}y + a_{33}z = b_3 & \textcircled{3} \end{cases}$$

Step: ~~Assume~~ $a_{11} \neq 0$. $\textcircled{2} \times (a_{11}) - \textcircled{1} \times (a_{21})$, $\textcircled{3} \times (a_{11}) - \textcircled{1} \times (a_{31})$

$$\begin{cases} (a_{22}a_{11} - a_{12}a_{21})y + (a_{23}a_{11} - a_{13}a_{21})z = b_2a_{11} - b_1a_{21} \end{cases}$$

$$\begin{cases} (a_{32}a_{11} - a_{12}a_{31})y + (a_{33}a_{11} - a_{13}a_{31})z = b_3a_{11} - b_1a_{31} \end{cases}$$

$$\frac{1}{2} \begin{vmatrix} a_{22}a_{11} - a_{12}a_{21} & a_{23}a_{11} - a_{13}a_{21} \\ a_{32}a_{11} - a_{12}a_{31} & a_{33}a_{11} - a_{13}a_{31} \end{vmatrix} \neq 0 \text{ or } \Delta \neq 0$$

$$\Delta = \begin{vmatrix} a_{22}a_{11} - a_{12}a_{21} & a_{23}a_{11} - a_{13}a_{21} \\ a_{32}a_{11} - a_{12}a_{31} & a_{33}a_{11} - a_{13}a_{31} \end{vmatrix}$$

$$\Delta = a_{22}a_{33}a_{11}^2 - \underbrace{a_{22}a_{11}a_{13}a_{31}} - a_{12}a_{21}a_{33}a_{11} + \cancel{a_{12}a_{21}a_{13}a_{31}} - \underbrace{a_{32}a_{11}^2a_{23}} + \underbrace{a_{32}a_{11}a_{13}a_{21}} + a_{12}a_{31}a_{23}a_{11} - \cancel{a_{12}a_{31}a_{13}a_{21}}$$

$$= (a_{11}) (a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + \underline{a_{12}a_{31}a_{23}} - a_{22}a_{13}a_{31} - a_{12}a_{21}a_{33} - a_{32}a_{11}a_{23})$$

$$y = \frac{\Delta_1}{\Delta} \quad \Delta_1 = a_{11} (a_{11}b_2a_{33} + a_{13}a_{21}b_3 + b_1a_{31}a_{23} - b_2a_{13}a_{31} - b_1a_{21}a_{33} - b_3a_{11}a_{23})$$

$$b_2 \Leftrightarrow a_{22}$$

$$b_1 \Leftrightarrow a_{12}$$

$$b_3 \Leftrightarrow a_{32}$$

定义 $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{a_{11} a_{22} a_{33}} + \underbrace{a_{12} a_{23} a_{31}} + \underbrace{a_{13} a_{21} a_{32}} - \underbrace{a_{22} a_{13} a_{31}} - \underbrace{a_{12} a_{21} a_{33}} - \underbrace{a_{32} a_{11} a_{23}}$

Note: 若 i 与 j 不同, 则 $a_{ij} = 0$. 不同列, 有符号差别.

2. $\begin{cases} x = \frac{\Delta_0}{\Delta} \\ y = \frac{\Delta_1}{\Delta} \\ z = \frac{\Delta_2}{\Delta} \end{cases}$

$a_{11} a_{22} a_{33}$ 列: 1. 2. 3. +
 $a_{12} a_{23} a_{31}$ 列: 2. 3. 1. +
 $a_{13} a_{21} a_{32}$ 3. 1. 2. +
 $a_{13} a_{22} a_{31}$ 列: 3. 2. 1. -

3. 若 n 个方程, n 个未知数, 则 $\{x_i\}$ 有唯一解.

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2 \\ \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n = b_n \end{cases}$$

4. 排列与逆序.

定义: 由 $1, 2, \dots, n$ 组成的有序数组称为一个 n 阶排列. 记为 $j_1 j_2 \dots j_n$. 共 $n!$ 种方式.

定义: 在 $j_1 j_2 \dots j_n$ 中, 若大数在小数前面, 则这两个数构成一个逆序. 逆序之总数称为逆序数 $\tau(j_1 j_2 \dots j_n)$.

$\tau(23541) = 5$. $\tau(12345) = 0$

定义: 奇排列. 偶排列.

5. 对换与奇偶性:

对换: 对换改变排列的奇偶性.

$$\tau(2341) = 3 \quad \tau(2143) = 2.$$

证明: 先考虑对换: 两个数在 n -排列中相邻: 4种情况

$$\dots i j \dots \xrightarrow{i \leftrightarrow j} \dots j i \dots$$

一般情况 $\dots i k_1 \dots k_s j \dots$

$\dots k_1 \dots k_s i j$

s 次

$\dots k_1 \dots k_s j i$

1 次

$\vdots$

$\dots j k_1 \dots k_s i \dots$

s 次

$(2s+1)$ 次..

$$\tau(1432) \text{ 奇.}$$

定理2: 在全部 n 阶排列中. 奇排列个数等于偶排列个数.

设有 s 个偶排列 t 个奇排列.

将所有的偶排列做一次对换. $\Rightarrow s \leq t$

$$\text{类似地 } t \leq s \quad \text{例} \quad s=t=\frac{1}{2}n!$$

6. 定义: n 阶行列式

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{\substack{\{j_1, \dots, j_n\} \\ = \{1, 2, \dots, n\}}} (-1)^{\tau(j_1, \dots, j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

Note ① $n!$ 在代数中

② 不同行不同列

③ 行指标为自然排列时, 符号取决于列指标的排列

④ $A = (a_{ij})_{n \times n}$

$f: A \rightarrow \mathbb{R}$ n^2 元的连续函数. $f(x) = 2x$

$f(x, y) = xy$

$f(x, y, t) = xyx$

Note: $f: A \rightarrow B$

$A \subseteq \mathbb{R}, A \neq \emptyset, B = \mathbb{R}$ 函数.

A 线性空间 B (线性空间) 算子.

A (线性空间) $B = \mathbb{R}$ 泛函

$A=B$ (线性空间) 变换

例1: 写出下列各式中含 a_3 与 a_{32} 项并确定
系数

$$+ a_{13} a_{21} a_{32} a_{44} - a_{13} a_{24} a_{32} a_{41}$$

12/2: $f(x) = \begin{vmatrix} x & 1 & 1 & 2 \\ 1 & x & 1 & -1 \\ 3 & 2 & x & 4 \\ 1 & 1 & 2x & 0 \end{vmatrix}$ 求 x^3 系数

$$1x^3: \quad + \quad x \cdot x \cdot 2x \cdot 4 \cdot (-1) \quad \tau(1243)$$

$$= x^3 - 8x^3 = -7x^3.$$

常数项 $= f(0)$

(a): $g(x) = \begin{vmatrix} \textcircled{2} & \triangle -1 & \underbrace{x} & \textcircled{2x} \\ 1 & \textcircled{1} & \textcircled{x} & \triangle -1 \\ 0 & \textcircled{x} & \textcircled{2} & 0 \\ \textcircled{x} & \textcircled{0} & -1 & \textcircled{3+x} \end{vmatrix}$

第 22 题 $2 \times 1 \times 2 \times 3 (-1)^{\tau(1234)} -$
 $+ 2 \times (-1) \times 2 \times 0 (-1)^{\tau(1432)}.$

$$+ (-1) \times 1 \times 2 \times 3 \cdot (-1)^{\tau(2134)} \cdot \underline{\underline{+ (-1) \times (-1) \times (-1) \times (-1)}}$$

$$\text{1a/4: } \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots \\ & & & a_{nn} \end{vmatrix} = a_{11} a_{22} \cdots a_{nn}$$

$$\begin{vmatrix} a_{11} & & * \\ & a_{22} & \\ & & \ddots \\ & & & a_{nn} \end{vmatrix} = a_{11} a_{22} \cdots a_{nn}$$

$$\begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & * & \\ & & \ddots \\ & & & a_{nn} \end{vmatrix} = a_{11} a_{22} \cdots a_{nn}$$

$$\text{1a/5: } \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\text{1a/6: } \begin{vmatrix} a_1 & (b_1) & & & \\ & a_2 & b_2 & & \\ & & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} \\ (b_n) & & & & a_n \end{vmatrix}$$

$$\forall n=1 \text{ of: } |a_1| = a_1$$

$$\hookrightarrow \{j_1=1\}$$

$$\forall n \geq 2 \text{ of: } j_1=1 \quad j_2=2 \cdots j_n=n \quad a_1 a_2 \cdots a_n$$

$$j_1=2, j_2=3, \dots, j_{n-1}=n, j_n=1, b_1 b_2 \dots b_{n-1} b_n$$

$$\begin{aligned} \Delta &= a_1 a_2 \dots a_n (-1)^{\tau(1, 2, \dots, n)} + b_1 b_2 \dots b_n (-1)^{\tau(2, 3, \dots, n, 1)} \\ &= a_1 a_2 \dots a_n + (-1)^{(n-1)} b_1 b_2 \dots b_n \end{aligned}$$

$$\text{ibm: } \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} b^{1-1} & a_{12} b^{1-2} & \dots & a_{1n} b^{1-n} \\ a_{21} b^{2-1} & a_{22} b^{2-2} & \dots & \underline{a_{2n} b^{2-n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} b^{n-1} & a_{n2} b^{n-2} & \dots & a_{nn} b^{n-n} \end{vmatrix} \quad (b \neq 0)$$

$$\text{ibm: } \Delta_2 = \sum_{\{j_1, \dots, j_n\} = \{1, 2, \dots, n\}} (-1)^{\tau(j_1, \dots, j_n)} a_{1j_1} a_{2j_2} \dots a_{nj_n} b^{1-j_1} b^{2-j_2} \dots b^{n-j_n}$$

$$= \sum_{\{j_1, \dots, j_n\} = \{1, 2, \dots, n\}} (-1)^{\tau(j_1, \dots, j_n)} a_{1j_1} a_{2j_2} \dots a_{nj_n} b^{(1+2+\dots+n)-(j_1+\dots+j_n)}$$

$b^0 = 1$

$$= \Delta$$

行列式の性質.

① 一行列式中某行全为零. 则行列式之值为零

② 行列式中各行互换. 其值不变.

$$(1) |A| = \sum (-1)^{\tau(j_1 \dots j_n)} \underline{a_{1j_1}} \underline{a_{2j_2}} \dots \underline{a_{nj_n}}$$

将 $a_{1j_1} a_{2j_2} \dots a_{nj_n}$ 换成 $a_{i_1 k_1} a_{i_2 k_2} \dots a_{i_n k_n}$

$$(-1)^{\tau(j_1 \dots j_n)} = (-1)^{\tau(i_1 \dots i_n) + \tau(k_1 \dots k_n)}$$

参考: P30 1-6.

§3 $\{j_1, j_2, \dots, j_n\}$ 的行列式:

Note Title

2015/10/13

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{\{j_1, \dots, j_n\} = \{1, 2, \dots, n\}} (-1)^{\tau(j_1, j_2, \dots, j_n)} a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

1. $\{j_1, j_2, \dots, j_n\}$ 中 $\{j_1, j_2, \dots, j_n\}$ 为排列. (有 $n!$ 个).

$$\begin{aligned} & \begin{matrix} a_{12} & a_{23} & a_{31} \\ \tau(231) = 2 \\ a_{23} & a_{12} & a_{31} \end{matrix} \end{aligned}$$

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix}$$

$$\tau(213) + \tau(321)$$

$$a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

$$\tau(j_1, \dots, j_n)$$

||

交换

$$a_{1k_1} a_{2k_2} \dots a_{nk_n}$$

$$\frac{\tau(i_1, \dots, i_n) + \tau(k_1, \dots, k_n)}{\text{行列式}}$$

$$|A| = \sum_{\{j_1, \dots, j_n\} = \{1, \dots, n\}} (-1)^{\tau(j_1, \dots, j_n)} a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

$$= \sum_{\{j_1, \dots, j_n\} = \{1, \dots, n\}} (-1)^{\tau(j_1, \dots, j_n) + \tau(k_1, \dots, k_n)} a_{1j_1} a_{2j_2} \dots a_{nj_n}$$

$$= \sum_{\{l_1, \dots, l_n\} = \{1, \dots, n\}} (-1)^{\tau(l_1, \dots, l_n)} a_{l_1 1} a_{l_2 2} \dots a_{l_n n}$$

$$= \begin{vmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{nn} \end{vmatrix}$$

Note 1: 对行或列作操作 对行列也成立.

2. 可把~~这~~行列~~这~~一个数~~这~~些数~~这~~乘~~这~~于~~这~~不同~~这~~行.

2. 行列式中某行有公因子, 可提出来

$$\begin{vmatrix} a_{11} & a_{12} \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ ka_{i1} & ka_{i2} \dots & ka_{in} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \dots & a_{nn} \end{vmatrix} = \sum (-1)^{\tau(j_1, \dots, j_n)} a_{j_1} \dots a_{i-1, j_{i-1}} (ka_{ij_i}) \dots a_{jn} \\ = k \sum (-1)^{\tau(j_1, \dots, j_n)} a_{j_1} \dots a_{jn} \\ = k \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \quad \text{对应初等行变换}$$

③ 加法法则

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \dots & a_{in}+b_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix} = \sum (-1)^{\tau(j_1, \dots, j_n)} a_{j_1} \dots a_{i-1, j_{i-1}} (a_{ij_i} + b_{ij_i}) \dots a_{jn} \\ \dots a_{jn}$$

$$= \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ b_{i1} & \dots & b_{in} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

④ 对行列式中的两行，值为两行之相反数。

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \begin{matrix} \rightarrow i \\ \rightarrow k \end{matrix} = \sum_{j_1 \dots j_n} (-1)^{\tau(j_1 \dots j_i \dots j_k \dots j_n)} \underbrace{a_{1j_1} \dots a_{ij_i} \dots a_{kj_k} \dots a_{nj_n}}_{\tau(j_1 j_2 \dots j_k \dots j_i \dots j_n)}$$

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \begin{matrix} \rightarrow i \\ \rightarrow k \end{matrix} = \sum_{j_1 \dots j_n} (-1)^{\tau(j_1 j_2 \dots j_k \dots j_i \dots j_n)} \underbrace{a_{1j_1} \dots a_{kj_k} \dots a_{ij_i} \dots a_{nj_n}}_{\tau(j_1 \dots j_k \dots j_i \dots j_n)}$$

$$- (-1)^{\tau(j_1 \dots j_k \dots j_i \dots j_n)} = (-1)^{\tau(j_1 \dots j_i \dots j_k \dots j_n)}$$

对两初子行交换

⑤ 行列式中有两行相同，值为零。

..... 对应成比例，值为零。

⑥ 行列式中某行元素都加到其它行, 值不变.

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{j1} + ka_{i1} & \dots & a_{jn} + ka_{in} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \underbrace{\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ ka_{i1} & \dots & ka_{in} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}}_{\text{对应初等行变换.}} // \quad \bigcirc$$

例 1. $\begin{vmatrix} 1 & 2 & -2 & 3 \\ -1 & -2 & 4 & -2 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & -3 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & 1 & 14 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 2 & -2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & 3 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

$$= -3$$

$$(2) 2: \begin{vmatrix} a_1+b_1 & b_1+c_1 & c_1+a_1 \\ a_2+b_2 & b_2+c_2 & c_2+a_2 \\ a_3+b_3 & b_3+c_3 & c_3+a_3 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$\frac{1}{2} m.$

||

①

②

$$\begin{vmatrix} a_1 & b_1+c_1 & c_1+a_1 \\ a_2 & b_2+c_2 & c_2+a_2 \\ a_3 & b_3+c_3 & c_3+a_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_1+c_1 & c_1+a_1 \\ b_2 & b_2+c_2 & c_2+a_2 \\ b_3 & b_3+c_3 & c_3+a_3 \end{vmatrix}$$

③

④

$m=2.$

$$(2) 3: \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2. \quad (2) \begin{vmatrix} 3a_{31} & a_{11}+2a_{21} & a_{11} \\ 3a_{32} & a_{12}+2a_{22} & a_{12} \\ 3a_{33} & a_{13}+2a_{23} & a_{13} \end{vmatrix} = ?$$

①

②

③

$= -12$

$$(2) 4. \quad D_n = \begin{vmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{vmatrix}$$

① $n=1$ of: $D_1 = a$

② $n \geq 2$ of:

$$n \geq 2 \text{ 时: } D_n = [a + (n-1)b] \cdot \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{vmatrix}$$

$$= (a + (n-1)b) \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & a-b & 0 & \dots & 0 \\ 0 & 0 & a-b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a-b \end{vmatrix}$$

$$= (a + (n-1)b) \cdot (a-b)^{n-1}$$

例5: a, b, c 为 $x^3 + 2x - 4 = 0$ 的根.

$$\text{求 } \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = ?$$

轮换式

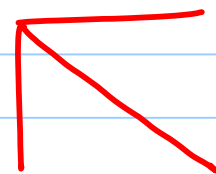
$$\text{解: } x^3 + 2x - 4 = (x-a)(x-b)(x-c) \\ = x^3 - (a+b+c)x^2 + \dots$$

$$a+b+c=0 \quad \sim \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & \dots \\ \vdots & \vdots & \vdots \end{vmatrix}$$

$$= 0$$

Ex 6: $\sum_{i=1}^n a_i x_i = 0$ $a_i \neq 0$ ($i=1 \dots n$)

$$\begin{vmatrix} a_0 & b_1 & b_2 & \dots & b_n \\ c_1 & a_1 & & & \\ c_2 & & a_2 & & \\ \vdots & & & \ddots & \\ c_n & & & & a_n \end{vmatrix}$$



$c_2 \times (-\frac{c_1}{a_1}) + c_1$

$$\begin{vmatrix} a_0 - \frac{b_1 c_1}{a_1} & b_1 & b_2 & \dots & b_n \\ 0 & a_1 & & & \\ c_2 & & a_2 & & \\ \vdots & & & \ddots & \\ c_n & & & & a_n \end{vmatrix}$$

$$= \dots = \begin{vmatrix} a_0 - \frac{b_1 c_1}{a_1} - \frac{b_2 c_2}{a_2} - \dots - \frac{b_n c_n}{a_n} & b_1 & \dots & b_n \\ & a_1 & & \\ & & a_2 & \\ & & & \ddots & \\ & & & & a_n \end{vmatrix}$$

$$= a_1 a_2 \dots a_n \cdot \left(a_0 - \sum_{i=1}^n \frac{b_i c_i}{a_i} \right)$$

Ex 7: $x_1, x_2, \dots, x_n \neq 0$

$$D_n = \begin{vmatrix} 1+x_1 & 2 & \dots & n \\ 1 & 2+x_2 & \dots & n \\ \vdots & & \ddots & \vdots \\ 1 & 2 & \dots & n+x_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & \dots & n \\ 1 & 2+x_2 & \dots & n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \dots & n+x_n \end{vmatrix} + \begin{vmatrix} x_1 & -2 & \dots & -n \\ 0 & 2+x_2 & \dots & n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 2 & \dots & n+x_n \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & \dots & n \\ x_2 & x_3 & \dots & x_n \end{vmatrix} + 2 \begin{vmatrix} 2+x_2 & \dots & n \\ 2 & 3+x_3 & \dots & n \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 3 & \dots & n+x_n \end{vmatrix}$$

例8: 证明: 奇数阶反对称行列式值为零.

$$a_{ij} = -a_{ji}$$

$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix}$$

例9: 求 n 阶行列式

$$D_n = \begin{vmatrix} 2a & a^2 & & \\ 1 & 2a & a^2 & \\ & 1 & 2a & a^2 \\ & & \ddots & \ddots \\ & & & 1 & 2a & a^2 \\ & & & & 1 & 2a \end{vmatrix}$$

三对角

$$n=1 \text{ 时 } D_1 = 2a$$

$n \geq 2$ 时: $D_n = \begin{vmatrix} 2a & 0 & & \\ & \frac{3}{2}a & a^2 & \\ & & 2a & a^2 \\ & & & 2a & a^2 \\ & & & & 2a \end{vmatrix}$ $2a - \frac{2}{3}a$

$= \begin{vmatrix} 2a & 0 & & \\ & \frac{3}{2}a & 0 & \\ & & \frac{4}{3}a & a^2 \\ & & & 2a & a^2 \\ & & & & 2a & a^2 \\ & & & & & 2a \end{vmatrix}$ $2 - \frac{3}{4}a$

$= \begin{vmatrix} 2a & 0 & 0 & 0 \\ & \frac{3}{2}a & 0 & 0 \\ & & \frac{4}{3}a & 0 \\ & & & \frac{5}{4}a \\ & & & & \ddots \\ & & & & & \frac{n+1}{n}a \end{vmatrix} = 2a \cdot \frac{3}{2}a \cdot \frac{4}{3}a \cdots \frac{n+1}{n}a$

$= (n+1)a^n$

• 三阶行列式按第一行展开:

$= \text{按第一行展开:}$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \underline{a_{11} a_{22} a_{33}} + \underline{a_{12} a_{23} a_{31}} + \underline{a_{13} a_{21} a_{32}} - \underline{a_{13} a_{22} a_{31}} - \underline{a_{12} a_{21} a_{33}} - \underline{a_{11} a_{23} a_{32}}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

第 i 行第 j 列之子的余子式 记作 M_{ij}

$(-1)^{i+j} M_{ij}$ 称为第 i 行第 j 列之子的代数余子式 用 A_{ij} 表示.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} A_{11} + a_{12} A_{12} + \cdots + a_{1n} A_{1n}$$

Note: 代数余子式 A_{ij} 与位置有关. 与 a_{ij} 无关

$D = \begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 0 & 2 & 1 \\ 3 & -1 & 0 & 1 \end{vmatrix}$ 求 A_{11} A_{12} A_{13} .

$$A_{12} = - \begin{vmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \\ 3 & 0 & 1 \end{vmatrix} = \underline{\hspace{2cm}}$$

例 2: 求 $\begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 0 & 2 & 1 \\ 3 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -2 \\ -1 & 2 & 2 & 2 \\ 3 & -7 & 0 & -2 \end{vmatrix}$

$$= 1 \cdot \begin{vmatrix} 1 & 1 & -2 \\ 2 & 2 & 2 \\ -7 & 0 & -2 \end{vmatrix}$$

$$\checkmark \text{Ans: } P_{31} \quad 7-18.$$

§4. 行列式展开, Cramer 法则

一. 行列式按一行展开

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{a_{11}a_{22}a_{33}} + \underbrace{a_{12}a_{23}a_{31}} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - \underbrace{a_{12}a_{21}a_{33}} - \underbrace{a_{11}a_{23}a_{32}}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j_2 j_3 \cdots j_n} (-1)^{\tau(j_1, \dots, j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

$$= a_{11} \sum_{j_2 j_3 \cdots j_n} (-1)^{\tau(1 j_2 j_3 \cdots j_n)} a_{2j_2} \cdots a_{nj_n}$$

$$+ a_{12} \sum_{j_2 j_3 \cdots j_n} (-1)^{\tau(2 j_2 \cdots j_n)} a_{2j_2} \cdots a_{nj_n}$$

$$+ \cdots + a_{1n} \sum_{j_2 \cdots j_n} (-1)^{\tau(n j_2 \cdots j_n)} a_{2j_2} \cdots a_{nj_n}$$

$$= a_{11} \sum_{j_2 \dots j_n} (-1)^{\tau(j_2 \dots j_n)} a_{2j_2} \dots a_{nj_n} + a_{12} (-1) \sum_{j_2 \dots j_n} (-1)^{\tau(j_2 \dots j_n)} a_{2j_2} \dots a_{nj_n} \\ + \dots + a_{1n} (-1)^{n+1} \sum_{j_2 \dots j_n} (-1)^{\tau(j_2 \dots j_n)} a_{2j_2} \dots a_{nj_n}.$$

定义: n 阶行列式中, 划掉第 i 行第 j 列元素, 将余下元素按原来次序构成 $(n-1)$ 阶行列式称为

第 i 行第 j 列元素的余子式 M_{ij} .

称 $(-1)^{i+j} M_{ij} = A_{ij}$ 为 a_{ij} 的代数余子式.

Note 1: $A = (a_{ij})_{n \times n}$

$$|A| = a_{11} A_{11} + a_{12} A_{12} + \dots + a_{1n} A_{1n}$$

Note 2: A_{ij} 只与位置有关, 与 a_{ij} 无关

例 1: 求 $\begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 0 & 2 & 1 \\ 3 & -1 & 0 & 1 \end{vmatrix}$ 的代数余子式 A_{11}, A_{12}, A_{13}

$$A_{11} = \begin{vmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix}$$

$$A_{12} = - \begin{vmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \\ 3 & 0 & 1 \end{vmatrix}$$

$$10) 2. \begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & -1 \\ -1 & 0 & 2 & 1 \\ 3 & -1 & 0 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 1 & -1 \\ -1 & 2 & 1 \\ 3 & 0 & 1 \end{vmatrix} \\ + 0 \times \begin{vmatrix} 1 & 3 & 1 \\ -1 & 0 & 2 \\ 3 & -1 & 0 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & 3 & 1 \\ -1 & 0 & 2 \\ 3 & -1 & 0 \end{vmatrix}$$

結果:
 $= -42.$

$$10) 3: \text{求} \begin{vmatrix} 5 & 3 & -1 & 2 & 0 \\ 1 & 7 & 2 & 5 & 2 \\ 0 & -2 & 3 & 1 & 0 \\ 0 & -4 & -1 & 4 & 0 \\ 0 & 2 & 3 & 5 & 0 \end{vmatrix} \\ = \begin{vmatrix} 0 & 5 & 3 & -1 & 2 \\ 2 & 1 & 7 & 2 & 5 \\ 0 & 0 & -2 & 3 & 1 \\ 0 & 0 & -4 & -1 & 4 \\ 0 & 0 & 2 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 0 & 5 & 3 & -1 & 2 \\ 5 & 1 & 7 & -2 & -4 \\ 3 & 7 & -2 & -4 & 2 \\ -1 & 2 & 3 & -1 & 3 \\ 2 & 5 & 1 & 4 & 5 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 5 & 0 & 0 & 0 \\ 3 & -2 & -4 & 2 \\ -1 & 3 & -1 & 3 \\ 2 & 1 & 4 & 5 \end{vmatrix} = -2 \times 5 \begin{vmatrix} -2 & -4 & 2 \\ 3 & -1 & 3 \\ 1 & 4 & 5 \end{vmatrix}$$

$$= -10 \begin{vmatrix} 0 & 0 & 3 \\ 6 & 5 & 3 \\ 6 & 14 & 5 \end{vmatrix}$$

$$= -10 \times 2 \begin{vmatrix} 6 & 5 \\ 6 & 14 \end{vmatrix} = -20 \times 9 \times 6 = -1080$$

$$\begin{aligned}
 \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \end{vmatrix} &= \sum_{j_1 j_2 \dots j_n} (-1)^{\tau(j_1 \dots j_n)} a_{ij_1} a_{2j_2} \dots a_{ij_i}^{\uparrow i} \dots a_{nj_n} \\
 &= a_{i1} \sum_{j_1 \dots j_{i-1} 1 j_{i+1} \dots j_n} (-1)^{\tau(j_1 \dots j_{i-1} 1 j_{i+1} \dots j_n)} a_{1j_1} \dots a_{i-1,j_{i-1}} a_{i+1,j_{i+1}} \dots a_{nj_n} \\
 &\quad + a_{i2} \sum_{j_1 \dots j_{i-1} 2 j_{i+1} \dots j_n} (-1)^{\tau(j_1 \dots j_{i-1} 2 j_{i+1} \dots j_n)} a_{1j_1} \dots a_{i-1,j_{i-1}} a_{i+1,j_{i+1}} \dots a_{nj_n} \\
 &\quad + \dots + a_{in} \sum_{j_1 \dots j_{i-1} n j_{i+1} \dots j_n} (-1)^{\tau(j_1 \dots j_{i-1} n j_{i+1} \dots j_n)} a_{1j_1} \dots a_{i-1,j_{i-1}} a_{i+1,j_{i+1}} \dots a_{nj_n} \\
 &= a_{i1} (-1)^{i+1} M_{i1} + a_{i2} (-1)^{i+2} M_{i2} + \dots + a_{in} (-1)^{i+n} M_{in} \\
 &= a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{in} A_{in}
 \end{aligned}$$

Note: જો 3x3 નો હોય.

$$|A| = a_{1i} A_{1i} + a_{2i} A_{2i} + \dots + a_{ni} A_{ni}.$$

$$\text{Ex 4: } \begin{vmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 3 & 1 \\ 0 & 2 & 0 & -2 \\ 1 & 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 2 \\ -2 & 1 & 3 & 1 \\ 0 & 2 & 0 & -2 \\ 1 & 3 & 4 & 1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & -1 & 2 \\ -2 & 3 & 2 \\ 1 & 4 & 1 \end{vmatrix} = \dots = 62.$$

$$\text{例 5: } \begin{vmatrix} a-6 & 2 & -2 \\ 2 & a-3 & -4 \\ -2 & -4 & a-3 \end{vmatrix} =$$

$$\text{例 6: } \begin{vmatrix} a & b & 0 & \dots & 0 \\ 0 & a & b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a \end{vmatrix}_n = a \begin{vmatrix} a & b & & \\ & b & & \\ & & \ddots & \\ & & & a \end{vmatrix}_{n-1} + (-1)^{n+1} b \begin{vmatrix} b & & & \\ a & & & \\ & \ddots & & \\ & & a & \end{vmatrix}_{n-1}$$

$$= \begin{cases} a^n + (-1)^{n+1} b^n & (n \geq 2) \\ a & n=1 \end{cases}$$

$$\text{例 7: } D_n = \begin{vmatrix} \alpha+\beta & -\alpha\beta & 0 & \dots & 0 & 0 & 0 \\ 1 & \alpha+\beta & \alpha\beta & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \alpha+\beta & \alpha\beta \\ 0 & 0 & 0 & \dots & 0 & 1 & \alpha+\beta \end{vmatrix}$$

$$D_1 = \alpha+\beta, \quad D_2 = \begin{vmatrix} \alpha+\beta & \alpha\beta \\ 1 & \alpha+\beta \end{vmatrix} = \alpha^2 + \alpha\beta + \beta^2$$

$$\text{当 } n \geq 3 \text{ 时: } D_n = (\alpha+\beta) D_{n-1} - \alpha\beta \begin{vmatrix} 1 & \alpha\beta & & \\ 0 & \alpha+\beta & \alpha\beta & \\ \vdots & 1 & \alpha+\beta & \alpha\beta \\ 0 & & & 1 & \alpha+\beta \end{vmatrix}_{n-1}$$

$$D_n = (\alpha+\beta) D_{n-1} - \alpha\beta D_{n-2}$$

$$D_n - \alpha D_{n-1} = \beta (D_{n-1} - \alpha D_{n-2})$$

$$D_n - \alpha D_{n-1} = (D_2 - \alpha D_1) \beta^{n-2} = \beta^n$$

$$\beta = 0 \text{ 时 } D_n = \alpha D_{n-1} = D_1 \alpha^{n-1} = \alpha^n$$

$$\beta \neq 0 \text{ 时 } \frac{D_n}{\beta^n} - \frac{\alpha}{\beta} \frac{D_{n-1}}{\beta^{n-1}} = 1$$

$$\text{令 } b_n = \frac{D_n}{\beta^n} \quad b_n = \frac{\alpha}{\beta} b_{n-1} + 1$$

$$b_n + A = \frac{\alpha}{\beta} (b_{n-1} + A)$$

$$\Rightarrow \left(\frac{\alpha}{\beta} - 1\right) A = 1 \quad \Rightarrow A = \frac{\beta}{\alpha - \beta}$$

$$\text{当 } \alpha = \beta \text{ 时: } b_n = b_1 + (n-1) \cdot 1$$

$$\frac{D_n}{\beta^n} = \frac{\alpha + \beta}{\beta^1} + (n-1) = n+1, \quad D_n = (n+1) \beta^n$$

$$\text{当 } \alpha \neq \beta \text{ 时: } b_n + A = (b_1 + A) \left(\frac{\alpha}{\beta}\right)^{n-1}$$

$$\frac{D_n}{\beta^n} + \frac{\beta}{\alpha - \beta} = \left(\frac{\alpha + \beta}{\beta} + \frac{\beta}{\alpha - \beta}\right) \left(\frac{\alpha}{\beta}\right)^{n-1}$$

$$= \frac{\alpha^2}{\beta(\alpha - \beta)} \left(\frac{\alpha}{\beta}\right)^{n-1} = \frac{\alpha^{n+1}}{\beta^n(\alpha - \beta)}$$

$$\frac{D_n}{\beta^n} = \frac{\alpha^{n+1}}{\beta^n(\alpha - \beta)} - \frac{\beta^{n+1}}{(\alpha - \beta)\beta^n} = \frac{\alpha^{n+1} - \beta^{n+1}}{(\alpha - \beta)\beta^n}$$

$$D_n = \begin{cases} (n+1) \alpha^n & \alpha = \beta \\ \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} & \alpha \neq \beta \end{cases}$$

当 $\beta = 0, n=1, 2$ 时

均成立

$$\text{例 8. } \begin{vmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & x \end{vmatrix}_n = \begin{vmatrix} 1 & a & a & \cdots & a \\ 0 & x & a & \cdots & a \\ 0 & a & x & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a & a & \cdots & x \end{vmatrix}_{n+1}$$

$$= \begin{vmatrix} 1 & a & a & \cdots & a \\ -1 & x-a & & & \\ -1 & & x-a & & \\ \vdots & & & \ddots & \\ -1 & & & & x-a \end{vmatrix} = \begin{vmatrix} 1+n\frac{a}{x-a} & a & a & \cdots & a \\ 0 & x-a & & & \\ 0 & & x-a & & \\ \vdots & & & \ddots & \\ 0 & & & & x-a \end{vmatrix}$$

$$= (x-a)^n \left(1 + n \frac{a}{x-a} \right) = (x-a)^{n-1} (x + (n-1)a)$$

"加j"

$$\text{例 9. } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \times A_{11} + 2 \times A_{12} + 3 \times A_{13}$$

$$\text{计算: } A_{11} - A_{12} + A_{13} = \begin{cases} \text{计算} \\ \begin{vmatrix} 1 & -1 & 1 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \end{cases}$$

$$4A_{11} + 5A_{12} + 6A_{13} = \begin{vmatrix} \underline{4} & 5 & 6 \\ 4 & 5 & 6 \\ \underline{7} & 8 & 9 \end{vmatrix} = 0$$

$$\text{结论: } a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} |A| & i=j \\ 0 & i \neq j \end{cases}$$

$$M_{11} + 3M_{12} + 4M_{13} = \underline{A_{11} - 3A_{12} + 4A_{13}}$$

二. 求解线性方程组: Cramer 法

n 个方程 in n 个未知数的方程组

$$(*) \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases} \quad \text{let } A = (a_{ij})_{n \times n} \\ \tilde{A} = \begin{pmatrix} A & \begin{smallmatrix} b_1 \\ \vdots \\ b_n \end{smallmatrix} \end{pmatrix}$$

$$\tilde{A} = \left(A \mid \begin{smallmatrix} b_1 \\ \vdots \\ b_n \end{smallmatrix} \right) \longrightarrow \left[\begin{array}{cccc|cccc} c_{11} & \dots & c_{1j-1} & c_{1j} & \dots & c_{1j+1} & \dots & c_{1n} & d_1 \\ 0 & \dots & 0 & c_{2j} & \dots & c_{2j+1} & \dots & c_{2n} & d_2 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & d_r \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & & 0 & 0 & & 0 & & 0 & 0 \end{array} \right]$$

$$|A| = \det J \quad \det J \neq 0 \quad \tilde{J} = \left[J \mid \begin{smallmatrix} d_1 \\ \vdots \\ d_n \end{smallmatrix} \right]$$

Case 1: 若 $|A| \neq 0$ 时: $|J| \neq 0 \Rightarrow J$ 没有零行

$$A \longrightarrow \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ & c_{22} & \dots & c_{2n} \\ & & \ddots & \\ & & & c_{nn} \end{bmatrix} \quad c_{ii} \neq 0$$

方程组有唯一解。

Case 2: $|A|=0$ 时: $|J|=0$ (必有重根) $\Rightarrow r < n$

$a_{r+1} \neq 0$ 无解.

$a_{r+1} = 0$ 无穷多解

定理: n 个方程 n 元线性方程组 当 $|A| \neq 0$ 时有唯一解. 当 $|A| = 0$ 时无解. 或无穷多解

推论: n 个方程 n 元齐次方程组

有唯一解 (只有零解) $\Leftrightarrow |A| \neq 0$

有无穷多解. $\Leftrightarrow |A| = 0$

补充: 行列式的求导法则. 设 f, g 可导 则

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

若 $f_1, f_2, \dots, f_n$ 均可导 则

$$(f_1 f_2 \dots f_n)' = f_1' f_2 \dots f_n + f_1 f_2' f_3 \dots f_n + \dots + f_1 f_2 \dots f_{n-1} f_n'$$

若 $a_{ij}(x)$ 为可导函数

$$f(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{vmatrix} = \sum_{j_1 \cdots j_n} a_{1j_1}(x) a_{2j_2}(x) \cdots a_{nj_n}(x)$$

$$\frac{df(x)}{dx} = \sum_{j_1 \cdots j_n} a'_{1j_1}(x) a_{2j_2}(x) \cdots a_{nj_n}(x) + \sum_{j_1 \cdots j_n} a_{1j_1}(x) a'_{2j_2}(x) a_{3j_3}(x) \cdots a_{nj_n}(x)$$

$$+ \cdots + \sum_{j_1 \cdots j_n} a_{1j_1}(x) \cdots a_{n-1j_{n-1}}(x) a'_{nj_n}(x)$$

$$= \begin{vmatrix} a'_{11}(x) & a'_{12}(x) & \cdots & a'_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{vmatrix} + \begin{vmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a'_{21}(x) & a'_{22}(x) & \cdots & a'_{2n}(x) \\ \vdots & \vdots & & \vdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{vmatrix}$$

$$+ \cdots + \begin{vmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \vdots & \vdots & & \vdots \\ a'_{n1}(x) & a'_{n2}(x) & \cdots & a'_{nn}(x) \end{vmatrix}$$

1.

$$D_n = \begin{vmatrix} x & b & b & \dots & b \\ c & x & b & \dots & b \\ c & c & x & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c & c & c & \dots & x \end{vmatrix} = \begin{vmatrix} x-c & b & b & \dots & b \\ 0 & x & b & \dots & b \\ 0 & c & x & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & c & c & \dots & x \end{vmatrix} +$$

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$$\begin{vmatrix} c & b & b & \dots & b \\ c & x & b & \dots & b \\ c & c & x & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c & c & c & \dots & x \end{vmatrix}$$

$$= (x-c) D_{n-1} + \begin{vmatrix} c & b & b & \dots & b \\ 0 & x-b & 0 & \dots & 0 \\ 0 & c-b & x-b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & c-b & c-b & \dots & x-b \end{vmatrix}.$$

$$\left. \begin{aligned} D_n &= (x-c) D_{n-1} + c(x-b)^{n-1} \\ D_n &= (x-b) D_{n-1} + b(x-c)^{n-1} \end{aligned} \right\}$$

$$b=c \quad \checkmark$$

$$b \neq c$$

2. 范德蒙行列式.

$$a_1, a_2, \dots, a_n \in \mathbb{R}$$

$$f(a_1, a_2, \dots, a_n) = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-2} & a_2^{n-2} & a_3^{n-2} & \dots & a_n^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \dots & a_n^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & a_2 - a_1 & a_3 - a_1 & \dots & a_n - a_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_2^{n-2} - a_1 a_2^{n-3} & a_3^{n-2} - a_1 a_3^{n-3} & \dots & a_n^{n-2} - a_1 a_n^{n-3} \\ 0 & a_2^{n-1} - a_1 a_2^{n-2} & a_3^{n-1} - a_1 a_3^{n-2} & \dots & a_n^{n-1} - a_1 a_n^{n-2} \end{vmatrix}$$

$$= \begin{vmatrix} a_2 - a_1 & a_3 - a_1 & \dots & a_n - a_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_2^{n-2} - a_1 a_2^{n-3} & a_3^{n-2} - a_1 a_3^{n-3} & \dots & a_n^{n-2} - a_1 a_n^{n-3} \\ a_2^{n-1} - a_1 a_2^{n-2} & a_3^{n-1} - a_1 a_3^{n-2} & \dots & a_n^{n-1} - a_1 a_n^{n-2} \end{vmatrix}$$

$$= \underbrace{(a_2 - a_1)(a_3 - a_1) \dots (a_n - a_1)}_{\text{Factor out } (a_j - a_1)} \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_2 & a_3 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_2^{n-2} & a_3^{n-2} & \dots & a_n^{n-2} \end{vmatrix}_{n-1}$$

$$= \left[\prod_{j=2}^n (a_j - a_1) \right] \left[\prod_{j=3}^n (a_j - a_2) \right] \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_3 & a_4 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_3^{n-3} & a_4^{n-3} & \dots & a_n^{n-3} \end{vmatrix}_{n-2}$$

$$= \prod_{1 \leq i < j \leq n} (a_j - a_i)$$

$$\begin{vmatrix} 1 & 1 \\ a_{n-1} & a_n \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix} = (a_3 - a_1)(a_3 - a_2)(a_2 - a_1)$$

$$3. \begin{array}{|cc|} \hline A_k & O_{k \times t} \\ \hline C_{t \times k} & B_t \\ \hline \end{array} \quad (k+t)$$

$$= \begin{vmatrix} * & D_1 \\ * & D_2 \end{vmatrix} = |D_1| |D_2|$$

$$= |A| |B|$$

P_{14} 例 10.
 P_{19} 例 12
 P_{21} 例 14, 15

$\left. \begin{array}{l} P_{14} \\ P_{19} \\ P_{21} \end{array} \right\} A''$

第二章 矩阵

1. 由 $m \times n$ 个数排成数表 $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

$$A = (a_{ij})_{m \times n} \quad A \in M_{m \times n}(\mathbb{R}) \quad A \in M_n(\mathbb{R})$$

$$A \in M_{m \times n}(\mathbb{C})$$

Note 1: $n \times 1$ 矩阵

$1 \times n$ 矩阵

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ 列向量.}$$

$$[b_1, b_2, \dots, b_m] \text{ 行向量.}$$

2. 零矩阵: $O_{m \times n}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq [0, 0]$$

二阶 $m=n$

$$\begin{cases} \text{上三角阵} \begin{bmatrix} a_{11} & \cdots & * \\ & & a_{nn} \end{bmatrix} \\ \text{下三角阵} \\ \text{对角阵} \begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & \ddots \\ & & & a_{nn} \end{bmatrix} = \text{diag}(a_{11}, a_{22}, \dots, a_{nn}) \\ \text{数量矩阵} \begin{bmatrix} c & & \\ & c & \\ & & \ddots \\ & & & c \end{bmatrix} = cI \\ \text{(数量矩阵)} \\ \text{单位矩阵} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} = I, E \end{cases}$$

初等矩阵 E_{ij} (基初等阵)

$$E_{ij} = \begin{bmatrix} & & \\ & 1 & \\ & & \\ & & \\ & & \\ & & \\ & & \\ & & \\ & & \\ & & \end{bmatrix} \begin{matrix} \rightarrow i \\ \downarrow j \end{matrix}$$

二. 矩阵运算:

1. (1) 同型矩阵: 两矩阵 E_j 的行数和列数分别相等. 则称为同型矩阵.

(2) 矩阵相等: 两个同型矩阵. 对应位置之元素相等

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ 0 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 & x+y \\ 1 & y+z & -1 \\ 2+x & 3 & 1 \end{bmatrix}$$

$A=B$ 求 $x+y+z$.

2. 加法: $A = (a_{ij})_{m \times n}$ $B = (b_{ij})_{m \times n}$

$C = A + B = (c_{ij})_{m \times n}$ 其中 $c_{ij} = a_{ij} + b_{ij}$

3. 数乘: $k \in \mathbb{R}$. $kA = (ka_{ij})_{m \times n}$ 23/12/21

4. 减法: $A - B = A + (-B) = (a_{ij} - b_{ij})_{m \times n}$

称 $-1 \cdot B$ 为 B 的负矩阵, 记为 $-B$.

$B + (-B) = (-B) + B = O$.

易证: { 加法: 交换律 $A + B = B + A$
结合律 $(A + B) + C = A + (B + C)$
零元 $A + O = O + A = A$
负元 $A + (-A) = O$
数乘 { $1 \cdot A = A$ $0 \cdot A = O$
 $k(lA) = (kl)A$

分配律 { $k(A + B) = kA + kB$
 $(k + l)A = kA + lA$

5. 转置: $A = (a_{ij})_{m \times n}$ T : transpose.

则 $A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$ $A^T \in M_{n,m}(\mathbb{R})$

① $(A^T)^T = A$ ③ $(kA)^T = kA^T$

② $(A + B)^T = A^T + B^T$

Note 1: 若 $A^T = A$ (A 为矩阵) 则 A 为对称矩阵

若 $A^T = -A$ $\dots\dots$ 反对称矩阵

反对称矩阵对角线全为 0

2. 对称矩阵中

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{bmatrix}$$

A 为 n 阶对称矩阵.

确定 A 需给出 $n + (n-1) + \dots + 1 = \frac{n(n+1)}{2}$

A 为反对称 n 阶矩阵. 确定 A 需给出 $(n-1) + (n-2) + \dots + 1 + 0 = \frac{(n-1)n}{2}$.

例 1: (反)对称矩阵和仍为 (反)对称矩阵

若 $A^T = A$, $B^T = B$, $A, B \in M_{m,n}(\mathbb{R})$

$$\underline{(A+B)^T = A^T + B^T = A + B}$$

2. 任何一矩阵均可分解为一个对称矩阵与一个反对称矩阵之和.

$$A = \underbrace{\frac{A+A^T}{2}}_B + \underbrace{\frac{A-A^T}{2}}_C$$

$$C^T = \frac{1}{2} (A - A^T)^T$$

$$= \frac{1}{2} (A^T - A) = -C.$$

$$B = \frac{A+A^T}{2}$$

$$\begin{aligned} B^T &= \frac{1}{2} (A+A^T)^T \\ &= \frac{1}{2} (A^T + A) = B \end{aligned}$$

6. 矩阵乘法

① 来源.

$$\begin{cases} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n = y_1 \\ b_{21}x_1 + b_{22}x_2 + \dots + b_{2n}x_n = y_2 \\ \vdots \\ b_{m1}x_1 + b_{m2}x_2 + \dots + b_{mn}x_n = y_m \end{cases}$$

$$B = (b_{ij})_{m \times n}$$

而 $y_1, y_2, \dots, y_m$ 由 Q 个方程组成

$$\begin{cases} a_{11}y_1 + a_{12}y_2 + \dots + a_{1m}y_m = l_1 \\ a_{21}y_1 + a_{22}y_2 + \dots + a_{2m}y_m = l_2 \\ \vdots \\ a_{t1}y_1 + a_{t2}y_2 + \dots + a_{tm}y_m = l_t \end{cases}$$

$$A = (a_{ij})_{t \times m}$$

$$a_{t1}y_1 + a_{t2}y_2 + \dots + a_{tm}y_m = l_t$$

$$\begin{cases} (a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1m}b_{m1})x_1 + (a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1m}b_{m2})x_2 \\ + \dots + (a_{11}b_{1n} + a_{12}b_{2n} + \dots + a_{1m}b_{mn})x_n = l_1 \\ \vdots \\ \dots \end{cases}$$

$$A = (a_{ij})_{t \times m}$$

$$B = (b_{ij})_{m \times n}$$

$$C = (AB)_{t \times n}$$

$$C_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{im}b_{mj}$$

Note: 1. 矩阵乘法不可交换.

$$\begin{bmatrix} \text{---} \\ \vdots \end{bmatrix}_i \begin{bmatrix} | \\ \vdots \end{bmatrix}_j = \begin{bmatrix} \text{---} \end{bmatrix}_i \begin{bmatrix} \downarrow \\ \vdots \end{bmatrix}_j$$

2. 矩阵乘法不可交换. $A_{1 \times 3} B_{3 \times 2}$

$$2. \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}$$

$$AX = b$$

$$\begin{cases} BX = Y \\ AY = L \end{cases} \quad \textcircled{AB}X = L$$

$$\text{例: } A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ 0 & 3 & 1 \end{bmatrix}_{3 \times 3} \quad B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}_{3 \times 1}$$

$$\text{求 } AB = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$$

$$\text{例: } A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{2 \times 1} \quad b = \begin{bmatrix} b_1 & b_2 \end{bmatrix}_{1 \times 2}$$

$$AB = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix}$$

$$BA = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1 b_1 + a_2 b_2$$

$$AA^T = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_2 a_1 & a_2^2 \end{bmatrix}$$

$$\text{例3: } A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Note: $AB = O$. A, B 不一定为零矩阵

$$BA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$$

Note: $AB \neq BA$

$$\text{例4: } A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 3 \\ -2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -7 & 1 \\ 1 & 2 \end{bmatrix}$$

求 AB, AC .

$$AB = \begin{bmatrix} -5 & 5 \\ -10 & 10 \end{bmatrix} = AC$$

Note: $AB = AC \not\Rightarrow B = C$

② 矩阵运算性质:

① 不满足交换律 $AB \neq BA$

② 不满足消去律 $AB = AC \not\Rightarrow B = C$

③ $AB = O \not\Rightarrow A = O$ 或 $B = O$

正常性质: $O \cdot A = O \quad A \cdot O = O$

$I \cdot A = A \quad A \cdot I = A$

反常
性质

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}_m \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$A(B+C) = AB + AC.$$

$$(B+C)A = BA + CA$$

$$\underline{A(BC) = (AB)C}$$

$$\underline{(AB)^T = B^T A^T}.$$

↓

证: 设 $A_{m \times n}$ $B_{n \times s}$ $C_{s \times t}$ $A = (a_{ij})$ $B = (b_{ij})$
 $C = (c_{ij}).$

$A(BC)$ 与 $(AB)C$ 均为 $m \times t$ 矩阵

$A(BC)$ 中第 i 行第 j 列元素

$$a_{i1}l_{1j} + a_{i2}l_{2j} + \cdots + a_{in}l_{nj} = \sum_{k=1}^n a_{ik}l_{kj}$$

其中 $l_{1j}, l_{2j}, \dots, l_{nj}$ 为 BC 中第 j 列元素

$$l_{kj} = b_{k1}c_{1j} + b_{k2}c_{2j} + \cdots + b_{ks}c_{sj}$$

$B_{n \times s}$

$C_{s \times t}$

$$= \sum_{p=1}^s b_{kp}c_{pj}$$

$A(BC)$ 中第 i 行第 j 列元素为 $\sum_{k=1}^n a_{ik} \sum_{p=1}^s b_{kp}c_{pj}$

$$= \sum_{k=1}^n \sum_{p=1}^s a_{ik}b_{kp}c_{pj}$$

$(AB)C$ 中第 i 行第 j 列元素

$A_{m \times n}$ $B_{n \times s}$

$$w_{i1}c_{1j} + w_{i2}c_{2j} + \cdots + w_{is}c_{sj}$$

$(AB)_{m \times s}$

$$= \sum_{p=1}^S w_{ip} C_{pj} \quad w_{i1} \dots w_{is} \text{ 为 } AB \text{ 第 } i \text{ 行}$$

$$w_{ip} = a_{i1}b_{1p} + a_{i2}b_{2p} + \dots + a_{in}b_{np} \quad \begin{matrix} A_{m \times n} \\ B_{n \times s} \end{matrix}$$

$$= \sum_{k=1}^n a_{ik} b_{kp}$$

$$(AB)C \text{ 第 } i \text{ 行第 } j \text{ 列元素为 } \sum_{p=1}^S \left(\sum_{k=1}^n a_{ik} b_{kp} \right) C_{pj}$$

$$= \sum_{p=1}^S \sum_{k=1}^n a_{ik} b_{kp} C_{pj}$$

$$(2) \quad (AB)^T = B^T A^T$$

$(A_{m \times n} B_{n \times s})^T$ 第 i 行第 j 列元素

为 AB 中第 j 行第 i 列元素 $\sum_{k=1}^n a_{jk} b_{ki}$

$B^T A^T$ 中第 i 行第 j 列元素 $\sum_{k=1}^n b_{ik} a_{kj}$

$\downarrow$ $s \times n$ $\downarrow$ $n \times m$

$\sum_{k=1}^n b_{ki} a_{jk}$

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$$(AB)^T = B^T A^T$$

便笺标题

2015/10/22

例: $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 7 & -1 \\ 4 & 2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$ 求 $(AB)^T$

$$(AB)^T = B^T A^T = \begin{bmatrix} 1 & 4 & 2 \\ 7 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 17 \\ 14 & 13 \\ -3 & 10 \end{bmatrix}$$

Note: ^(R) 对称矩阵的乘积不一定是对称 ^(R)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$CD = \begin{bmatrix} 0 & 0 & \textcircled{1} \\ 0 & \times & \times \\ \textcircled{1} & \times & \times \end{bmatrix}$$

(2) 上三角阵与上三角阵乘积仍为上三角

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ & b_{22} & \dots & b_{2n} \\ & & \ddots & \\ & & & b_{nn} \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$(4) \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} A_{n \times s} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ a_{21} & a_{22} & \dots & a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{ns} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 a_{11} & \lambda_1 a_{12} & \dots & \lambda_1 a_{1s} \\ \lambda_2 a_{21} & \lambda_2 a_{22} & \dots & \lambda_2 a_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n a_{n1} & \lambda_n a_{n2} & \dots & \lambda_n a_{ns} \end{bmatrix}$$

$$A \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_s \end{bmatrix}$$

1. 矩阵乘方:

$$A^n = \underbrace{A \cdot A \cdot \dots \cdot A}_{n \uparrow}$$

(见 P5)

$A_{m \times n} \quad A_{m \times n}$

设 $f(x)$ 为 x 的多项式: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

($a_n \neq 0$)

则 $f(A) = a_n \underline{A^n} + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I$

Note 1: $A^k \cdot A^l = A^{k+l} \quad (A^k)^l = A^{kl} \quad AI = IA = A$

$A^l \cdot A^k = A^{l+k} \quad k, l \in \mathbb{N}^*$

2. ~~交换律~~ AB 与 BA 为 $n \times n$ 矩阵. $AB \neq BA$

$(AB)^k = \underbrace{AB \cdot AB \cdot AB \dots AB}_{k \uparrow} \neq A^k B^k$

例: $f(A)g(A) = g(A)f(A)$

3. 若 $AB=BA$ 则 A, B 是可交换的.

此时 $f(A)g(B) = g(B)f(A)$

$$\cancel{(A+B)^n} \neq C_n^0 A^n + C_n^1 A^{n-1} B + C_n^2 A^{n-2} B^2 + \dots + B^n$$

$$(a+b)^n = (a+b)(a+b)\dots(a+b)$$

$$= C_n^0 a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \dots + b^n$$

$$(A+B)(A-B) = A^2 - AB + BA - B^2$$

2. 3. $P_{ij} \in \mathbb{R}^{n \times n}$. $A \in M_n(\mathbb{R})$

$\det: A \rightarrow \mathbb{R}$ — n^2 元の多項式

$$|A| = \det A.$$

$$\textcircled{1} |A+B| \neq |A| + |B|$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\textcircled{2} |kA| = k^n |A|$$

$$|A|B| = |A|^n |B|$$

例 1: $\alpha = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$ $A = E - \alpha\alpha^T$ $B = E + 2\alpha\alpha^T$

$\alpha_{4 \times 1}$ $\alpha_{1 \times 4}^T$ 求 AB

$\alpha \alpha^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$$AB = (E - \alpha\alpha^T)(E + 2\alpha\alpha^T) = E + 2\alpha\alpha^T - \alpha\alpha^T - 2\alpha\alpha^T \alpha\alpha^T$$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$
 4×1 1×4 4×1 1×4

$$= E + 2\alpha\alpha^T - \alpha\alpha^T - 2\alpha\alpha^T$$

$$= E - \alpha\alpha^T = E - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Ex 2: $\alpha = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $\beta = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}$ $A = \alpha \beta^T$ find A^4

$$A^4 = \underbrace{\alpha \beta^T \alpha}_{\beta^T \alpha} \beta^T \alpha \beta^T$$

$$= 8 \alpha \beta^T = \checkmark$$

$$\beta^T \alpha = [1 \ \frac{1}{2} \ 0] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 2$$

Ex 3: $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ find A^n .

$$A = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \cdots \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$= 7^{n-1} A = 7^{n-1} \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Ex 4: β is 3×1 vector $\beta \beta^T = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}$

find $\beta^T \beta = 6$.

$$\beta = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \beta^T = [x_1 \ x_2 \ x_3] \quad (\beta \beta^T)^T = \beta \beta^T$$

$$\beta \beta^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1 x_2 & x_1 x_3 \\ x_2 x_1 & x_2^2 & x_2 x_3 \\ x_3 x_1 & x_3 x_2 & x_3^2 \end{bmatrix}$$

$$\beta^T \beta = [x_1 \ x_2 \ x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1^2 + x_2^2 + x_3^2$$

4. $(x_1, x_2, x_3) \begin{bmatrix} 1 & 0 & -2 \\ 2 & 4 & -1 \\ 0 & -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ 求 $x_2 x_3$ 的值.

6. $\forall b \in M_{n \times 1}(\mathbb{R}) \quad Ab = 0$ 求证: $A = 0$

$$b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

7. $A = [\gamma, \alpha_1, \alpha_2, \alpha_3]$ $B = [\beta, \alpha_1, \alpha_2, \alpha_3]$
 $\alpha_1, \alpha_2, \alpha_3, \beta$ 为 4 个线性无关向量 $|A| = -1, |B| = -5$.
 求 $|A+B|$

解: $|A+B| = |\gamma + \beta, 2\alpha_1, 2\alpha_2, 2\alpha_3|$
 $= 8|\gamma + \beta, \alpha_1, \alpha_2, \alpha_3| = 8(|\gamma, \alpha_1, \alpha_2, \alpha_3| + |\beta, \alpha_1, \alpha_2, \alpha_3|)$
 $= 8(-1 - 5) = -48$

8. $\alpha_1, \alpha_2, \alpha_3, \beta, \gamma$ 为 5 个线性无关向量

若 $|\alpha_1, \alpha_2, \alpha_3, \gamma| = a$ $|\beta + \gamma, \alpha_1, \alpha_2, \alpha_3| = b$

求 $|2\beta, \alpha_1, \alpha_2, \alpha_3|$

$$|2\beta, \alpha_1, \alpha_2, \alpha_3| = 2 |\beta, \alpha_1, \alpha_2, \alpha_3| = 2(b+a).$$

$$|\gamma, \alpha_1, \alpha_2, \alpha_3| = -a.$$

§6. 逆矩阵.

1. 例: $ax = b \quad a \neq 0$

$$\left(\frac{1}{a}\right) \cdot a x = \left(\frac{1}{a}\right) \cdot b \Rightarrow x = \frac{b}{a}.$$

$$A_n X_{n \times 1} = b_{n \times 1}$$

1. 定义: $A \in M_n(\mathbb{R})$. 若 $\exists B \in M_n(\mathbb{R})$. s.t.

$AB = BA = I$ 则称 A 是可逆矩阵或非奇异矩阵, 称 B 是 A 的逆矩阵 记作 A^{-1}

否则称 A 不可逆或奇异

Note: ① 验证: $AB = BA = I, \quad AC = CA = I,$

$$B = BI = \underbrace{BAC}_{=I} = IC = C$$

② 逆矩阵是互逆的.

$$AB = BA = I \quad BA = AB = I.$$

$$\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \frac{1}{\lambda_2} & \\ & & \ddots \\ & & & \frac{1}{\lambda_n} \end{bmatrix}$$

$$\lambda_i \neq 0$$

2. 可逆三要素与逆矩阵

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 1 \end{bmatrix}$$

阵形式:

① 若 A 可逆. 则

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 10 \\ 1 \end{bmatrix}$$

$AX=b$ 有唯一解. $(A^{-1}AX=A^{-1}b)$

$\Rightarrow |A| \neq 0$

$X=A^{-1}b$

② $|A| \neq 0$ $AX=b$ 有唯一解

$$AX=b \Rightarrow x_j = \frac{|B_j|}{|A|} = \frac{b_1 A_{1j} + b_2 A_{2j} + \dots + b_n A_{nj}}{|A|}$$

$$B_j = \begin{bmatrix} a_{11} \dots a_{1j-1} & b_1 & a_{1j+1} \dots a_{1n} \\ a_{21} \dots a_{2j-1} & b_2 & a_{2j+1} \dots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} \dots a_{nj-1} & b_n & a_{nj+1} \dots a_{nn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} b_1 A_{11} + b_2 A_{21} + \dots + b_n A_{n1} \\ b_1 A_{12} + b_2 A_{22} + \dots + b_n A_{n2} \\ \vdots \\ b_1 A_{1n} + b_2 A_{2n} + \dots + b_n A_{nn} \end{bmatrix}$$

$$= \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$AX=b$

逆矩阵 $A^* = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}$

$|A| \neq 0$.

$$\begin{aligned}
 & A \cdot \frac{A^*}{|A|} \\
 &= \frac{1}{|A|} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & & A_{nn} \end{bmatrix} \\
 &= \frac{1}{|A|} \begin{bmatrix} |A| & 0 & \dots & 0 \\ 0 & |A| & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & |A| \end{bmatrix} = I
 \end{aligned}$$

Q2. $\frac{A^*}{|A|} \cdot A = I$ (結果)

A が可逆 $\iff |A| \neq 0$

A が可逆 $A^{-1} = \frac{A^*}{|A|}$

Note 1: A が可逆 A^* も可逆 $(A^*)^{-1} = \frac{A}{|A|}$

$\left(A \cdot \frac{A^*}{|A|} \right) = \frac{A^*}{|A|} \cdot A = I$

$\frac{A}{|A|} \cdot A^* = A^* \cdot \frac{A}{|A|} = I$

2. $A \in M_n(\mathbb{R})$. A が可逆

$AB = AC \implies \underline{A^{-1}AB} = A^{-1}AC$

$\downarrow$
 $B = C$

$AX = 0 \implies A^{-1}AX = A^{-1}0 \implies X = 0$

例1: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{A^*}{|A|} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
 $ad-bc \neq 0$

例2: $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & -2 \end{bmatrix}$ 求 $A^{-1} = \frac{A^*}{|A|}$ 熟练.

证: ① 若 A 可逆 $\Rightarrow A^{-1}$ 也可逆 $(A^{-1})^{-1} = A$

② 若 A, B 均可逆 则 AB 可逆 $(AB)^{-1} = B^{-1}A^{-1}$.

$$(AB) \cdot (B^{-1}A^{-1}) = A \underline{B B^{-1}} A^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = I$$

③ $A_1 \cdots A_s$ 均可逆 $A_1 \cdots A_s$ 也可逆.

$$(A_1 \cdots A_s)^{-1} = A_s^{-1} A_{s-1}^{-1} \cdots A_1^{-1}$$

$$(A^{-1})^k = \underbrace{A^{-1} \cdot A^{-1} \cdots A^{-1}}_{k \uparrow}$$

$$A^k = A \cdot A \cdots A$$

$$A^k \cdot (A^{-1})^k = A \cdots A \cdot \underbrace{A^{-1} \cdots A^{-1}}_{k \uparrow} = I$$

$$\Rightarrow (A^{-1})^k = (A^k)^{-1} \quad \text{这又} (A^{-1})^k = A^{-k}$$

④ A 可逆 $k \neq 0$.

$\Rightarrow kA$ 也可逆 $(kA)^{-1} = \frac{1}{k} A^{-1}$.

$$(kA) \cdot \frac{1}{k} A^{-1} = I.$$

⑤ A 可逆 $\Rightarrow A^T$ 也可逆. $(A^T)^{-1} = (A^{-1})^T$.

$$\underline{A A^{-1}} = \underline{A^{-1} A} = I$$

性质 2. $\underline{(A^{-1})^T} A^T = A^T (A^{-1})^T = I$

例 1: 若 $ABC = E$ 则 $\underline{BCA} = E, \quad CAB = E$

2: $A^2 + A - 2I = 0$ 证明: A 与 $A - 2I$ 可逆

$$A \cdot (A + I) = 2I$$

$$A \cdot \frac{A+I}{2} = I$$

$$(A+I)A = 2I$$

$$\frac{A+I}{2} \cdot A = I$$

$$A^{-1} = \frac{A+I}{2}$$

$$\underline{(A-2I)(A+3I)} = \underline{A^2 - 2A + 3A - 6I} = -4I$$

$$(A-2I) \cdot \frac{A+3I}{-4} = I$$

$$\Rightarrow (A-2I)^{-1} = \frac{A+3I}{-4}$$

例 3: A^* 为 $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ 的伴随

$A^*B = A$ 求 B 的第 3 行元素之和为多少?

$$\underline{A A^*B} = A \cdot A$$

$$|A|I B = A^2$$

$$B = \frac{A^2}{|A|} = \frac{1}{|A|} A \cdot A$$

2 -

例4: $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ 求 $(A^*)^{-1} = \frac{A}{|A|} = \checkmark$

$$AA^* = A^*A = |A|I$$

- 定义: $A \in M_n(\mathbb{R})$. 若 $\exists B \in M_n(\mathbb{R})$ 使
- $AB = I$ 则 A 可逆 $\wedge A^{-1} = B$.
($BA = I$)

证明: 若 $\exists B$. 使 $AB = I$

又因为 $BX = 0$ 等价于 $(AB)X = 0$ 等价于 $I \Rightarrow X = 0$

$\Rightarrow BX = 0$ 只有零解 即 $BX \Rightarrow$ 有 n 个非零解

$$B \text{ 可逆} \cdot AB = I \Rightarrow AB \cdot B^{-1} = IB^{-1}$$

$$\Rightarrow A = B^{-1}$$

$$AB = I. \quad BA = BB^{-1} = I$$

Spec $\begin{cases} \circ A, B \in M_n(\mathbb{R}) \\ \textcircled{2} AB = I \\ \textcircled{3} BA = I \end{cases}$ 是不独立.

证明: P77 18. 19(2) 20. 22.

初等变换与初等矩阵

便笺标题

2015/10/27

$$AX=b \quad [A:b] \rightarrow \left[\begin{array}{c|c} & \\ \hline & \end{array} \right]$$

① 倍法变换 $\lambda r_i (\lambda \neq 0)$

② 加法变换 $\lambda r_i + r_j (\lambda \in \mathbb{R})$

③ 交换变换 $r_i \leftrightarrow r_j$

1. 定理: 初等行变换相当于左乘, 左乘把单位矩阵作初等变换后得到的矩阵

$$\left[\begin{array}{c|c} 1 & \\ \hline & \\ \vdots & \\ \lambda & \\ \vdots & \\ 1 & \end{array} \right]_m \left[\begin{array}{c|c} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \dots a_{mn} \end{array} \right]_{m \times n} = \left[\begin{array}{c|c} a_{11} & a_{12} \dots a_{1n} \\ \lambda a_{21} & \lambda a_{22} \dots \lambda a_{2n} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \dots a_{mn} \end{array} \right]$$

$$\left[\begin{array}{c|c} 1 & \\ \vdots & \\ i & \lambda \\ \vdots & \\ j & \\ \vdots & \\ 1 & \end{array} \right] \left[\begin{array}{c|c} a_{11} & a_{12} \dots a_{1n} \\ \vdots & \vdots \\ a_{i1} & a_{i2} \dots a_{in} \\ \vdots & \vdots \\ a_{j1} & a_{j2} \dots a_{jn} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \dots a_{mn} \end{array} \right] = \left[\begin{array}{c|c} \hline \hline \lambda a_{i1} + a_{j1} & \dots \lambda a_{in} + a_{jn} \\ \hline \hline \end{array} \right]$$

$$\left[\begin{array}{c|c} 1 & \\ \vdots & \\ i & 0 \\ \vdots & \\ j & 0 \\ \vdots & \\ 1 & \end{array} \right] \left[\begin{array}{c|c} a_{11} & a_{12} \dots a_{1n} \\ \vdots & \vdots \\ a_{i1} & a_{i2} \dots a_{in} \\ \vdots & \vdots \\ a_{j1} & a_{j2} \dots a_{jn} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \dots a_{mn} \end{array} \right] = \left[\begin{array}{c|c} a_{j1} & a_{j2} \dots a_{jn} \\ \vdots & \vdots \\ a_{i1} & a_{i2} \dots a_{in} \\ \vdots & \vdots \end{array} \right]$$

初等矩阵: $E_i(\lambda) = \left[\begin{array}{c|c} 1 & \\ \vdots & \\ \lambda & \\ \vdots & \\ 1 & \end{array} \right] \rightarrow i$

$$E_{i,j} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 0 & \cdots \\ & & \vdots & \ddots \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 0 & \cdots \\ & & & & & & \vdots & \ddots \\ & & & & & & & & 1 \end{bmatrix} \begin{matrix} \rightarrow i \\ \rightarrow j \end{matrix}$$

$\downarrow \quad \downarrow$
 $i \quad j$

$$E_{i,j}^{-1} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{matrix} \rightarrow i \\ \rightarrow j \end{matrix}$$

$\downarrow \quad \downarrow$
 $i \quad j$

$$E_{i,j}(\lambda) = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \lambda & \cdots \\ & & \vdots & \ddots \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 0 & \cdots \\ & & & & & & \vdots & \ddots \\ & & & & & & & & 1 \end{bmatrix} \begin{matrix} \rightarrow i \\ \rightarrow j \end{matrix}$$

$\downarrow \quad \downarrow$
 $i \quad j$

• 2: 初等矩阵 = 可逆: 初等矩阵均可逆

$$\lambda \neq 0, (E_i(\lambda))^T = E_i(\frac{1}{\lambda})$$

$$E_i(\frac{1}{\lambda}) \cdot E_i(\lambda) I = I \Rightarrow E_i(\frac{1}{\lambda}) E_i(\lambda) = I$$

$$(E_{i,j}(\lambda))^T = E_{i,j}(-\lambda)$$

$$E_{i,j}(-\lambda) E_{i,j}(\lambda) I = I \Rightarrow E_{i,j}(-\lambda) \cdot E_{i,j}(\lambda) = I$$

$$(E_{i,j})^T = E_{i,j}$$

$$\bullet \det(E_{i,j} A) = - \det A = \det(E_{i,j}) \det A.$$

$$\lambda \neq 0 \quad \det(E_i(\lambda) A) = \lambda \det A = \det(E_i(\lambda)) \det A$$

$$\det(E_{i,j}(\lambda) A) = 1 \det A = \underbrace{\det(E_{i,j}(\lambda))}_1 \det A$$

对于初等矩阵 P

$$|PA| = |P| |A|$$

$$|AP| = |A| |P|$$

3. 相似关系: 矩阵B可以由A经过一系列初等变换得到. 称A, B相似或A, B等价

($\exists$ 一系列初等矩阵 $P_1, P_2, \dots, P_n, Q_1, Q_2, \dots, Q_m$).

$$\underline{B = P_n \dots P_2 P_1 A Q_1 Q_2 \dots Q_m \text{ 记作 } A \sim B}$$

① 自反性: $A \sim A$

② 对称性: $A \sim B$, 则 $B \sim A$.

$$A = P_1^{-1} P_2^{-1} \dots P_n^{-1} B Q_m^{-1} Q_{m-1}^{-1} \dots Q_1^{-1}$$

③ 传递性: $A \sim B, B \sim C \Rightarrow A \sim C$

$$A = P_n \dots P_1 B Q_1 \dots Q_m$$

$$B = R_s \dots R_1 C T_1 \dots T_t$$

$$\Rightarrow A = P_n \dots P_1 R_s \dots R_1 C T_1 \dots T_t Q_1 \dots Q_m$$

定理: 任何矩阵都与形如 $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ 的矩阵相似

$$\begin{bmatrix} -1 & 0 & 1 & 1 \\ 1 & -2 & -1 & -3 \\ 2 & -1 & -2 & -3 \end{bmatrix} \xrightarrow{\text{行变换}} \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{\text{列变换}} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{行变换}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Pr: } P_m \cdots P_2 P_1 A Q_1 Q_2 \cdots Q_k = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Note: } A \sim B \Leftrightarrow r_A = r_B.$$

秩相等

r_A, r_B 分别为 A, B 经初等变换后非零行的个数

$$\text{Note: } \forall A \in M_{m,n}(\mathbb{R}) \quad \exists P \in M_m(\mathbb{R}) \quad Q \in M_n(\mathbb{R}) \quad P, Q \text{ 可逆}$$

$$\text{s.t. } PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

4. 初等变换求逆矩阵

① $A \in M_n(\mathbb{R})$ 若 A 是可逆的 则

$$P_m \cdots P_2 P_1 A = \begin{bmatrix} a_{11} & * & * & * \\ & a_{22} & * & * \\ & & \ddots & * \\ & & & a_{nn} \end{bmatrix} \quad a_{ii} \neq 0$$

$$A \Rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \xrightarrow{a_{11} \neq 0} \begin{bmatrix} a_{11} & * & * & * & * \\ 0 & & & & \\ & A_1 & & & \end{bmatrix}$$

$$\rightarrow \cdots \rightarrow \begin{bmatrix} a_{11} & * & * & * & * \\ & a_{22} & * & * & * \\ & & \ddots & * & * \\ & & & a_{n-1,n-1} & * \\ & & & & a_{nn} \end{bmatrix}$$

P A 为可逆矩阵时 A 的 (逆 - id) 初等变换
即为单位阵

$$(P_t \cdots P_2 P_1) A = I$$

$$(A^{-1}) = P_t P_{t-1} \cdots P_1 I$$

$$\cdot [A | I] \xrightarrow{\{j\}} [I | A^{-1}]$$

$$A^{-1} = \frac{A^*}{|A|}$$

Ex: $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$ 求 A^{-1}

$$[A | I] = \left[\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow[r_1 \leftrightarrow r_2]{r_2 + r_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow[r_3 \times (-2) + r_1]{r_3 + r_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right]$$

$$\xrightarrow[r_2 \times \frac{1}{2}]{r_2 \times (-1) + r_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$$

Note: $P_t \cdots P_3 P_2 P_1 A = I$ A 可逆.

$$\Rightarrow A^{-1} = P_t P_{t-1} \cdots P_2 P_1$$

$$\Rightarrow A = (P_t P_{t-1} \cdots P_2 P_1)^{-1}$$

$$= P_1^{-1} P_2^{-1} \cdots P_{t-1}^{-1} P_t^{-1}$$

例 初等矩阵连乘积仍为初等矩阵

!!! A 可逆 $\Leftrightarrow A$ 可以表写成一系列初等矩阵乘积

*: $A, B \in M_n(\mathbb{R})$

B 不可逆时: $|B|=0$, $BX=0$ 无穷多解

($BX=0$ 解 全部是 $ABX=0$ 解)

从而 $ABX=0$ 也有无穷多解 AB 不可逆

从而 $|AB|=0$

此时 $|AB| = |A| |B|$

B 可逆时: $B = P_1 P_2 \cdots P_t$ P_i 为初等矩阵.

$$AB = A P_1 P_2 \cdots P_{t-1} P_t$$

$$\det(AB) = \det(A P_1 \cdots P_{t-1} P_t)$$

$$= \det(A P_1 \cdots P_{t-1}) \det P_t$$

$$= \det A \cdot \underbrace{\det P_1 \det P_2 \cdots \det P_t}$$

$$= \det A \cdot \det(P_1 P_2) \det P_3 \cdots \det P_t$$

$$= \cdots = \det A \det(P_1 P_2 \cdots P_t)$$

$$= \det A \det B$$

$$|AB| = |A| \cdot |B|$$

Note 1: A 可逆时 $AA^{-1} = I$

$$|AA^{-1}| = |I| = 1$$

$$|A| \cdot |A^{-1}|$$

$$\Rightarrow |A^{-1}| = \frac{1}{|A|}$$

2. A 可逆时 $|A| \neq 0$.

$$AA^* = |A|I = \begin{bmatrix} |A| & & \\ & \ddots & \\ & & |A| \end{bmatrix}$$

$$\Rightarrow |A| \cdot |A^*| = |A|^n$$

$$\text{当 } |A| \neq 0 \text{ 时 } \underline{|A^*| = |A|^{n-1}}$$

A 不可逆时: $(|A|=0)$

$$\underline{A^*A = AA^* = |A|I = 0}$$

$$A^* \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{bmatrix}$$

$$A^* \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} = 0 \quad A^* \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} = 0 \quad \dots \quad A^* \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$A^*X=0$ 有无数解. A^* 不可逆

$$|A^*| = 0$$

$$\text{总之: } |A^*| = |A|^{n-1}$$

例 1: $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ 求 $|A^{-1} + A^*|$

$$A^{-1} = \frac{A^*}{|A|}$$

$$|A^{-1} + A^*| = \left| \frac{1}{|A|} A^* + A^* \right| = \left| \left(\frac{1}{|A|} + 1 \right) A^* \right|$$

$$= \left(\frac{1}{|A|} + 1\right)^3 |A^*| = \left(\frac{1}{|A|} + 1\right)^3 |A|^2.$$

例: $A, B \in M_3(\mathbb{R})$

$$|A| = 2, |B| = 3.$$

$$\text{求 } \textcircled{1} \quad \left| \left(-\frac{1}{2}\right) A^{-1} - 2 A^* \right| \quad \textcircled{2} \quad |2 A^* B^{-1}|$$

$$\parallel$$

$$\left| \left(-\frac{1}{2}\right) A^{-1} - 4 A^{-1} \right|$$

$$= \left| \left(-\frac{9}{2}\right) A^{-1} \right|$$

$$= \left(-\frac{9}{2}\right)^3 |A^{-1}|$$

$$= \left(-\frac{9}{2}\right)^3 \cdot \frac{1}{2}$$

$$A^{-1} = \frac{A^*}{|A|}$$

$$A^* = |A| A^{-1}$$

$$= 2 A^{-1}$$

$$\parallel$$

$$8 |A^*| |B^{-1}|$$

$$= 8 \cdot |A|^2 \cdot \frac{1}{|B|}$$

* 例3: $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}^3$ (3个非0向量).

$$|\alpha_1, \alpha_2, \alpha_3| = 1 \quad \text{求}$$

$$|\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 4\alpha_3, \alpha_1 + 3\alpha_2 + 9\alpha_3| = ?$$

解:

$$B = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + 4\alpha_3, \alpha_1 + 3\alpha_2 + 9\alpha_3)$$

$$= (\alpha_1, \alpha_2, \alpha_3) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$|B| = |\alpha_1, \alpha_2, \alpha_3| \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = (2-1)(3-1)(3-2)$$

$$= 2.$$

• 分块矩阵.

当处理矩阵阶数较大时. 常用分块.

$$1. A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [\alpha_1, \alpha_2 \cdots \alpha_n] = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}$$

α_i 为 $m \times 1$ 的列向量

β_j 为 $1 \times n$ 的行向量

$$= \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

注意与代数公式符号差别

运算① A, B 为同型矩阵. 用相同划分方法分块.

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & \cdots & B_{1s} \\ \vdots & & \vdots \\ B_{r1} & \cdots & B_{rs} \end{bmatrix}$$

A_{ij} 与 B_{ij} 为同型矩阵

$$A+B = \begin{bmatrix} A_{11}+B_{11} & \cdots & A_{1s}+B_{1s} \\ \vdots & & \vdots \\ A_{r1}+B_{r1} & \cdots & A_{rs}+B_{rs} \end{bmatrix}$$

$$\lambda A = \begin{bmatrix} \lambda A_{11} & \cdots & \lambda A_{1s} \\ \vdots & & \vdots \\ \lambda A_{r1} & \cdots & \lambda A_{rs} \end{bmatrix}$$

转置

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1s} \\ A_{21} & \cdots & A_{2s} \\ \vdots & & \vdots \\ A_{r1} & \cdots & A_{rs} \end{bmatrix}$$

$$A^T = \begin{bmatrix} A_{11}^T & A_{21}^T & \cdots & A_{r1}^T \\ A_{12}^T & A_{22}^T & & \vdots \\ \vdots & A_{2s}^T & & A_{rs}^T \end{bmatrix}$$

例: P76. 14, 15, 16, 17

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{bmatrix}_{3 \times 2} X = \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$X = \underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 0 \\ -1 & 2 & -1 \end{bmatrix}}_A^{-1} \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$= A^{-1} \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix}}$$

$$P_t \cdots P_2 P_1 A = I$$

$$A^{-1} = P_t P_{t-1} \cdots P_1$$

$$\underbrace{\left[A, \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix} \right]}_{\equiv}$$

$$P_t \cdots P_2 P_1 A = I$$

$$\underbrace{P_t \cdots P_2 P_1}_{\cdot} \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix} = A^{-1} \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix}$$

分块矩阵乘法:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{bmatrix}$$

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1t} \\ B_{21} & B_{22} & \cdots & B_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ B_{s1} & B_{s2} & \cdots & B_{st} \end{bmatrix}$$

$$AB = C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1t} \\ C_{21} & C_{22} & \cdots & C_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{rt} \end{bmatrix}$$

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{is}B_{sj}$$

$$\text{例 1. } A = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ \hline -3 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \end{array} \right]$$

$$= \begin{bmatrix} A_{11} & 0 \\ A_{21} & I_2 \end{bmatrix}$$

$$B = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 0 & 0 \\ \hline 2 & 3 & 1 & 3 & 1 \\ 0 & 2 & 2 & 0 & 1 \end{array} \right]$$

$$= \left[\begin{array}{c|ccc} B_{11} & 0 & 0 & 0 \\ \hline B_{21} & B_{22} & 0 & 0 \end{array} \right]$$

$$AB = \left[\begin{array}{cc|ccccc} 2 & 4 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 \\ \hline 5 & 9 & 1 & 1 & 3 & 1 \\ 2 & 6 & 2 & 0 & 1 & 0 \end{array} \right]$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$$

$$\text{例 2: } A_{m \times n} B_{n \times s}$$

$$B = [\alpha_1, \alpha_2, \dots, \alpha_s]$$

$$\alpha_i \in \mathbb{R}^{n \times 1}$$

$$= A[\alpha_1, \alpha_2, \dots, \alpha_s]$$

$$= [A\alpha_1 \ A\alpha_2 \ \cdots \ A\alpha_s]$$

$$A_{m \times n} B_{n \times s} = O_{m \times s}$$

$$[A\alpha_1 \cdots A\alpha_s] = O_{m \times s}$$

$$B \sim \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_s \end{bmatrix} \rightarrow \text{化为}$$

$$AX = O \quad \begin{matrix} m \times n & n \times 1 & m \times 1 \end{matrix}$$

$$A_{m \times n} B_{n \times s}$$

$$= \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix} B$$

$$\beta_i \in \mathbb{R}^{1 \times n}$$

$$= \begin{bmatrix} \beta_1 B \\ \beta_2 B \\ \vdots \\ \beta_m B \end{bmatrix}$$

$$AB = O$$

$$\Rightarrow \beta_i B = O$$

$$\Rightarrow B^T \beta_i^T = O$$

$$A \text{ 的 } i \text{ 行位置为 } B^T X = O \text{ 的 } i \text{ 行。}$$

例: $\rightarrow$ 主上三角阵 (主元)

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ & A_{22} & \cdots & A_{2s} \\ & & \ddots & \vdots \\ & & & A_{ss} \end{bmatrix}$$

$\rightarrow$ 下三角阵。

A_{ii} 为主元

$$\rightarrow \text{对称阵} \begin{bmatrix} A_{11} & & \\ & A_{22} & \\ & & \ddots \\ & & & A_{ss} \end{bmatrix} = \text{diag}(A_{11}, A_{22}, \cdots, A_{ss})$$

A_{ii} 为主元

$$\begin{bmatrix} 1 & & \\ \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} & & \\ & \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix} & \end{bmatrix}$$

$$\text{例: } A = \begin{bmatrix} B & D \\ 0 & C \end{bmatrix} \quad B, C \text{ 均为可逆方阵。}$$

证: A 可逆, 求 A^{-1} 。

$$\text{证: 令分块矩阵 } \begin{bmatrix} X & Z \\ W & Y \end{bmatrix} \text{ 满足}$$

$$\begin{bmatrix} B_n & D_{n \times m} \\ 0_{m \times n} & C_m \end{bmatrix} \begin{bmatrix} X_n & Z \\ W & Y_m \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}$$

$$\begin{cases} BX + DW = I_n \\ BZ + DY = 0 \\ CW = 0 \\ CY = I_m \end{cases}$$

$$Y = C^{-1} \quad W = 0$$

$$X = B^{-1}$$

$$BZ = -DY = -DC^{-1}$$

$$Z = -B^{-1}DC^{-1}$$

$$\begin{bmatrix} B^{-1} & -B^{-1}DC^{-1} \\ 0 & C^{-1} \end{bmatrix}$$

$$\begin{bmatrix} B & D \\ 0 & C \end{bmatrix}^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}DC^{-1} \\ 0 & C^{-1} \end{bmatrix}$$

• 分块矩阵的初等变换

$$M = \begin{bmatrix} A_s & B \\ C & D_t \end{bmatrix}$$

① 用可逆矩阵 P 左乘 M 的某一行 (列)

② 用矩阵 Q 左乘 M 的某一行 (列) 加上另一行 (列)

③ 交换 M 的两行或两列

$$I = \begin{bmatrix} I_s & \\ & I_t \end{bmatrix}$$

$$\begin{bmatrix} P_m & \\ & I_s \end{bmatrix} \begin{bmatrix} A_{m \times n} & B_{m \times t} \\ C_{s \times n} & D_{s \times t} \end{bmatrix} = \begin{bmatrix} PA & PB \\ C & D \end{bmatrix} \quad Q_{s \times m}$$

$$\begin{bmatrix} I_m & \\ Q_{s \times m} & I_s \end{bmatrix} \begin{bmatrix} A_{m \times n} & B_{m \times t} \\ C_{s \times n} & D_{s \times t} \end{bmatrix} = \begin{bmatrix} A & B \\ QA+C & QB+D \end{bmatrix}$$

$$\begin{bmatrix} & I_s \\ I_m & \end{bmatrix} \begin{bmatrix} A_{m \times n} & B \\ C & D_{s \times t} \end{bmatrix} = \begin{bmatrix} C & D \\ A & B \end{bmatrix}$$

例 1: $A = \begin{bmatrix} B & D \\ 0 & C \end{bmatrix}$ B, C 可逆. 求 A^{-1}

$$[A, I] = \left[\begin{array}{cc|cc} B_n & D_{n \times m} & I_m & 0 \\ 0 & C_m & 0 & I_m \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cc|cc} I_n & B^{-1}D_{n \times m} & B^{-1} & 0 \\ 0 & I_m & 0 & C_m^{-1} \end{array} \right]$$

$$\begin{array}{l} \text{②} \xrightarrow{-B^{-1}D} \\ +\text{①} \end{array} \left[\begin{array}{cc|cc} I_n & 0 & B^{-1} & -B^{-1}DC^{-1} \\ 0 & I_m & 0 & C^{-1} \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} B^{-1} & -B^{-1}DC^{-1} \\ 0 & C^{-1} \end{bmatrix}$$

例: $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ A 可逆. D 为方阵

① 求 $|M|$

② 给出 M 可逆的条件并求出 M^{-1} .

$$M = \begin{bmatrix} A_n & B_{n \times m} \\ C & D_m \end{bmatrix} \rightarrow \begin{bmatrix} I_n & A^{-1}B \\ C_{m \times n} & D \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

$$\begin{bmatrix} I & \\ -C & I \end{bmatrix} \begin{bmatrix} A^{-1} & \\ & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

两边取行列式: $\underbrace{\begin{vmatrix} I & 0 \\ -C & I \end{vmatrix}} \cdot \underbrace{\begin{vmatrix} A^{-1} & \\ & I \end{vmatrix}} \cdot \underbrace{\begin{vmatrix} A & B \\ C & D \end{vmatrix}} = \begin{vmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{vmatrix}$

$$\begin{vmatrix} A_n & B \\ 0 & D_m \end{vmatrix} = |M| \cdot |P|$$

$$1 \cdot |A^{-1}| \cdot \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D - CA^{-1}B|$$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \frac{1}{|A^{-1}|} |D - CA^{-1}B| = |A| \cdot |D - CA^{-1}B|$$

② 当 $D - CA^{-1}B$ 可逆时 M 可逆.

$$\begin{bmatrix} I & \\ -C & I \end{bmatrix} \begin{bmatrix} A^{-1} & \\ & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^{-1} & \\ & I \end{bmatrix}^{-1} \begin{bmatrix} I & \\ -C & I \end{bmatrix}^{-1} \begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \underbrace{\begin{bmatrix} I & A^T B \\ 0 & D - CA^T B \end{bmatrix}^{-1}}_{\text{归结为 } 2 \times 2 \text{ 矩阵}} \begin{bmatrix} I & \\ -C & I \end{bmatrix} \begin{bmatrix} A^T & \\ & I \end{bmatrix}$$

$$\text{例 2): } \begin{vmatrix} A_n & 0 \\ * & B_m \end{vmatrix} = |A| |B|$$

$$\begin{vmatrix} A_n & * \\ & B_m \end{vmatrix} = \begin{vmatrix} A^T & 0 \\ *^T & B^T \end{vmatrix} = |A^T| |B^T| = |A| \cdot |B|$$

$$\begin{vmatrix} 0 & A_n \\ B_m & * \end{vmatrix} = (-1)^{mn} \begin{vmatrix} A_n & 0 \\ * & B_m \end{vmatrix} = (-1)^{mn} |A| |B|$$

$$\text{例: 构造 } M = \begin{bmatrix} A_n & 0 \\ -I & B_n \end{bmatrix}$$

$$\begin{bmatrix} A_n & 0 \\ -I & B \end{bmatrix} \xrightarrow{\text{行}} \begin{bmatrix} 0 & AB \\ -I & B \end{bmatrix}$$

$$\begin{bmatrix} I & A \\ & I \end{bmatrix} \begin{bmatrix} A & 0 \\ -I & B \end{bmatrix} = \begin{bmatrix} 0 & AB \\ -I & B \end{bmatrix} \quad (\text{两边乘以 } I)$$

$$1. \quad |A| \cdot |B| = \begin{vmatrix} 0 & AB \\ -I & B \end{vmatrix} = (-1)^{n^2} \begin{vmatrix} AB & 0 \\ B & -I \end{vmatrix} = (-1)^{n^2} (-1)^n |AB| = (-1)^{\overbrace{n(n+1)}^{\text{偶数}}} |AB| = |AB|$$

从而我们又一次证明了 $|AB| = |A| \cdot |B|$.

例①: $A \in \mathbb{R}^{m \times n}$ $B \in \mathbb{R}^{n \times m}$ w
 $|I_m - AB| = |I_n - BA|$.

$$\begin{bmatrix} I_m & A_{m \times n} \\ B_{n \times m} & I_n \end{bmatrix} \xrightarrow{\text{①}} \begin{bmatrix} I_m & A \\ 0 & I - BA \end{bmatrix}$$

$$\begin{bmatrix} I & \\ -B & I \end{bmatrix} \begin{bmatrix} I & A \\ B & I \end{bmatrix} = \begin{bmatrix} I_m & A \\ 0 & I - BA \end{bmatrix}$$

例② 例①の逆: $1 \cdot \begin{vmatrix} I & A \\ B & I \end{vmatrix} = \underline{|I_n - BA|}$

$$\begin{bmatrix} I_m & A_{m \times n} \\ B_{n \times m} & I_n \end{bmatrix} \xrightarrow{\text{②}} \begin{bmatrix} I_m - AB & A_{m \times n} \\ 0 & I_n \end{bmatrix}$$

$$\begin{bmatrix} I & A \\ B & I \end{bmatrix} \begin{bmatrix} I & \\ -B & I \end{bmatrix} = \begin{bmatrix} I_m - AB & A \\ 0 & I \end{bmatrix}$$

例③ 例②の逆: $\begin{vmatrix} I & A \\ B & I \end{vmatrix} \cdot 1 = \underline{|I_m - AB|}$.

w $|I_m - AB| = |I_n - BA|$.

例④: $M = \begin{bmatrix} & & & A_1 \\ & & A_2 & \\ & \ddots & & \\ A_5 & & & \end{bmatrix}$ A_i 9x9. 求 M^{-1}

$$\left[\begin{array}{ccc|ccc} & & A_1 & I & & \\ & A_2 & & & I & \\ A_S & \dots & & & & I \end{array} \right]$$

$$\rightarrow \left[\begin{array}{cccc|cccc} A_5 & & & & & & & \\ & A_4 & & & & & & \\ & & \ddots & & & & & \\ & & & A_1 & & & & \\ & & & & I & & & \\ & & & & & I & & \\ & & & & & & I & \\ & & & & & & & I \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} I & & & & & \\ & I & & & & \\ & & \ddots & & & \\ & & & I & & \\ & & & & A_1^{-1} & \\ & & & & A_2^{-1} & \\ & & & & & \ddots & A_S^{-1} \end{array} \right]$$

$$\begin{bmatrix} & & A_1 \\ & \ddots & A_2 \\ A_S & & \end{bmatrix}^{-1} = \begin{bmatrix} & & A_S^{-1} \\ & \ddots & A_2^{-1} \\ A_1^{-1} & & \end{bmatrix}$$

矩阵的秩.

便笺标题

2015/11/3

定义: $A_{m \times n}$ 存在一个 r 阶子式不为零, 所有 $(r+1)$ 阶子式均为零. 称 A 的秩为 r . $r(A) = r$.
 $\text{rank}(A) = r$.

Note 1: $r(A) \leq \min\{m, n\}$

$$r(O) = 0.$$

2. $r(A) = r$, $\forall$ A 中任意 $r+1$ 阶子式均为零.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad \underline{\underline{r(A)=2}}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}(A_{11}) + a_{12}(A_{12}) + a_{13}(A_{13}) + a_{14}(A_{14})$$

3. $r(A) \leq s \Leftrightarrow A$ 中所有 $(s+1)$ 阶子式均为零

$r(A) \geq s \Leftrightarrow A$ 中存在一个 s 阶子式不为零.

4. $r(A) = r(A^T)$.

5. $A \in M_n(\mathbb{R})$ A 可逆 $\Leftrightarrow |A| \neq 0$.
 $\Leftrightarrow r(A) = n$

Note: $A \in M_{m \times n}(\mathbb{R})$

若 $r(A) = m$ 称 A 行满秩
若 $r(A) = n$... 列满秩 ...

秩的求法

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 5 & 6 & 7 & 0 \\ 0 & 0 & 8 & 9 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4 \times 5}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 8 \end{vmatrix} = (1 \times 5 \times 8)$$

阶梯型矩阵的秩 = 非零行行数
= 主元个数

定理: 初等变换不改变矩阵的秩.

证明: 设 A 经过一系列初等变换后变成矩阵 B.

证: $r(B) \leq r(A)$ 设 $r(A) = r$.

首先经过一次初等变换后得 B_1 $r(B_1) \leq r(A)$

对 A 分块 $A = (\alpha_1, \alpha_2, \dots, \alpha_n)$

$$\textcircled{1} A \rightarrow B_1 = (\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \lambda \alpha_i, \alpha_{i+1}, \dots, \alpha_n)$$

$\lambda \neq 0$

设 M 为 B_1 的一个 $(r+1)$ 阶子式 若 M 不含第 i 列则

M 也为 A 的一个 $(r+1)$ 阶子式.

若 M 含第 i 列 $M = \lambda \cdot M_1$ M_1 是 A 的 $(r+1)$ 阶子式.

即 B_1 所有 $(r+1)$ 阶子式均为零.

$$r(B_1) \leq r = r(A).$$

$$\textcircled{2} A \rightarrow B_1 = (\alpha_1, \alpha_2, \dots, \alpha_i, \dots, \alpha_j + \lambda \alpha_i, \dots, \alpha_n)$$

当 M 不含第 j 列时

当 M 含第 j 列时: M 可以写成两列的列和.

有一个是 A 的 $(r+1)$ 阶子式.

另一个是两列对应成比例列是 A 的 $(r+1)$ 阶子式. 因此

$$r(B_1) \leq r = r(A)$$

$$(3) \quad A \rightarrow B_1 = (\alpha_1 \cdots \underbrace{\alpha_j \cdots \alpha_i}_{\text{两列成比例}} \cdots \alpha_n)$$

$$r(B_1) \leq r = r(A).$$

$$\text{综上: } r(B_1) \leq r(A) \quad r(B) \leq r(A)$$

$$\text{初等变换过程} \quad r(A) \leq r(B).$$

$$\Rightarrow r(A) = r(B).$$

例 1: $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & b \\ 2 & 3 & a & 4 \\ 3 & 5 & 1 & 7 \end{bmatrix} \quad r(A) = 3. \text{ 求 } a, b.$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & b \\ 0 & 1 & a-2 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & b \\ 0 & 0 & a-1 & 2-b \\ 0 & 0 & 0 & 4-2b \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & b \\ 0 & 0 & a-1 & 0 \\ 0 & 0 & 0 & 4-2b \end{bmatrix}$$

$$\begin{cases} a-1 \neq 0 \\ 4-2b = 0 \end{cases} \quad \text{或} \quad \begin{cases} 4-2b \neq 0 \\ a-1 = 0 \end{cases}$$

2. 矩阵秩的性质:

$$\textcircled{1} \quad P \in M_m(\mathbb{R}) \quad Q \in M_n(\mathbb{R}) \\ A \in M_{m,n}(\mathbb{R}). \quad P, Q \text{ 可逆}$$

$$\therefore r(PAQ) = r(A).$$

$$\textcircled{2} \quad \text{若 } B = PAQ, \quad P, Q \text{ 可逆 (相抵关系)}.$$

$$\text{推论: } A, B \in M_{m,n}(\mathbb{R}), \quad A, B \text{ 相抵} \Leftrightarrow r(A) = r(B)$$

$$\text{若 } r(A) = r, \quad A \cong \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \\ r(B) = r, \quad B \cong \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

3. 秩的重要性质:

$$\textcircled{1} \quad r \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = r(A) + r(B)$$

证: A 经过初等行变换化为阶梯形矩阵即非零行

个为 $r(A)$. $B \dots \dots r(B)$

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \rightarrow \begin{bmatrix} \text{阶梯形} & 0 \\ 0 & \text{阶梯形} \end{bmatrix}$$

$$\textcircled{2} \quad r \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \geq r(A) + r(B)$$

证明: 令 $r(A) = r, \quad r(B) = s$. A 中存在一个 r 阶子式不为零. B 中存在一个 s 阶子式不为零.

将 $(r+s)$ 行 $(r+s)$ 列 子块 或 $(r+s)$ 行 $(r+s)$ 列 子块,

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \rightsquigarrow r \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \geq r(A) + r(B)$$

Note: 有时 $\overline{A} \neq \overline{B}$

$$\begin{vmatrix} A_n & 0 \\ 0 & B_m \end{vmatrix} = |A| \cdot |B|$$

$$\begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \rightarrow \begin{bmatrix} \text{diag} & 0 \\ * & \text{diag} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 5 & 6 & 7 & 8 & 0 & 1 & 0 & 0 \\ 7 & 8 & 9 & 10 & 0 & 0 & 0 & 1 \\ 10 & 11 & 12 & 13 & 0 & 0 & 0 & 0 \end{bmatrix}$$

③. $A \in M_{m,n}(\mathbb{R})$. $B \in M_{t,n}(\mathbb{R})$.

$$\max\{r(A), r(B)\} \leq r \begin{bmatrix} A \\ B \end{bmatrix} \leq \underline{r(A) + r(B)}$$

证: $\begin{bmatrix} A \\ B \end{bmatrix} \xrightarrow{\text{行}} \begin{bmatrix} \text{diag} \\ \text{diag} \end{bmatrix}$ 取子: $\{A \text{ 的行子块}\} \cap \{B \text{ 的行子块}\} = \emptyset$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow r \begin{bmatrix} A \\ B \end{bmatrix} = r(A)$$

B 可被 A 利用初等行变换消掉.

$$\textcircled{4} \quad r(A+B) \leq r \begin{bmatrix} A+B \\ B \end{bmatrix} = r \begin{bmatrix} A \\ B \end{bmatrix} \leq r(A) + r(B)$$

⑤ $A \in M_{m,n}(\mathbb{R}), B \in M_{n,s}(\mathbb{R})$

$$r(A) + r(B) - n \leq \underline{r(AB)} \leq \min\{r(A), r(B)\}$$

证: 令 $r(A) = r$. $\exists$ 可逆阵 P, Q s.t.
 such that.

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} r(AB) &= r(\underline{PA}B) = r(\underline{PAQ}Q^{-1}B) \\ &= r\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}B\right) \end{aligned}$$

$$\begin{aligned} PAB &= PAQQ^{-1}B = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \underbrace{(Q^{-1}B)}_{\substack{\text{抽行} \\ \downarrow}} \\ &= \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \begin{matrix} \rightarrow r \text{ 行} \\ \rightarrow n-r \text{ 行} \end{matrix} \end{aligned}$$

$$\begin{aligned} \text{即 } r(AB) &= r(PAB) = \underline{r\left(\begin{bmatrix} C_1 \\ 0 \end{bmatrix}\right)} = r(PAQQ^{-1}B) = r\begin{bmatrix} C_1 \\ 0 \end{bmatrix} \\ &= r(C_1) \leq r = r(A) \end{aligned}$$

$$\text{即 } \underline{r(AB)} \leq \underline{r(A)}$$

$$\underline{r(AB)} = r((AB)^T) = r(\underline{B^T A^T}) \leq r(\underline{B^T}) \stackrel{||}{=} r(B)$$

$r(AB) \leq r(A), \quad r(AB) \leq r(B)$

证: 设 $r(A)=r$.

$$r(AB) = r \begin{bmatrix} C_1 \\ 0 \end{bmatrix} = r(C_1) \stackrel{\textcircled{1}}{\geq} r \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} - (n-r)$$

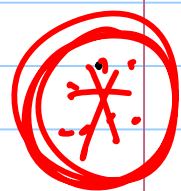
$$\stackrel{\text{||}}{=} r(Q^T B) + r - n$$

$$\stackrel{\text{||}}{=} r(B) + r(A) - n$$

$$\left[\begin{array}{l} \textcircled{1}: r \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \leq r(C_1) + \underbrace{r(C_2)}_{\downarrow r(C_1)} \leq \underbrace{r(C_1) + n - r}_{r(B) + r(A) - n} \\ r(C_1) \geq r \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} - (n-r) \end{array} \right]$$

$$A_{m,n} \quad B_{n,s}$$

$$r(A) + r(B) - n \leq r(AB) \leq \min\{r(A), r(B)\}$$



① 若 $AB = O$, 则 $r(A) + r(B) \leq n$

② 若 $r(A) = n$ $r(AB) = r(B)$

若 $r(B) = n$, $r(AB) = r(A)$

③ 方程组 $AX = \underline{0}$ ($A_{m,n} \quad X_{n,1}$)

若 $r(A) = n$ 则

$$r(AX) = r(X)$$

$$\stackrel{\text{||}}{=} r(\underline{0})$$

$$\Rightarrow r(X) = 0$$

$$X = 0$$

④ $AB = O \not\Rightarrow B = O$

$$\text{若 } A \text{ 可逆} \text{ 且 } A \text{ 可逆} \quad AB = O \Rightarrow \underline{\underline{A^{-1}AB = A^{-1}O}}$$

$$\Rightarrow \underline{\underline{B = O}}$$

若 A 不是一行一列时: $r(A) = n$ $AB = O$ $r(AB) = r(B)$

$$\Rightarrow r(B)=0 \Rightarrow B=O$$

$$AB=O \xrightarrow{r(A)=n} B=O$$

类似地 $AB=AC$
若 $r(A)=n$
 $\Rightarrow B=C.$

例 1: $A \in M_n(\mathbb{R})$

$$r(A^*) = \begin{cases} n & r(A)=n \\ 1 & r(A)=n-1 \\ 0 & r(A) < n-1 \end{cases}$$

① $r(A)=n \Leftrightarrow A$ 可逆 $\Leftrightarrow |A| \neq 0$

$$AA^* = A^*A = |A|I.$$

$$\Rightarrow \frac{1}{|A|}AA^* = A^* \cdot \frac{A}{|A|} = I \Rightarrow A^* \text{ 也可逆}.$$

② $r(A)=n-1, \Leftrightarrow |A|=0$

$$AA^* = A^*A = |A|I = O$$

$$\Rightarrow \underbrace{r(A)}_{n-1} + r(A^*) \leq n$$

$$\Rightarrow r(A^*) \leq 1 \quad r(A^*)=1 \text{ 或 } 0$$

$$r(A^*) \neq 0. \text{ 即 } A^* \text{ 不是零矩阵 从而 } r(A^*)=1$$

例 2: $AX=b$ 线性方程组 $A_{m,n} X_{n,1} \mid \begin{smallmatrix} b \\ m \end{smallmatrix}$
若 $b=0$ 齐次方程组 $AX=0$

$$A \rightarrow \begin{bmatrix} \text{阶梯形} \end{bmatrix}$$

$$AX=0 \text{ 有非零解} \Leftrightarrow r(A) < n$$

$$AX=b \quad [A|b] \xrightarrow{\text{增广矩阵}} \left[\begin{array}{cccc|c} c_{11} & \dots & c_{12} & \dots & c_{1r} & \dots & c_{1n} & d_1 \\ 0 & \dots & c_{22} & \dots & c_{2r} & \dots & c_{2n} & d_2 \\ \vdots & & & & & & & \\ 0 & \dots & 0 & \dots & c_{r1} & \dots & c_{rn} & d_r \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 & d_{r+1} \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 & 0 \end{array} \right]$$

$$c_{11}, c_{22}, \dots, c_{rr} \neq 0$$

$$AX=b \text{ 有解} \Leftrightarrow r(A) = r(A|b)$$

$$\begin{aligned} \text{若 } r(A) = r(A|b) = n & \text{ 唯一解} \\ < n & \text{ 无穷多解 } \quad n-r \text{ 个自由变量} \end{aligned}$$

习题课.

便笺标题

2015/11/5

$$1. r \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = r(A) + r(B)$$

$$2. r(A+B) \leq r(A) + r(B)$$

$$3. r(A) + r(B) - n \leq r(AB) \leq \min\{r(A), r(B)\}$$

$\Downarrow$

$$\textcircled{1} AB=0 \Rightarrow r(A) + r(B) \leq n$$

$$\textcircled{2} A_{m \times n}, B_{n \times s} \quad r(A)=n \Rightarrow r(AB)=r(B)$$

$$\textcircled{3} \text{若 } AB=0, \text{ 且 } r(A)=n \Rightarrow B=0$$

$$\text{若 } AB=AC \text{ 且 } r(A)=n \Rightarrow B=C$$

$$\textcircled{4} AX=0 \text{ 有非零解} \Leftrightarrow r(A) < n$$

$$AX=b \text{ 有解} \quad r(A)=r(A:b) = \begin{cases} =n & \text{唯一解} \\ <n & \text{无穷多解} \end{cases}$$

$$\textcircled{5} A \in M_n(\mathbb{R}) \quad r(A^*) = \begin{cases} n & r(A)=n \\ 1 & r(A)=n-1 \\ 0 & r(A) \leq n-2 \end{cases}$$

$$\textcircled{6} A \in M_n(\mathbb{R}) \quad A \text{ 可逆} \Leftrightarrow |A| \neq 0 \Leftrightarrow r(A)=n$$

秩与行列式

$$1. r(A)=2, |A|=0 \quad \begin{vmatrix} 2 & -3 & 1 \\ 1 & a & 1 \\ 5 & 0 & 3 \end{vmatrix} = 0$$

$$6a - 15 - 5a + 9 = 0$$

$$a = 6$$

284: $A \rightarrow \begin{bmatrix} 2 & -3 & 1 \\ 0 & a+\frac{3}{2} & \frac{1}{2} \\ 0 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & 1 \\ 0 & \frac{5}{2} & \frac{1}{2} \\ 0 & a+\frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{matrix} \textcircled{1} \times (-\frac{1}{2}) + \textcircled{2} \\ \textcircled{1} \times (-\frac{5}{2}) + \textcircled{3} \end{matrix}$

$$a + \frac{3}{2} = \frac{15}{2}$$

2. $A \rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 0 & t-4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad t \neq 4$

3. $AB=0 \Rightarrow \textcircled{r(A)} + r(B) \leq n$

4. $A_{3 \times 2} \quad B_{2 \times 3} \quad \textcircled{AB_{3 \times 3}} \quad (BA)_{2 \times 2}$

$$r(AB) \leq r(A) \leq 2$$

$$|AB| = 0$$

5. $\alpha_{3 \times 1} \quad \beta_{1 \times 3} \quad A = (\alpha\beta)_{3 \times 3} \quad \alpha = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{r(A) = r(\alpha\beta) \leq r(\alpha) = 1}$$

$$r(AB-B) = r[\underline{(A-I)B}] = r(B)$$

$$A-I = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & -1 \\ 2 & 4 & 4 \end{bmatrix} \quad |A-I| \neq 0 \quad A-I \text{ 可逆} \quad r(A-I) = 3$$

$$8. \quad r(A) + r(B) \leq 3$$

$$r(A) \leq 2 \quad \begin{bmatrix} 1 & -2 & 2 \\ -2 & 6 & x \\ 3 & 0 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & \underline{2} & \underline{x+4} \\ 0 & 6 & -12 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & x+8 \end{bmatrix}$$

$$x = -8 \quad r(A) = 2 \quad \underline{r(A) + r(B) \leq 3}$$

9. 略

$$10. \quad r(AB+B) = r((A+I)B) = 2$$

$$A = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} [1 \ -1 \ 0] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

$$A+I = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & \textcircled{1} \end{bmatrix} \quad |A+I| = 1 \cdot \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \neq 0$$

$$r((A+I)B) = r(B) = 2$$

$$12. \quad \underline{n \geq 3} \quad a = \frac{1}{1-n} \quad n=2 \quad \textcircled{a = \pm 1.} \Rightarrow |A| = 0$$

$$\begin{vmatrix} 1 & a & a & \cdots & a \\ a & 1 & a & \cdots & a \\ a & a & 1 & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & 1 \end{vmatrix} = (1 + (n-1)a) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & 1 & a & \cdots & a \\ a & a & 1 & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & 1 \end{vmatrix}$$

$$= (1 + (n-1)a) \begin{vmatrix} 1 & \cdots & 1 \\ 1-a & 1-a & \cdots & 1-a \end{vmatrix}$$

$$= (1 + (n-1)a) (1-a)^{n-1}$$

$$= 0.$$

$$a=1 \text{ 或 } a = -\frac{1}{n-1}$$

$$a=1 \text{ 时 } r(A)=1$$

$$a = -\frac{1}{n-1}$$

$$\begin{aligned} \text{12: } \begin{bmatrix} 1 & a & a & \dots & a \\ a & 1 & a & \dots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \dots & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1+(n-1)a & 1+(n-1)a & \dots & 1+(n-1)a \\ a & 1 & \dots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \dots & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1+(n-1)a & 0 & 0 & \dots & 0 \\ a & 1-a & 0 & \dots & 0 \\ \vdots & \vdots & 1-a & \ddots & \vdots \\ a & 0 & 0 & \dots & 1-a \end{bmatrix} \end{aligned}$$

13.

$$r(AB) \leq r(A)$$

$$\text{若 } r(A)=n \Rightarrow r(AB)=r(B)$$

$$r(AB) < r(B) \Rightarrow r(A) < n$$

$$|A|=0, |B|=0.$$

$$\text{14. } AX=0 \quad r(A) < n$$

$$A = \begin{bmatrix} 1 & 0 & k \\ k & 1 & 5 \\ 1 & 1 & -1 \end{bmatrix} \quad |A|=0$$

$$-1 + k^2 - k - 5 = 0$$

$$k=3 \text{ 或 } -2. \quad k^2 - k - 6 = 0. \quad (k-3)(k+2) = 0$$

15. $\underline{r(A)} = \underline{r(A \mid b)}_{m \times (n+1)}$

$4 \times 2 \quad (A \mid b) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$r(A) = m$
 $\Downarrow$
 $r(A \mid b) = m.$

$[A \mid b] \rightarrow \begin{bmatrix} a_{11} & & & & \\ & a_{22} & & & \\ & & \ddots & & \\ & & & a_{mm} & \dots \\ & & & & \vdots \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ & 7 & 8 & 9 & 10 & 11 \\ & & 12 & 13 & 14 & 15 \\ & & & 16 & 17 & 18 \end{bmatrix}$
 4×5

16. $(AB)_{m \times m}$

1. $r(A) = n \quad \underline{r(AB) = r(B) = m}$
 $AB \text{ is invertible.}$

$X = (AB)^{-1}b.$

2. $r(A) = n \leq m$
 $r(B) = m \leq n. \Rightarrow m = n. \Rightarrow A, B \text{ are invertible}$

$X = B^{-1}A^{-1}b$

$$1. \quad \left(\frac{1}{a} + \frac{1}{b}\right)^{-1} = \left(\frac{a+b}{ab}\right)^{-1} = \frac{ab}{a+b}$$

$$\underline{A(A^{-1} + B^{-1})B} \\ = (I + AB^{-1})B = B + A = A+B$$

$$A^{-1} + B^{-1} = A^{-1}(A+B)B^{-1} \\ (A^{-1} + B^{-1})^{-1} = B(A+B)^{-1}A$$

$$|AB| = |A||B| = 0$$

$$ab = ba$$

$$\frac{1}{a+b} \neq \frac{1}{a} + \frac{1}{b}$$

$$|kA| = k^n |A|$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A^2 = 0. \quad \text{Jordan form}$$

$$(A^2)^T = (A \cdot A)^T = A^T \cdot A^T = A \cdot A = A^2$$

$$AA^* = A^*A = |A|I$$

$$|A| \neq 0. \quad \frac{A}{|A|} \cdot A^* = A^* \cdot \frac{A}{|A|} = I.$$

$$(A^*)^{-1} = \frac{A}{|A|} = |A|^{-1} A$$

$$(A^{-1})^T = (A^T)^{-1}$$

$$A A^{-1} = I$$

$$\underbrace{(A^{-1})^T}_{\text{wavy}} \underbrace{(A^T)}_{\text{circle}} = I$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$D: \left((A^{-1})^T \right)^{-1} = \left((A^{-1})^{-1} \right)^T = A^T$$

$$A+B = (\alpha_1 + \beta_1, 2\alpha_2, 2\alpha_3, 2\alpha_4)$$

$$|A+B| = 8 \begin{vmatrix} \alpha_1 + \beta_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix}$$

$$= 8 \left(\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix} + \begin{vmatrix} \beta_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{vmatrix} \right)$$

9

$$A^2 = I \quad (A-I)(A+I) = \mathbf{0}$$

$$\Rightarrow |A-I| \cdot |A+I| = 0$$

10.

$$\underbrace{AB \cdot AB}_{\text{circle}} = I$$

$$B^{-1} = ABA$$

$$A^{-1} = BAB$$

$$B^{-1}A^{-1} = \underbrace{ABA BAB}_{\text{wavy}} = AB$$

$$(BAB)A = I$$

$$= \left| -\frac{2}{3} A^{-1} \right| = \left(-\frac{2}{3} \right)^3 \cdot |A^{-1}| = -\frac{8}{27} \cdot 2 = -\frac{16}{27}$$

21. $\because A=0 \Rightarrow A^T A = 0$

$\Leftarrow A^T A = 0$

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$a_{11}^2 + a_{21}^2 + \dots + (a_{n1})^2 = 0$$

23. $r(A) = r(A^T A)$

秩3种语言:

① 矩阵的秩

② 向量组的秩

③ 方程组语言的

$$AX=0 \Rightarrow A^T AX=0$$

$n \times m \quad m \times n \quad \rightarrow \quad n \times 1$

$\Leftarrow: A^T AX=0$

$n \times m$

$$\Rightarrow \underbrace{X^T}_{1 \times n} A^T AX = 0$$

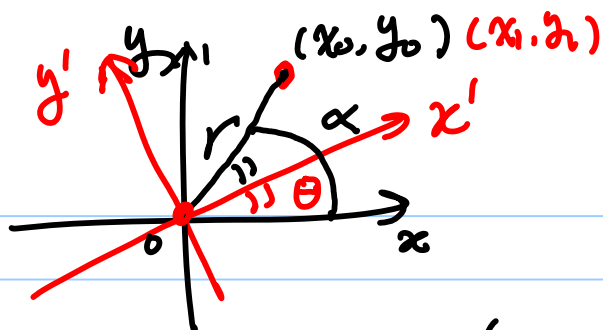
$$\Rightarrow (AX)^T AX = 0 \quad \text{由 21 题}$$

$$\Rightarrow AX=0$$

$\text{即 } AX=0 \text{ 与 } A^T AX=0 \text{ 同解}$

秩和维数: $n - r(A) = n - r(A^T A)$

20.



$$\begin{cases} x_0 = r \cos \alpha \\ y_0 = r \sin \alpha \end{cases}$$

$$\begin{cases} x_1 = r \cos(\alpha - \theta) = \underline{r (\cos \alpha \cos \theta + \sin \alpha \sin \theta)} \\ y_1 = r \sin(\alpha - \theta) = \underline{r (\sin \alpha \cos \theta - \cos \alpha \sin \theta)} \end{cases}$$

$$\begin{cases} x_1 = x_0 \cos \theta + y_0 \sin \theta \\ y_1 = -x_0 \sin \theta + y_0 \cos \theta \end{cases}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}}_{\text{记为 } \tilde{A}} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^{-1}}_{\tilde{A}^{-1}} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{\tilde{A}^T}$$

$$\tilde{A}^{-1} \tilde{A} = \tilde{A}^T \tilde{A} = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \sin \theta \cos \theta - \cos \theta \sin \theta \\ \sin \theta \cos \theta - \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{A}^{-1} = \tilde{A}^T$$

即 $\tilde{A}$ 为 正交矩阵

第三章 空间解析几何(I)

便笺标题

2015/11/24

§3.1. 向量记号.

1. 定义: 既有大小, 又有方向.

表示: $\alpha, \beta, \gamma, \overrightarrow{AB}$

模长: $|\overrightarrow{AB}|$

代数表示 (x, y, z)

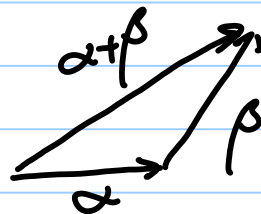
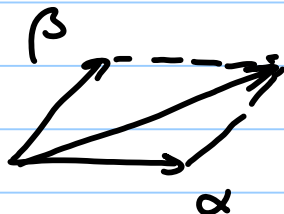
向量相等: $\alpha = \beta \Leftrightarrow$ 模长相等, 方向相同.

反向量: $-\alpha$: 仅与 α 方向相反, 模长相等.

单位向量: $|\alpha| = 1$

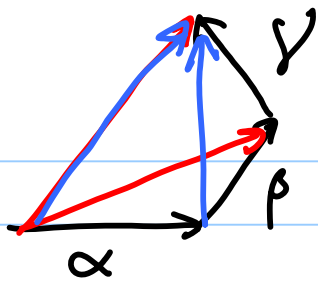
2. 运算

加法:



性质: $\alpha + \beta = \beta + \alpha$
 $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

$$\alpha + 0 = 0 + \alpha = \alpha$$
$$\alpha + (-\alpha) = 0.$$



规定 $\alpha - \beta = \alpha + (-\beta)$

数乘: $k\alpha$. 长度 $|k||\alpha|$

方向 $k > 0$ 与 α 相同 $k < 0$ 与 α 相反

零向量: $|\alpha| = 0$ 方向: 任意.

3. 共线与共面向量.

① 共线: α, β 同向或反向.

② 共面. α, β 均与同一平面平行.

$\forall k. k\alpha \parallel \alpha.$

定义1: α, β 共线 $\Leftrightarrow \exists$ 不全为零数 x, y 使

$$x\alpha + y\beta = 0$$

必要42: α, β 共线 $\exists k: \alpha = k\beta$ 或 $\beta = k\alpha. \Rightarrow x\alpha + y\beta = 0.$

充分43. 不妨设 $x \neq 0$

$$\alpha = -\frac{y}{x}\beta. \quad \alpha, \beta \text{ 共线.}$$

定义2: $\alpha_1, \alpha_2, \alpha_3$ 共面 $\Leftrightarrow$ 存在不全为零 k_1, k_2, k_3

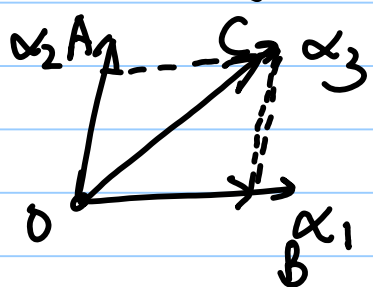
$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0$$

证明: (1) $\alpha_1, \alpha_2, \alpha_3$ 中存在平行的向量: 设 $\alpha_1 \parallel \alpha_2$

$\exists$ 不全为零 k_1, k_2 . $k_1\alpha_1 + k_2\alpha_2 = 0$

$$k_1\alpha_1 + k_2\alpha_2 + 0 \cdot \alpha_3 = 0.$$

(2) 两两均不平行.

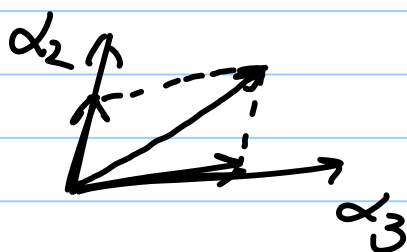


$$\alpha_3 = k_1\alpha_1 + k_2\alpha_2.$$

$$k_1\alpha_1 + k_2\alpha_2 + (-1)\alpha_3 = 0.$$

充分性: $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0$. 不妨设 $k_1 \neq 0$.

$$\alpha_1 = -\frac{k_2}{k_1}\alpha_2 - \frac{k_3}{k_1}\alpha_3$$



推论: α, β 不共线 $\Leftrightarrow \begin{cases} k_1\alpha + k_2\beta = 0 \\ \Rightarrow k_1 = k_2 = 0 \end{cases}$

$\alpha_1, \alpha_2, \alpha_3$ 不共面 $\Leftrightarrow \begin{cases} k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0 \\ \Rightarrow k_1 = k_2 = k_3 = 0 \end{cases}$

教材. P84 例1. 例2.

§3.2 仿射坐标系.

$\alpha = k e$


 $\alpha = k_1 e_1 + k_2 e_2$. 表示法叫作 - .
 $= k'_1 e_1 + k'_2 e_2$

$$(k_1 - k'_1) e_1 + (k_2 - k'_2) e_2 = 0$$

e_1, e_2 不共线 $\Leftrightarrow k_1 = k_1', k_2 = k_2'$



$$\alpha = k_1 e_1 + k_2 e_2 + k_3 e_3$$

$$\alpha = \overrightarrow{OP'} + \overrightarrow{P'P} = \overrightarrow{OP'} +$$

$$\alpha = \overrightarrow{op'} + \overrightarrow{p'p} = \underbrace{\overrightarrow{op'}}_{k_1 e_1 + k_2 e_2} + k_3 e_3$$

定义: O 及不共面的向量 e_1, e_2, e_3 可以构成一个仿射坐标系 $\{O; e_1, e_2, e_3\}$. O 称为原点.

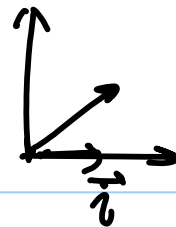
e_i 基底

$$\vec{op} = x e_1 + y e_2 + z e_3.$$

(x, y, z) 为 $\vec{OP}$ 在 $\{0; e_1, e_2, e_3\}$ 下的一组坐标。

右手系. 左手系.

$$2. \quad \vec{OP} = x e_1 + y e_2 + z e_3 \\ = \underbrace{[e_1 \ e_2 \ e_3]} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



$$[0; e_1, e_2, e_3] \quad \alpha = x_1 e_1 + x_2 e_2 + x_3 e_3 \\ = [e_1, e_2, e_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

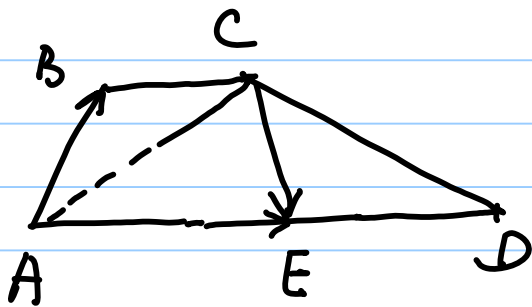
$$\alpha = (x_1, x_2, x_3)$$

定义: M 是仿射坐标系中的点, 称 $\vec{OM}$ 在仿射系为 M 在

$$[0; e_1, e_2, e_3] \text{ 下的坐标} \quad \vec{OM} = x_1 e_1 + x_2 e_2 + x_3 e_3$$

$$\text{记 } M(x_1, x_2, x_3).$$

例: 梯形 $ABCD$ 中 $AD \parallel BC$. $AD = 3BC$ E 为 AD 中点. 求 C 及向量 $\vec{CE}$ 在 $[A; \vec{AE}, \vec{AB}]$ 下的坐标.



$$\vec{AC} = \vec{AB} + \vec{BC} \\ = \vec{AB} + \frac{1}{3} \vec{AD} \\ = \vec{AB} + \frac{2}{3} \vec{AE}$$

$$C = \left(\frac{2}{3}, 1 \right)$$

$$\vec{CE} = \vec{CA} + \vec{AE} = -\vec{AB} - \frac{2}{3} \vec{AE} + \vec{AE} \\ = -\vec{AB} + \frac{1}{3} \vec{AE}$$

$$\vec{CE} = \left(\frac{1}{3}, -1 \right)$$

2. 向量运算: $\{0; e_1, e_2, e_3\}$. $\alpha = (x_1, x_2, x_3)$

$$\beta = (y_1, y_2, y_3) \text{ 则 } \alpha \pm \beta = (x_1 \pm y_1, x_2 \pm y_2, x_3 \pm y_3)$$

$$k\alpha = (kx_1, kx_2, kx_3).$$

证: $\alpha = x_1 e_1 + x_2 e_2 + x_3 e_3$

$$\beta = y_1 e_1 + y_2 e_2 + y_3 e_3.$$

$$\alpha + \beta = (x_1 e_1 + x_2 e_2 + x_3 e_3) + (y_1 e_1 + y_2 e_2 + y_3 e_3)$$

$$= (x_1 + y_1) e_1 + (x_2 + y_2) e_2 + (x_3 + y_3) e_3$$

$$\alpha + \beta = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \quad \text{证.}$$

例: $\alpha = (1, -2, 5)$ $\beta = (-3, 6, 1)$

$$\alpha + \beta = (-2, 4, 6). \quad 3\alpha - 5\beta = (18, -36, 10)$$

例. $A(3, 1, 2)$ $B(1, -1, 5)$ 求 $\vec{OA}$ 及 $\vec{AB}$

$$\vec{OA} = (3, 1, 2).$$

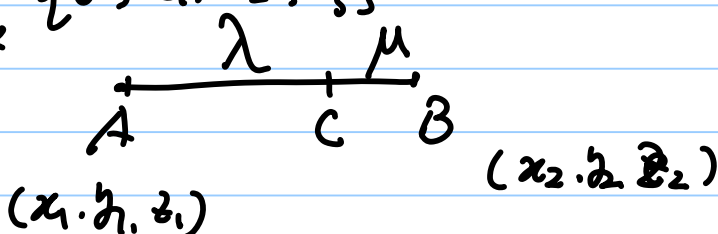
$$\vec{OB} = (1, -1, 5)$$

$$\vec{AB} = \vec{AO} + \vec{OB} = \vec{OB} - \vec{OA} = (-2, -2, 3)$$

$$A(x_1, y_1, z_1). \quad \vec{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

$$B(x_2, y_2, z_2)$$

例: $\{0; e_1, e_2, e_3\}$



(x_1, y_1, z_1) (x_2, y_2, z_2)

$$\mu \vec{AC} = \lambda \vec{CB}$$

$$\vec{AC} = \frac{\lambda}{\lambda + \mu} \vec{AB}$$

$$(x_c - x_1, y_c - y_1, z_c - z_1) = \frac{\lambda}{\lambda + \mu} (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

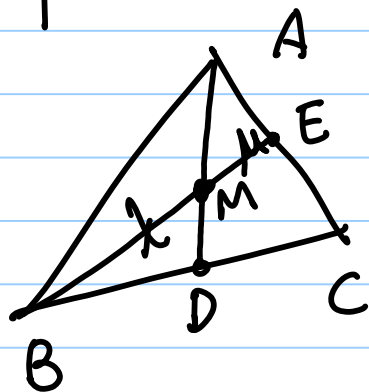
$$x_c = \frac{\lambda(x_2 - x_1)}{\lambda + \mu} + x_1 = \frac{\mu x_1 + \lambda x_2}{\lambda + \mu}$$

$$y_c =$$

$$z_c.$$

$$\text{中点公式: } \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

例: 证明: $\triangle ABC$ 重心分中线比为 $2:1$



$$\vec{AD} = \frac{1}{2}(\vec{AB} + \vec{AC})$$

$$\begin{aligned} \vec{AM} &= \vec{AB} + \vec{BM} \\ &= \vec{AB} + \frac{\lambda}{\lambda + \mu} \vec{BE} \end{aligned}$$

$$= \vec{AB} + \frac{\lambda}{\lambda + \mu} \cdot \frac{1}{2}(\vec{BA} + \vec{BC})$$

$$= \vec{AB} + \frac{\lambda}{2(\lambda + \mu)}(\vec{AC} - 2\vec{AB})$$

$$= \frac{\mu}{\lambda + \mu} \vec{AB} + \frac{\lambda}{2(\lambda + \mu)} \vec{AC}$$

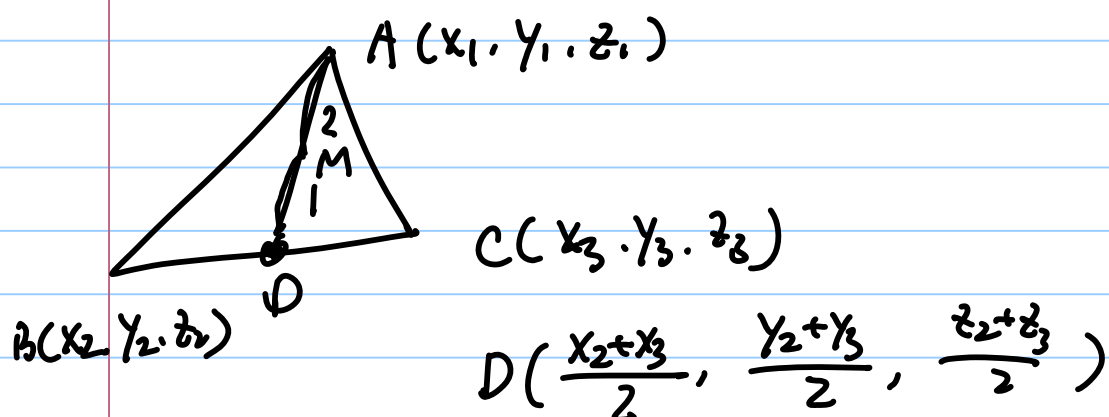
$$\vec{AM} \parallel \vec{AD} \cdot \exists k \text{ 使 } k\vec{AD} = \vec{AM}.$$

$$\frac{k}{2}(\vec{AB} + \vec{AC}) = \frac{\mu}{\lambda + \mu} \vec{AB} + \frac{\lambda}{2(\lambda + \mu)} \vec{AC}$$

$$\left(\frac{\mu}{\lambda + \mu} - \frac{k}{2} \right) \vec{AB} + \left(\frac{\lambda}{2(\lambda + \mu)} - \frac{k}{2} \right) \vec{AC} = \vec{0}$$

$$\vec{AB}, \vec{AC} \text{ 不共线, } \frac{\mu}{\lambda + \mu} = \frac{\lambda}{2(\lambda + \mu)} = \frac{k}{2}.$$

$$\text{pp } \lambda = 2\mu,$$



$$M(x_m, y_m, z_m)$$

$$x_m = \frac{x_1 + 2 \cdot \frac{x_2+x_3}{2}}{3} = \frac{x_1+x_2+x_3}{3}$$

$$y_m = \frac{y_1+y_2+y_3}{3}$$

$$z_m = \frac{z_1+z_2+z_3}{3}$$

3. 向量共线 共面 坐标表示.

$$\alpha = (x_1, x_2, x_3) \quad \beta = (y_1, y_2, y_3) \quad \gamma = (z_1, z_2, z_3)$$

α, β 共线 $\Leftrightarrow$ 存在不为零常数 λ, μ 使

$$\lambda\alpha + \mu\beta = 0$$

$$\lambda x_1 + \mu y_1 = 0 \quad \lambda x_2 + \mu y_2 = 0 \quad \lambda x_3 + \mu y_3 = 0$$

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} \quad (y_i \neq 0 \Leftrightarrow x_i = 0)$$

α, β, γ 共面 $\Leftrightarrow$ 存在不全为零的 k_1, k_2, k_3

$$k_1\alpha + k_2\beta + k_3\gamma = 0$$

$$\begin{cases} k_1x_1 + k_2y_1 + k_3z_1 = 0 \\ k_1x_2 + k_2y_2 + k_3z_2 = 0 \\ k_1x_3 + k_2y_3 + k_3z_3 = 0 \end{cases}$$

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{有非零解.}$$

$$\Leftrightarrow \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

4. 空间直角坐标系

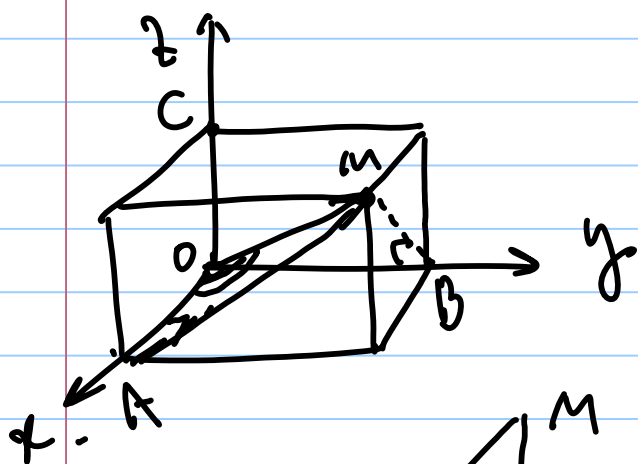
仿射坐标系 $\hat{+}$ $\begin{cases} e_1, e_2, e_3 \text{ 右手系} \\ e_1, e_2, e_3 \text{ 两两垂直} \\ |e_1| = |e_2| = |e_3| = 1. \end{cases} \Rightarrow$ 空间直角坐标系.

$$e_1 = \vec{i}, e_2 = \vec{j}, e_3 = \vec{k}.$$

距离公式: ① (两点间) 距离公式 $P(x_1, y_1, z_1)$
 $Q(x_2, y_2, z_2)$

$$|PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

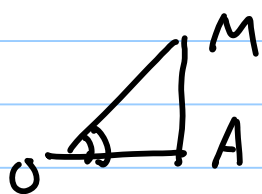
② 方向角: $\vec{OM}$ 与 $\vec{i}, \vec{j}, \vec{k}$ 的夹角称为 $\vec{OM}$ 的方向角.



$MA \perp OA$

$$\cos \alpha = \frac{OA}{|OM|} \rightarrow \text{数量}$$

$$\cos \beta = \frac{OB}{|OM|} \rightarrow \text{数量}$$



$$\cos \gamma = \frac{OC}{|OM|} \rightarrow \text{数量}$$

$\{\cos \alpha, \cos \beta, \cos \gamma\}$ 称为 $\vec{OM}$ 的方向余弦

例: $A(1, -1, 2) \quad B(3, 1, 1)$

$$\vec{AB} = (2, 2, -1) \quad |\vec{AB}| = 3$$

$$\cos \alpha = \frac{2}{3}, \quad \cos \beta = \frac{2}{3}, \quad \cos \gamma = -\frac{1}{3}$$

单位化: 给定 α , 称 $\frac{\alpha}{|\alpha|}$ 为 α 的单位化向量.
 $\alpha \neq 0$.

$$\vec{OM} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= |\vec{OM}| (\cos \alpha \vec{i} + \cos \beta \vec{j} + \cos \gamma \vec{k})$$

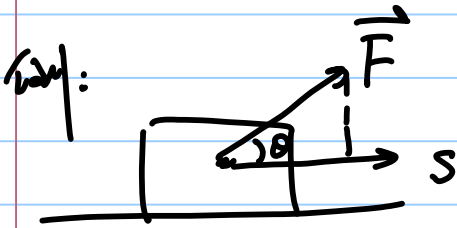
记 $\frac{\alpha}{|\alpha|} = \alpha_0$

§ 3.4. 数量积, 向量积, 混合积.

(内积, 点积.)

(外积, 叉积.)

一. 数量积:



$$W = |\vec{F}| \cdot \cos \theta \cdot S$$

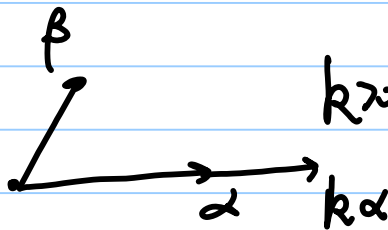
$[0, \pi]$

定义: $\alpha \cdot \beta = |\alpha| \cdot |\beta| \cos \langle \alpha, \beta \rangle$

性质:

(1) $\alpha \cdot \beta = \beta \cdot \alpha$

(2) $(k\alpha) \cdot \beta = \alpha \cdot (k\beta) = k(\alpha \cdot \beta)$



证: $(k\alpha) \cdot \beta = |k\alpha| \cdot |\beta| \cos \langle k\alpha, \beta \rangle$

$$= |k| |\alpha| \cdot |\beta| \cos \langle \alpha, \beta \rangle$$

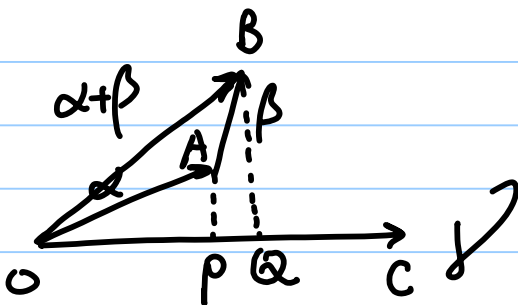
$$= k |\alpha| \cdot |\beta| \cos \langle \alpha, \beta \rangle = k(\alpha \cdot \beta)$$

$$k(\alpha \cdot \beta) = k |\alpha| |\beta| \cos \langle \alpha, \beta \rangle$$

$k=0$ ✓

$k < 0$

(3) $(\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$



$$(\alpha + \beta) \cdot \gamma = |\vec{OB}| \cdot |\vec{OC}| \cdot \cos \angle BOC$$

$$= |\vec{OA}| \cdot |\vec{OC}|$$

$$= \vec{OA} \cdot \vec{OC}$$

$$\alpha \cdot \gamma = \vec{OA} \cdot \vec{OC}$$

$$\beta \cdot \gamma = \vec{PA} \cdot \vec{OC}$$

$$\frac{1}{2} \quad \overrightarrow{op} = \lambda \overrightarrow{oc} \quad \overrightarrow{pq} = \mu \overrightarrow{oc}$$

$$\alpha \cdot \gamma = \overrightarrow{op} \cdot \overrightarrow{oc} = \lambda (\overrightarrow{oc} \cdot \overrightarrow{oc})$$

$$\beta \cdot \gamma = \mu (\overrightarrow{oc} \cdot \overrightarrow{oc})$$

$$(\alpha + \beta) \cdot \gamma = (\overrightarrow{op} + \overrightarrow{pq}) \cdot \overrightarrow{oc} = (\lambda + \mu) \overrightarrow{oc} \cdot \overrightarrow{oc}$$

$$\text{即} \quad (\alpha + \beta) \cdot \gamma = \alpha \cdot \gamma + \beta \cdot \gamma$$

$$(4) \quad \alpha^2 = \alpha \cdot \alpha = |\alpha| \cdot |\alpha| \cos 0 = |\alpha|^2$$

$$\alpha^2 \geq 0, \quad \alpha^2 = 0 \Leftrightarrow \alpha = 0.$$

2. 内积的坐标运算:

$$\{0, e_1, e_2, e_3\}, \quad \alpha = (x_1, x_2, x_3) \\ \beta = (y_1, y_2, y_3)$$

$$\alpha \cdot \beta = (x_1 e_1 + x_2 e_2 + x_3 e_3) \cdot (y_1 e_1 + y_2 e_2 + y_3 e_3)$$

$$= x_1 y_1 e_1 \cdot e_1 + x_1 y_2 e_1 \cdot e_2 + x_1 y_3 e_1 \cdot e_3$$

$$+ x_2 y_1 e_2 \cdot e_1 + x_2 y_2 e_2 \cdot e_2 + x_2 y_3 e_2 \cdot e_3$$

$$+ x_3 y_1 e_3 \cdot e_1 + x_3 y_2 e_3 \cdot e_2 + x_3 y_3 e_3 \cdot e_3$$

$$= [x_1, x_2, x_3] \begin{bmatrix} y_1 e_1 \cdot e_1 + y_2 e_1 \cdot e_2 + y_3 e_1 \cdot e_3 \\ y_1 e_2 \cdot e_1 + y_2 e_2 \cdot e_2 + y_3 e_2 \cdot e_3 \\ y_1 e_3 \cdot e_1 + y_2 e_3 \cdot e_2 + y_3 e_3 \cdot e_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} e_1 \cdot e_1 & e_1 \cdot e_2 & e_1 \cdot e_3 \\ e_2 \cdot e_1 & e_2 \cdot e_2 & e_2 \cdot e_3 \\ e_3 \cdot e_1 & e_3 \cdot e_2 & e_3 \cdot e_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

例 $A = (e_i e_j)_{3 \times 3}$ 为 $\{0; e_1, e_2, e_3\}$ 度量矩阵.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

A 是对称矩阵.

$$\alpha \cdot \beta = X^T A Y$$

直角坐标系下 $A = I_3$. $\alpha \cdot \beta = X^T Y$
 $= x_1 y_1 + x_2 y_2 + x_3 y_3$

例 1: $\{0; e_1, e_2, e_3\}$, $|e_1| = 1$, $|e_2| = 1$, $|e_3| = 2$.

$$\langle e_1, e_2 \rangle = 60^\circ \quad \langle e_1, e_3 \rangle = \langle e_2, e_3 \rangle = 45^\circ$$

$$\alpha = (2, 0, -\sqrt{2}) \quad \beta = (3, \sqrt{2}, 1) \quad \text{求 } \alpha \cdot \beta.$$

$$e_1 \cdot e_1 = 1, \quad e_2 \cdot e_2 = 1, \quad e_3 \cdot e_3 = 4$$

$$e_1 \cdot e_2 = e_2 \cdot e_1 = \frac{1}{2}, \quad e_1 \cdot e_3 = e_3 \cdot e_1 = \sqrt{2},$$

$$e_2 \cdot e_3 = e_3 \cdot e_2 = \sqrt{2}.$$

度量矩阵为

$$\begin{bmatrix} 1 & \frac{1}{2} & \sqrt{2} \\ \frac{1}{2} & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 4 \end{bmatrix}$$

$$\alpha \cdot \beta = [2, 0, -\sqrt{2}] \begin{bmatrix} 1 & \frac{1}{2} & \sqrt{2} \\ \frac{1}{2} & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 4 \end{bmatrix} \begin{bmatrix} 3 \\ \sqrt{2} \\ 1 \end{bmatrix} = \underline{\hspace{2cm}}.$$

3. 内积应用:

$$① |\alpha| = \sqrt{\alpha^2}.$$

$$② \cos \theta = \frac{\alpha \cdot \beta}{|\alpha| \cdot |\beta|} = \frac{\alpha \cdot \beta}{\sqrt{\alpha^2} \cdot \sqrt{\beta^2}},$$

在 $\mathbb{R}^3$ 坐标下 $\alpha = (x_1, y_1, z_1), \beta = (x_2, y_2, z_2).$

$$\cos \theta = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2}}.$$

$$③ \text{垂直. } \alpha \perp \beta \Leftrightarrow \langle \alpha, \beta \rangle = \frac{\lambda}{2} \Leftrightarrow \alpha \cdot \beta = 0$$

例: 求上例中, $e_1 + e_2$ 与 e_3 夹角

$$(e_1 + e_2) \cdot e_3 = e_1 \cdot e_3 + e_2 \cdot e_3 = 2\sqrt{2}.$$

$$(e_1 + e_2) \cdot e_3 = |e_1 + e_2| \cdot |e_3| \cos \theta = \sqrt{3} \cdot 2 \cos \theta$$

$$|e_1 + e_2|^2 = e_1^2 + 2e_1 \cdot e_2 + e_2^2 = 3$$

$$\cos \theta = \frac{\sqrt{6}}{3} \quad \theta = \arccos \frac{\sqrt{6}}{3}$$

例: Cauchy-Schwarz 不等式.

$$\alpha = (a_1, a_2, a_3).$$

$$\beta = (b_1, b_2, b_3)$$

$$\alpha \cdot \beta = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\alpha \cdot \beta = |\alpha| \cdot |\beta| \cos \angle \alpha, \beta$$

$$|\alpha \cdot \beta| \leq |\alpha| \cdot |\beta|$$

绝对值

$$|a_1 b_1 + a_2 b_2 + a_3 b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}$$

$$(a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

取等号 $\Leftrightarrow \cos \langle \alpha, \beta \rangle = \pm 1$

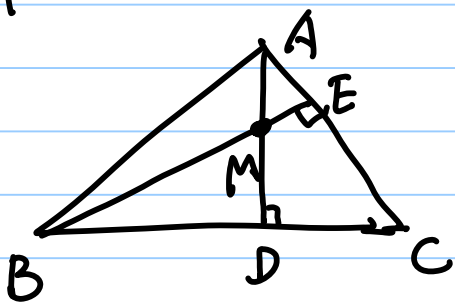
$$\Leftrightarrow \alpha, \beta \text{ 共线}$$

$$\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3},$$

例: $\forall \beta, \alpha \perp \beta$ 则 $\alpha = 0$.

证: $\alpha \perp \alpha, \alpha \cdot \alpha = 0, |\alpha|^2 = 0, \alpha = 0$.

例 1. 证明: $\triangle ABC$ 中, 三条高交于一点.



$$\overrightarrow{AM} \cdot \overrightarrow{BC} = 0$$

$$\overrightarrow{BM} \cdot \overrightarrow{AC} = 0$$

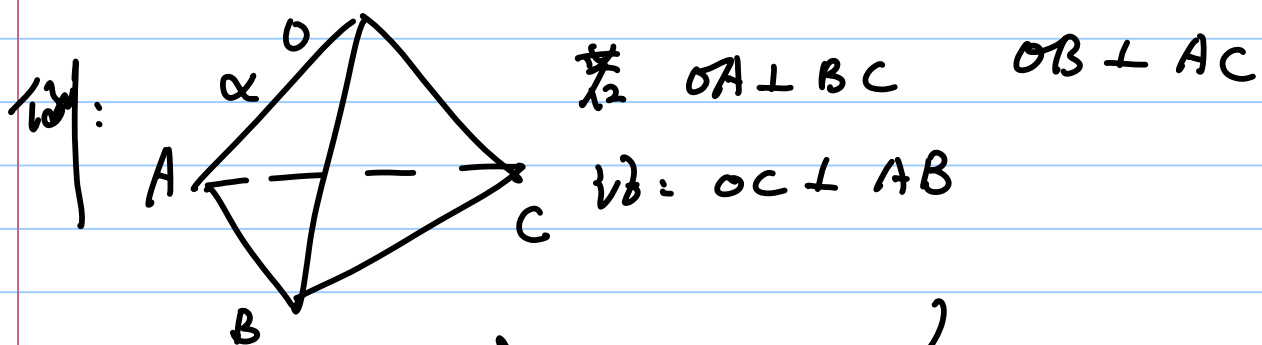
$$\text{求证: } \overrightarrow{CM} \cdot \overrightarrow{AB} = 0$$

证: $\overrightarrow{AM} \cdot \overrightarrow{BC} = 0 \Rightarrow (\overrightarrow{AB} + \overrightarrow{BM}) \cdot \overrightarrow{BC} = 0$

$$\overrightarrow{BM} \cdot \overrightarrow{AC} = 0 \Rightarrow \overrightarrow{BM} \cdot (\overrightarrow{AB} + \overrightarrow{BC}) = 0$$

$$\overrightarrow{AB} \cdot \overrightarrow{BC} = \overrightarrow{BM} \cdot \overrightarrow{AB}$$

$$\overrightarrow{AB} \cdot \overrightarrow{CM} = 0$$



Let: $\vec{OA} = \alpha$, $\vec{OB} = \beta$, $\vec{OC} = \gamma$

$\vec{AB} = \vec{OB} - \vec{OA} = \beta - \alpha$, $\vec{BC} = \gamma - \beta$.

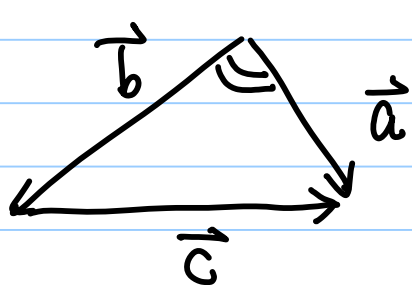
$\vec{CA} = \alpha - \gamma$.

$$\alpha \cdot (\gamma - \beta) = 0 \quad \beta \cdot (\alpha - \gamma) = 0$$

$$\Rightarrow \alpha \cdot \gamma = \beta \cdot \alpha \quad (\alpha - \beta) \cdot \gamma = 0$$

$$\boxed{AB \perp OC}.$$

Ex: 向量法证明余弦定理.



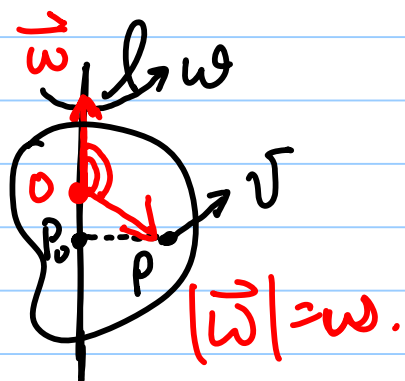
$$\vec{c} = \vec{a} - \vec{b}$$

$$\vec{c}^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$

$$= \vec{a} \cdot \vec{a} - 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$$

∴ $|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\angle(\vec{a}, \vec{b})$

二、向量积 (外积, 叉乘)



$$|v| = w \cdot |OP| = w \cdot r$$

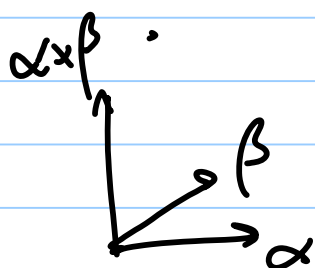
$$= |\vec{w}| \cdot |\vec{OP}| \sin \langle \vec{OP}, \vec{w} \rangle$$

定义: $\alpha \times \beta$ 是这样一·个向量.

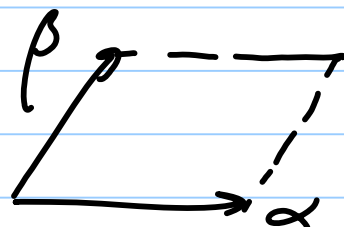
① 大小, $|\alpha \times \beta| = |\alpha| |\beta| \sin \langle \alpha, \beta \rangle$

② 方向: $\alpha \times \beta$ 与 α, β 均垂直

要求 $\alpha, \beta, \alpha \times \beta$ 成右手系,



Note: $|\alpha \times \beta|$ 为 α, β 构成



二·倍[?]面积 = 面积..

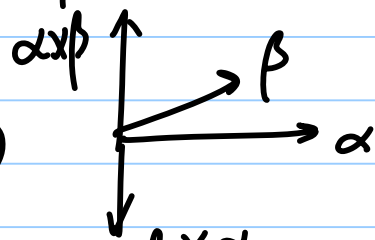
2. 外积性质:

1) $\alpha \times \beta = -\beta \times \alpha$

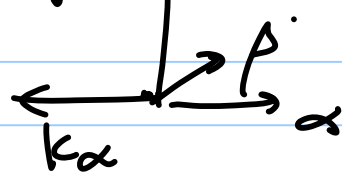
2) $(k\alpha) \times \beta = \alpha \times (k\beta) = k(\alpha \times \beta)$

(3) $\alpha \times (\beta + \gamma) = \alpha \times \beta + \alpha \times \gamma$

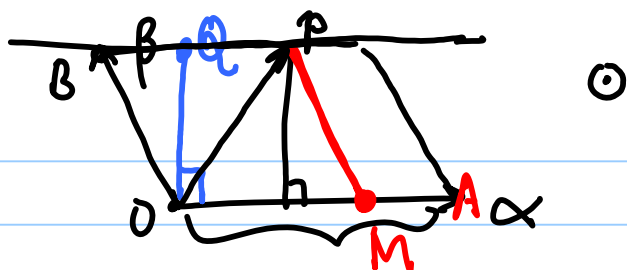
$\alpha \times \beta, \beta \times \alpha$



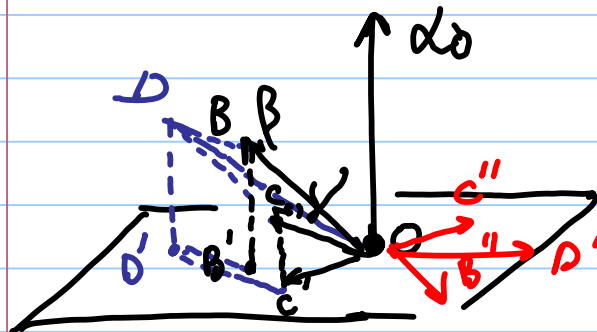
$\beta \times \alpha$



$$\begin{aligned}
 \textcircled{1} \quad \alpha \times \beta &= \alpha \times \overrightarrow{OP} \\
 &= \alpha \times (\overrightarrow{OB} + \overrightarrow{OM}) \\
 &= \alpha \times (\beta + k\alpha) \\
 \alpha \times \beta &= \alpha \times \overrightarrow{OQ} \rightarrow \alpha_{\perp}
 \end{aligned}$$



$$|\alpha_0 \times \alpha_{\perp}| = |\alpha_0| |\alpha_{\perp}| \cdot \sin \angle \alpha_0, \alpha_{\perp} = |\alpha_{\perp}|$$



$$\sum \alpha_0 \times (\beta + \gamma) = \alpha_0 \times \beta + \alpha_0 \times \gamma$$

$$\alpha_0 \times \beta = \alpha_0 \times \overrightarrow{OB'} = \overrightarrow{OB''}$$

$$\alpha_0 \times \gamma = \alpha_0 \times \overrightarrow{OC'} = \overrightarrow{OC''}$$

$$\alpha_0 \times (\beta + \gamma) = \alpha_0 \times \overrightarrow{OD'} = \overrightarrow{OD''}$$

$$\overrightarrow{OB''} + \overrightarrow{OC''} = \overrightarrow{OD''}$$

$$\beta + \gamma = \overrightarrow{OD'}$$

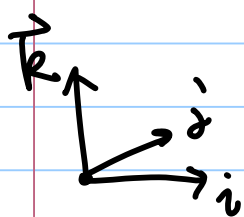
$$\alpha = |\alpha| \alpha_0$$

直角坐标系中：

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$

$$\vec{i} \times \vec{j} = \vec{k}$$

$$\vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}$$



$$\alpha = (x_1, y_1, z_1) \quad \beta = (x_2, y_2, z_2)$$

$$\begin{aligned} \alpha \times \beta &= (x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k}) \times (x_2 \vec{i} + y_2 \vec{j} + z_2 \vec{k}) \\ &= (x_1 y_2 - y_1 x_2) \vec{i} \times \vec{j} + (y_1 z_2 - y_2 z_1) \vec{j} \times \vec{k} \\ &\quad + (z_1 x_2 - z_2 x_1) \vec{k} \times \vec{i} \\ &= (y_1 z_2 - y_2 z_1) \vec{i} + (z_1 x_2 - z_2 x_1) \vec{j} \\ &\quad + (x_1 y_2 - y_1 x_2) \vec{k} \end{aligned}$$

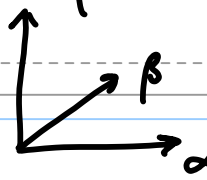
$$\alpha \times \beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

17.4: 习题 3

1-20 \setminus \{9\}

$\alpha \times \beta$. $\alpha \times \beta$ 大小 $|\alpha \times \beta| = |\alpha||\beta|\sin\langle\alpha, \beta\rangle$

且 $\alpha, \beta, \alpha \times \beta$ 两两垂直.



α, β 两两垂直

$$\alpha = x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k}$$

$$\beta = y_1 \vec{i} + y_2 \vec{j} + y_3 \vec{k}$$

$$\alpha \times \beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

$$\alpha \times \beta = -\beta \times \alpha$$

例: $(\vec{i} + 2\vec{j}) \times 2\vec{k} = -2\vec{j} + 4\vec{i}$

例: $\alpha \parallel \beta$ 时 $k\alpha + 9\beta$ 与 $4\alpha + k\beta$ 垂直

$$k\alpha + 9\beta \perp 4\alpha + k\beta$$

$$(k\alpha + 9\beta) \times (4\alpha + k\beta) = \vec{0}$$

$$k^2 \alpha \times \beta - 36 \alpha \times \beta = \vec{0}$$

$$(k^2 - 36) \alpha \times \beta = \vec{0}$$

$$\Rightarrow k^2 = 36. \quad k = \pm 6$$

$$\vec{i} \cdot (\vec{j} \cdot \vec{k})$$

$$\begin{matrix} \vec{k} & \vec{i} & \vec{j} \\ \vec{j} & \vec{k} & \vec{i} \end{matrix}$$

$$\alpha \parallel \beta$$

$$\Leftrightarrow \alpha \times \beta = \vec{0}$$

例: α, β, γ 不共线 且 $\alpha + \beta + \gamma = \vec{0} \Leftrightarrow \alpha \times \beta = \beta \times \gamma = \gamma \times \alpha$

若 $\alpha + \beta + \gamma = \vec{0} \Rightarrow \alpha = -(\beta + \gamma)$

$$\alpha \times \beta = -(\beta + \gamma) \times \beta = -\gamma \times \beta = \beta \times \gamma$$

...

$$\text{证: } \alpha \times \beta = \beta \times \gamma = \gamma \times \alpha$$

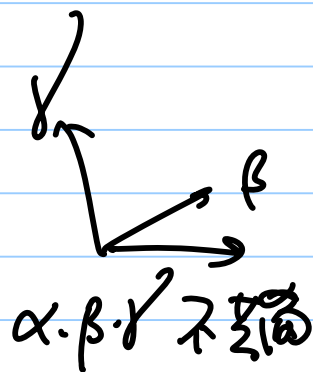
$$\Rightarrow \alpha \times (\alpha + \beta + \gamma) = \alpha \times \beta + \alpha \times \gamma = \vec{0}$$

$$\Rightarrow \alpha \text{ 与 } \alpha + \beta + \gamma \text{ 共线}$$

$$\text{类似推出 } \beta \text{ 与 } \alpha + \beta + \gamma \text{ 共线}$$

$$\gamma \text{ 与 } \alpha + \beta + \gamma \text{ 共线}$$

$$\Rightarrow \alpha + \beta + \gamma = \vec{0}$$



α, β, γ 不共面

1. 混合积

α, β, γ 混合积 记为: $(\alpha \times \beta) \cdot \gamma$

$$\text{记为 } (\alpha, \beta, \gamma) = (\alpha \times \beta) \cdot \gamma$$

取坐标: $\alpha = (x_1, x_2, x_3) \quad \beta = (y_1, y_2, y_3)$

$$\gamma = (z_1, z_2, z_3)$$

$$\alpha \times \beta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2) \vec{i} + (x_3 y_1 - x_1 y_3) \vec{j} + (x_1 y_2 - x_2 y_1) \vec{k}$$

$$\gamma = z_1 \vec{i} + z_2 \vec{j} + z_3 \vec{k}$$

$$(\alpha \times \beta) \cdot \gamma = (x_2 y_3 - x_3 y_2) z_1 + (x_3 y_1 - x_1 y_3) z_2$$

$$+ (x_1 y_2 - x_2 y_1) z_3$$

$$= \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

• 混合积公式 (τ) .

$$① (\alpha, \beta, \gamma) = (\beta, \gamma, \alpha) = (\gamma, \alpha, \beta)$$

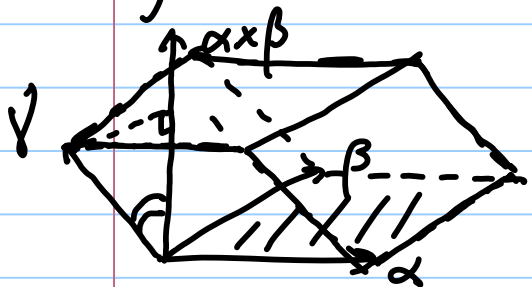
$$② (\alpha, \beta, \gamma) = -(\beta, \alpha, \gamma)$$

$$③ (k\alpha, \beta, \gamma) = (\alpha, k\beta, \gamma) = (\alpha, \beta, k\gamma) \\ = k(\alpha, \beta, \gamma)$$

$$④ (\alpha_1 + \alpha_2, \beta, \gamma) = (\alpha_1, \beta, \gamma) + (\alpha_2, \beta, \gamma)$$

几何意义:

α, β, γ 构成平行六面体.



底面积为 $|\alpha \times \beta|$

$$h = |\gamma| |\cos \langle \alpha \times \beta, \gamma \rangle|$$

若 α, β, γ 成右手系, $\cos \langle \alpha \times \beta, \gamma \rangle > 0$

$$V = |\alpha \times \beta| |\gamma| \cos \langle \alpha \times \beta, \gamma \rangle$$

$$= (\alpha, \beta, \gamma)$$

$$\text{若成左手系 } V = -(\alpha, \beta, \gamma)$$

混合积, 几何意义 有向体积.

例1: $\alpha = \vec{i} - 3\vec{j} + \vec{k}, \beta = 2\vec{i} - \vec{j} + 3\vec{k}$

$$\gamma = \vec{i} + 2\vec{j} + 3\vec{k}$$

$$(\alpha, \beta, \gamma) = \begin{vmatrix} 1 & -3 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & 3 \end{vmatrix} = \dots$$

例2: α, β, γ 共面 $\Leftrightarrow (\alpha, \beta, \gamma) = 0$

$$\Leftrightarrow \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0.$$

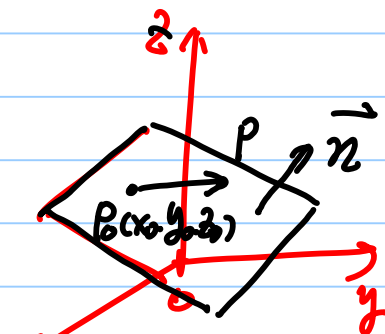
02. 平面与直线方程. 空间中二位置关系. 角度. 距离

(一) 面的方程

① 点法式: $\vec{n} = \{a, b, c\}$

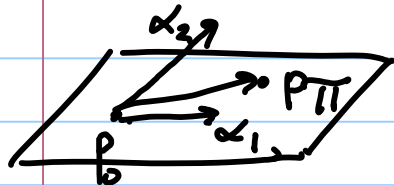
$$\vec{P_0P} \perp \vec{n}$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$



② 参数式: 平面 π 过 $P_0(a, b, c)$ 平行于向量 α_1, α_2 .

$$\alpha_1 = (x_1, y_1, z_1), \alpha_2 = (x_2, y_2, z_2), \alpha_1 \neq \alpha_2$$



$\vec{P_0P}, \alpha_1, \alpha_2$ 共面.

$$\vec{P_0P} = k_1 \alpha_1 + k_2 \alpha_2$$

$$x-a = k_1 x_1 + k_2 x_2$$

$$y-b = k_1 y_1 + k_2 y_2$$

$$z-c = k_1 z_1 + k_2 z_2$$

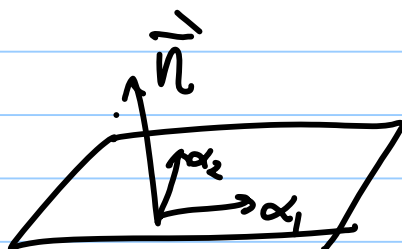
$$\begin{cases} x = a + k_1 x_1 + k_2 x_2 \\ y = b + k_1 y_1 + k_2 y_2 \\ z = c + k_1 z_1 + k_2 z_2 \end{cases} \quad k_1, k_2 \text{ 为任意实数}$$

③ 对称式: $\vec{P_0P}, \alpha_1, \alpha_2$ 共面

$$\begin{vmatrix} x-a & y-b & z-c \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$$

④ 一般式: $Ax + By + Cz + D = 0$

$$\vec{n} = \{A, B, C\}$$

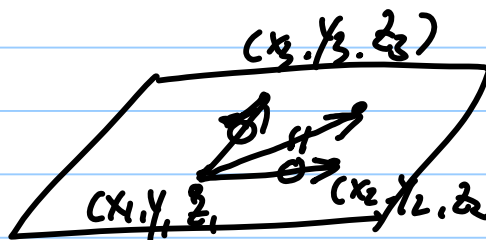


例1: 求过3点 $(1, 2, 3)$ $(-1, 4, 2)$
 $(0, 1, -1)$ 的平面方程.

设1: $Ax + By + Cz + D = 0$

$$\begin{cases} A + 2B + 3C + D = 0 \\ -A + 4B + 2C + D = 0 \\ B - C + D = 0 \end{cases}$$

设2: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$



$$\begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 0 & x-x_1 & y-y_1 & z-z_1 \\ 0 & x_2-x_1 & y_2-y_1 & z_2-z_1 \\ 0 & x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

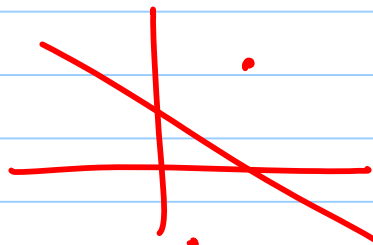
$$= \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x & y & z \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

$$\begin{cases} Ax + By + Cz + D = 0 \\ Ax_1 + By_1 + Cz_1 + D = 0 \\ Ax_2 + By_2 + Cz_2 + D = 0 \\ Ax_3 + By_3 + Cz_3 + D = 0 \end{cases} \begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

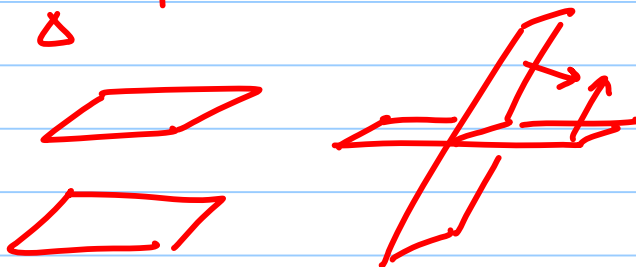
位置关系与度量

① 点与平面 $\begin{cases} \text{点在面上} \\ \text{点在面外} \end{cases}$



$$2x + 3y + 4z + 1 = 0$$

② 平面与平面



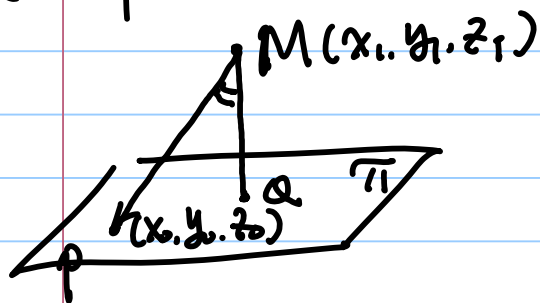
$\begin{cases} \text{相交} \\ \text{平行} \end{cases}$

$\vec{n}_1$ 与 $\vec{n}_2$ 是否平行

$$\pi_1 \perp \pi_2 \Leftrightarrow \vec{n}_1 \perp \vec{n}_2$$

点、线、面距离

$$|\vec{MQ}|$$



$$Ax + By + Cz + D = 0$$

$$\vec{n} = \left\{ \frac{A}{\sqrt{A^2+B^2+C^2}}, \frac{B}{\sqrt{A^2+B^2+C^2}}, \frac{C}{\sqrt{A^2+B^2+C^2}} \right\}$$

$\vec{n}$ 为单位向量

$$|\overrightarrow{MQ}| = |\overrightarrow{MP}| |\cos \langle \overrightarrow{MP}, \vec{n} \rangle| \cdot |\vec{n}|$$

$$= |\overrightarrow{MP} \cdot \vec{n}|$$

↙ 点法式

$$= \left| (x_0 - x_1) \frac{A}{\sqrt{A^2 + B^2 + C^2}} + (y_0 - y_1) \frac{B}{\sqrt{A^2 + B^2 + C^2}} + (z_0 - z_1) \frac{C}{\sqrt{A^2 + B^2 + C^2}} \right|$$

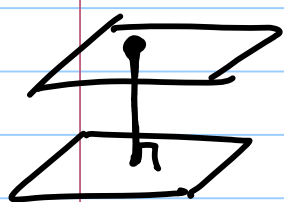
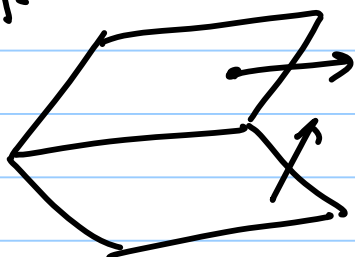
$$= \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$\pi_1: A_1x + B_1y + C_1z + D_1 = 0$$

$$\pi_2: A_2x + B_2y + C_2z + D_2 = 0$$

$$\pi_1 \perp \pi_2 \Leftrightarrow A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

夹角:



$$Ax + By + Cz + D_1 = 0$$

$$Ax + By + Cz + D_2 = 0 \quad \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

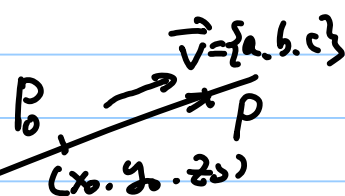
例 过点 $P(2, -3, 1)$ 与 $\pi: 2x + 3y - 1 = 0$ 平行的平面方程

$$2(x - 2) + 3(y + 3) + 0 \cdot (z - 1) = 0.$$

(二) 直线的方程.

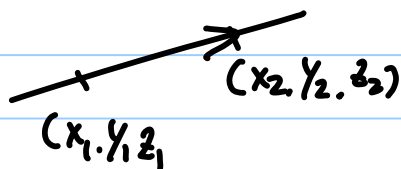
1. 点向式方程: $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

(对称式)



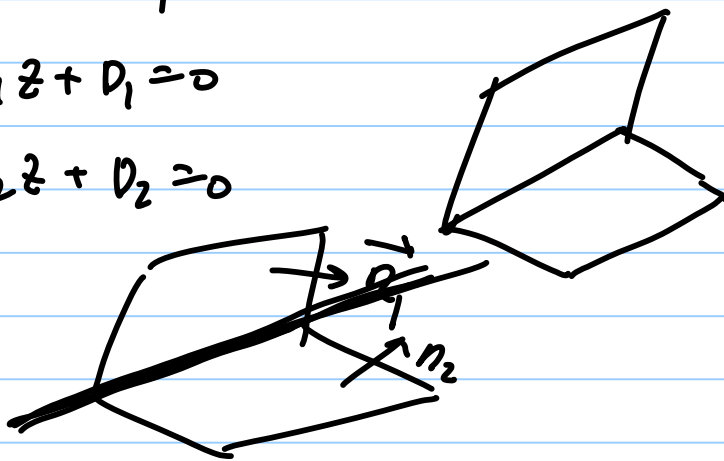
$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

例:
$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$



2. 一般式:
$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

例: 一般式化对称式:



解:

$$\vec{n} = \vec{n}_1 \times \vec{n}_2$$

例: 将
$$\begin{cases} x - 2y + z = 1 \\ 2x + y - 2z = -2 \end{cases}$$
 化为对称式方程

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 2 & 1 & -2 & -2 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 5 & -4 & -4 \end{array} \right]$$

$$z=1 \quad y=0 \quad x=0 \quad (3, 4, 6)$$

$$z=6 \quad y=4 \quad x=3 \quad (0, 0, 1)$$

$$\frac{x-3}{3} = \frac{y-4}{4} = \frac{z-6}{5}$$

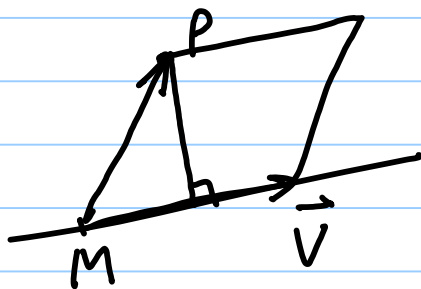
直线的基元问题: $\begin{cases} \text{点向式, 两点式.} \\ \text{过点且垂直于已知直线 (垂直)} \\ \text{过点且垂直于给定平面} \\ \text{直线方程互化.} \end{cases}$

3. 位置关系与度量.

① 点与直线位置关系 $\begin{cases} \text{点在线上} \\ \text{点在线外} \end{cases}$

点到直线距离

$$d = \frac{|\vec{MP} \times \vec{v}|}{|\vec{v}|}$$



例: 求 $A(2, -3, -1)$ 到 $\begin{cases} x-1 \\ -2 \\ y+1 \\ -1 \\ z \\ 1 \end{cases}$ 的距离

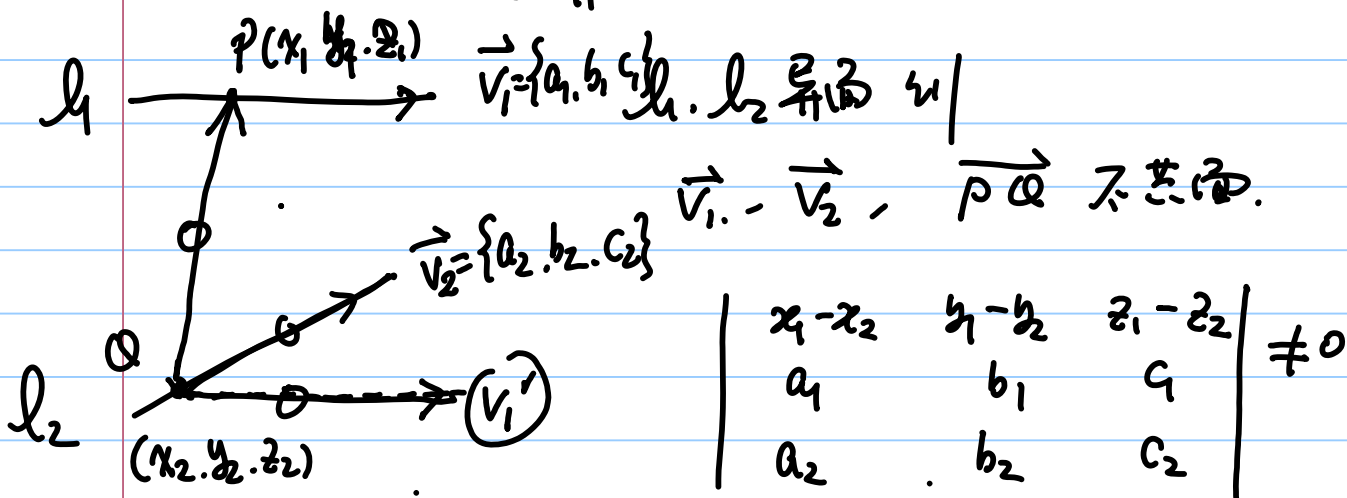
$$\begin{aligned} \vec{MA} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -1 \\ -2 & -1 & 1 \end{vmatrix} \quad \begin{matrix} A(2, -3, -1) \\ M(1, -1, 0) \\ \{ -2, -1, 1 \} \end{matrix} \\ &= -3\vec{i} + \vec{j} - 5\vec{k} \end{aligned}$$

$$|\vec{MA} \times \vec{v}| = \sqrt{35}$$

$$|\vec{v}| = \sqrt{6}$$

$$d = \frac{\sqrt{35}}{\sqrt{6}}$$

② 直线与直线 $\begin{cases} \text{平行} \\ \text{相交} \\ \text{异面} \end{cases} > \text{共面}.$



$$\text{其面} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

$$\text{例: 两直线} \frac{x+2}{2} = \frac{y}{-3} = \frac{z-1}{4} \rightarrow \frac{x-3}{m} = \frac{y-1}{4} = \frac{z-7}{2}$$

$$\underline{(-2, 0, 1) (3, 1, 7)}$$

$$\begin{vmatrix} 5 & 1 & 6 \\ 2 & -3 & 4 \\ m & 4 & 2 \end{vmatrix} = 0$$

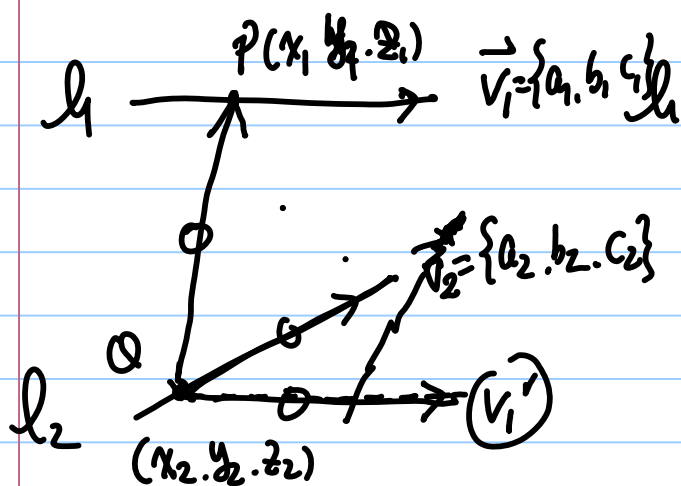
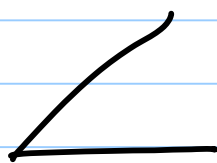
$$\text{例: 证明: } l_1: \frac{x-3}{3} = \frac{y-2}{2} = \frac{z+1}{-2}$$

$$\text{与 } l_2: \frac{x+2}{2} = \frac{y-3}{-3} = \frac{z+3}{4} \text{ 共面.}$$

且求出该平面方程

$$(3, 2, -1)$$

$$\begin{vmatrix} x-3 & y-2 & z+1 \\ 3 & 2 & -2 \\ 2 & -3 & 4 \end{vmatrix} = 0$$



异面直线距离

$$d = \frac{|(\vec{PQ}, \vec{v}_1, \vec{v}_2)|}{|\vec{v}_1 \times \vec{v}_2|}$$

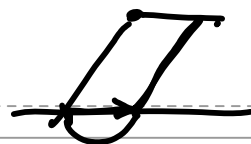
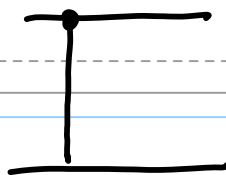
P₁₀₄ 21. (1-3) , 22-29.

30. B) 31(2) 32.(1)(3)

33 (1) 35. 36(1) 37(1)

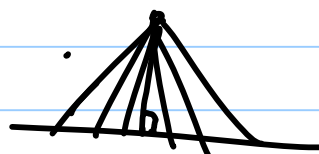
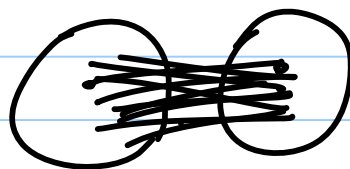
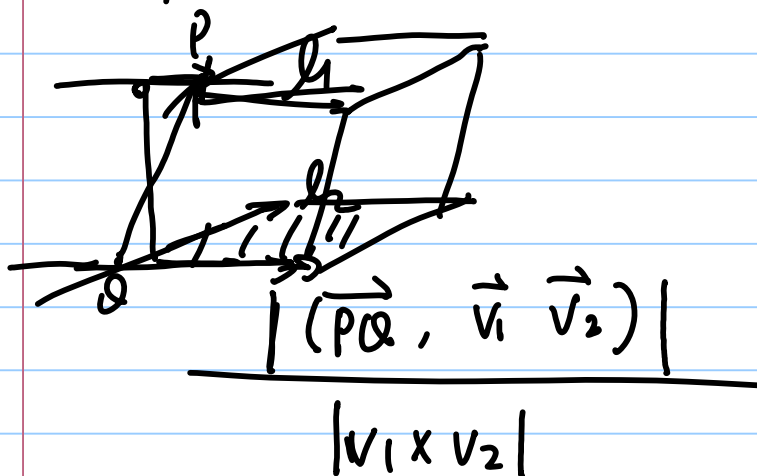
38(1) 39 (1) 40-45.

度量: ① 平行线间距离

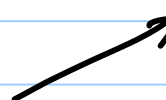
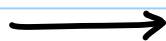


即为点与直线的距离

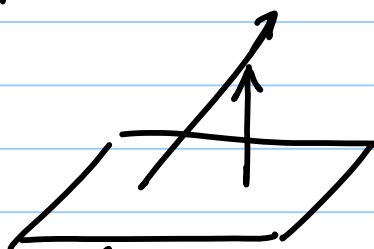
② 异面直线的距离



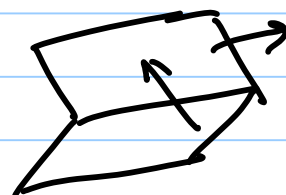
表角



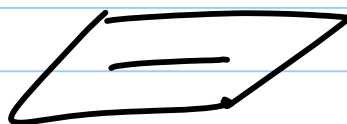
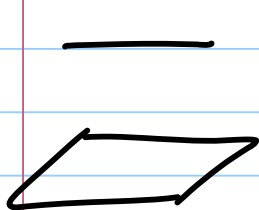
$[0, \frac{\pi}{2}]$



$[0, \frac{\pi}{2}]$

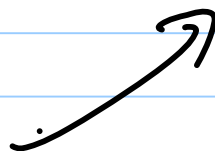


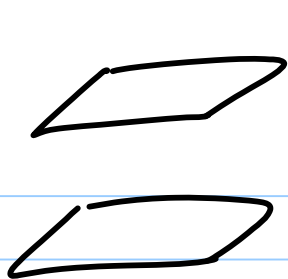
直线与平面的位置关系:



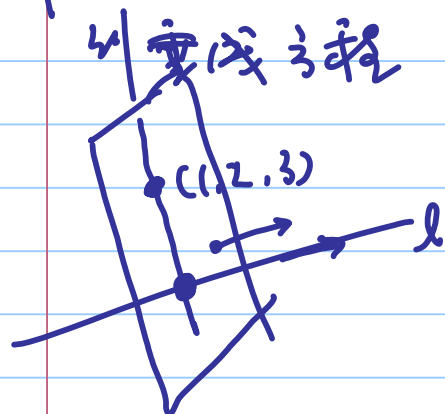
$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t \quad Ax+By+Cz+D=0$$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$





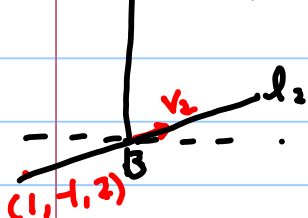
例: 过点 $(1, 2, 3)$ 作 $x-1 = 1-y = \frac{z+3}{2}$ 垂线



例: 求 $l_1: \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ 与 $l_2: x-1 = y+1 = z-2$ 公垂线

方程

$(0,0,0)$ A l_1



解: 设公垂线为 AB. 将 AB 看成 l_1 与 AB 确定的平面 α 和 l_2 与 AB 确定的平面 β 的交线.

$\vec{v}_1 = \{1, 2, 3\}$ $\vec{v}_2 = \{1, 1, 1\}$ AB 的方向为 $\vec{v}_1 \times \vec{v}_2$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \{-1, 2, -1\}$$

α 过点 $(0,0,0)$. 有 2 个方向 $\vec{v}_1, \vec{v}_1 \times \vec{v}_2$. α 方程为 $\begin{vmatrix} x-0 & y-0 & z-0 \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix} = 0$

β 过点 $(1,-1,2)$ 有 2 个方向 $\vec{v}_2, \vec{v}_1 \times \vec{v}_2$ β 方程为 $\begin{vmatrix} x-1 & y+1 & z-2 \\ 1 & 1 & 1 \\ -1 & 2 & -1 \end{vmatrix} = 0$

从而公垂线所在直线方程为 $\begin{cases} \begin{vmatrix} x-0 & y-0 & z-0 \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix} = 0 \\ \begin{vmatrix} x-1 & y+1 & z-2 \\ 1 & 1 & 1 \\ -1 & 2 & -1 \end{vmatrix} = 0 \end{cases}$

第4章. 向量空间 $\mathbb{R}^n$

空间: 集合. 具有一定结构之集合

$\mathbb{R}^n$: $\forall \alpha, \beta, \gamma$ 是 n 维向量: $\forall k, l \in \mathbb{R}$ $+$ $\cdot$

① $\alpha + \beta = \beta + \alpha$

② $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

③ $\alpha + 0 = \alpha$ 零元

④ $\alpha + (-\alpha) = 0$ 负元

⑤ $1 \cdot \alpha = \alpha$ 单位元

⑥ $k(l\alpha) = (kl)\alpha$

⑦ $(k+l)\alpha = k\alpha + l\alpha$

⑧ $k(\alpha + \beta) = k\alpha + k\beta$

V 集合, 非空, 定义两种运算 $+$, $\cdot$ $k \in F$ (数域)

若 ① $\forall \alpha, \beta \in V, \alpha + \beta \in V, k\alpha \in V$

② $\forall \alpha, \beta \in V, \alpha + \beta = \beta + \alpha$

③ $\forall \alpha, \beta, \gamma \in V, (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

④ $\exists 0 \in V, \forall \alpha \in V, \alpha + 0 = \alpha$

⑤ $\forall \alpha \in V, \alpha + (-\alpha) = 0$

⑥ $\exists 1 \in F, 1\alpha = \alpha$

⑦ $k(l\alpha) = (kl)\alpha$

⑧ $k(\alpha + \beta) = k\alpha + k\beta$

⑨ $(k+l)\alpha = k\alpha + l\alpha$

或之. 称 V 为一个线性空间

$\mathbb{R}^n$ 空间: 研究向量之间的运算是否

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases} \quad \underline{\underline{AX=b}}$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$A = [\alpha_1 \ \alpha_2 \ \dots \ \alpha_n] \quad \text{分块}$$

$$x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n = \underline{\underline{b}}$$

1. 线性组合: $\alpha_1, \alpha_2, \dots, \alpha_s \in \mathbb{R}^n$
 $k_1, k_2, \dots, k_s \in \mathbb{R}$

称 $k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s$ 为 $\alpha_1, \dots, \alpha_s$ 的一个线性组合
 (linear combination)

若 $\beta = k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s$ 称 β 可由 $\alpha_1, \dots, \alpha_s$ 线性表示

$AX=b$ 有解 $\Leftrightarrow b$ 可由 A 的列向量组线性表示

Note: ① 零向量可由任何向量组表示

$$\vec{0} = 0 \cdot \alpha_1 + 0 \cdot \alpha_2 + \dots + 0 \cdot \alpha_s$$

② $\alpha_1, \dots, \alpha_s$ 可表示自己

$$\alpha_1 = 1 \cdot \alpha_1 + 0 \cdot \alpha_2 + \dots + 0 \cdot \alpha_s$$

$$\underline{\underline{AX=b}}$$

$$\textcircled{3} e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \dots e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i \dots e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\forall \alpha \in \mathbb{R}^n \quad \alpha = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \alpha = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

$$\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_t \in \mathbb{R}^n$$

$$\textcircled{4} \alpha_1, \dots, \alpha_s \text{ 可由 } \beta_1, \dots, \beta_t \text{ 线性表出.}$$

$$\alpha_1 = c_{11} \beta_1 + c_{21} \beta_2 + \dots + c_{t1} \beta_t$$

$$\alpha_2 = c_{12} \beta_1 + c_{22} \beta_2 + \dots + c_{t2} \beta_t$$

$$\vdots$$

$$\alpha_s = c_{1s} \beta_1 + c_{2s} \beta_2 + \dots + c_{ts} \beta_t$$

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_s \end{bmatrix}_{n \times s} = \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_t \end{bmatrix}_{n \times t} \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1s} \\ c_{21} & c_{22} & \dots & c_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ c_{t1} & c_{t2} & \dots & c_{ts} \end{bmatrix}_{t \times s}$$

$$A = B C$$

2. $AX=0$ 有非零解

$\exists x_1, x_2, \dots, x_n$ 不全为 0. 设

$$x_1 \alpha_1 + x_2 \alpha_2 + \dots + x_n \alpha_n = 0$$

线性相关: $\mathbb{R}^n$ 中, $\alpha_1, \alpha_2, \dots, \alpha_s$ 满足: 存在不全为 0 的 k_1

$$k_2, \dots, k_s \text{ 使 } k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_s \alpha_s = 0$$

称 $\alpha_1, \dots, \alpha_s$ 线性相关 (linearly dependent)

反之, 若为线性无关.

$$\text{线性无关} \iff \begin{aligned} &k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_s \alpha_s = 0 \\ &\text{若且仅当 } k_1 = k_2 = \dots = k_s = 0 \end{aligned}$$

$AX=0$ 有非零解 $\Leftrightarrow \alpha_1 \dots \alpha_n$ 线性相关

例1: $\alpha_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ $\alpha_3 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ 线性相关.

$$(-2)\alpha_1 + \alpha_2 + 0 \cdot \alpha_3 = 0$$

例2: $A \in M_n(\mathbb{R})$, $\alpha \in \mathbb{R}^n$ $A^{m-1}\alpha \neq 0$
 $A^m\alpha = 0$

证明: $\alpha, A\alpha, A^2\alpha, \dots, A^{m-1}\alpha$ 线性无关

证: $k_1\alpha + k_2A\alpha + k_3A^2\alpha + \dots + k_mA^{m-1}\alpha = 0$

左乘 A^{m-1} : $k_1 \cancel{A^{m-1}}\alpha = 0$ $k_1 = 0$.

$$k_2A\alpha + k_3A^2\alpha + \dots + k_mA^{m-1}\alpha = 0$$

左乘 $A^{m-2} \dots$

结论: ① 含有零向量: 向量组线性相关

$$\alpha_1, \alpha_2, \dots, \alpha_s, \vec{0}$$

$$0 \cdot \alpha_1 + 0 \cdot \alpha_2 + \dots + 0 \cdot \alpha_s + 1 \cdot \vec{0} = \vec{0}$$

② 向量组中两个向量对应成比例, 则向量组线性相关

$$\alpha_1, k\alpha_1, \alpha_3, \dots, \alpha_s \quad (k \neq 0)$$

$$(-k)\alpha_1 + 1(k\alpha_1) + 0 \cdot \alpha_3 + \dots + 0 \cdot \alpha_s = 0$$

③ $\alpha_1 \dots \alpha_s$ 中有部分组 $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}$ 线性相关 $\Rightarrow \alpha_1 \dots \alpha_s$ 也相关

证: 存在不全为0 的 $k_1 \dots k_r$

$$k_1 \alpha_{i_1} + k_2 \alpha_{i_2} + \dots + k_r \alpha_{i_r} = 0$$

部分相关 $\Rightarrow$ 整体相关

整体无关 $\Rightarrow$ 部分无关.

④ $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ 0 & a_{22} & & a_{2s} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & a_{ss} \\ 0 & 0 & & 0 \end{bmatrix} \xrightarrow{s} \begin{matrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_s \end{matrix} \quad \begin{matrix} n \times s \\ a_{ii} \neq 0 \end{matrix}$

$$k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_s \alpha_s = 0.$$

$$[\alpha_1 \alpha_2 \dots \alpha_s] \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_s \end{bmatrix} = 0 \quad AK=0 \quad K \in \mathbb{R}^s$$

$r(A)=s$ 只有零解.

$\alpha_1 \dots \alpha_s$ 线性无关 即 矩阵型矩阵的列向量是无关的

3. 线性相关与线性表出: 关系

证: $\alpha_1 \alpha_2 \dots \alpha_s \in \mathbb{R}^n$ 线性相关

1) $\exists$ 向量可由其余向量线性表出

证: $\exists k_1, k_2, \dots, k_s$ 不全为零. $k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_s \alpha_s = 0$

$$\text{不妨设 } k_1 \neq 0. \quad \alpha_1 = -\frac{k_2}{k_1} \alpha_2 - \frac{k_3}{k_1} \alpha_3 - \dots - \frac{k_s}{k_1} \alpha_s$$

Note: $\exists$ 向量. 不是 V 向量

定理: $\alpha_1, \alpha_2, \dots, \alpha_s$ 线性无关. 则 $\beta, \alpha_1, \dots, \alpha_s$ 线性相关. $\forall \beta$ 由 $\alpha_1, \dots, \alpha_s$ 唯一表示

证: 由于 $\beta, \alpha_1, \alpha_2, \dots, \alpha_s$ 相关.

$\exists$ 不全为零 $k_0, k_1, \dots, k_s$ 使

$$k_0 \beta + k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_s \alpha_s = 0$$

断言: $k_0 \neq 0$. 否则若 $k_0 = 0$. $\forall k_1 \alpha_1 + \dots + k_s \alpha_s = 0$

$\alpha_1, \dots, \alpha_s$ 无关. $\forall k_1 = k_2 = \dots = k_s = 0$

矛盾.

$$\beta = -\frac{k_1}{k_0} \alpha_1 - \frac{k_2}{k_0} \alpha_2 - \dots - \frac{k_s}{k_0} \alpha_s$$

$$\text{假设 } \beta = k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_s \alpha_s \\ = l_1 \alpha_1 + l_2 \alpha_2 + \dots + l_s \alpha_s$$

$$\Rightarrow (k_1 - l_1) \alpha_1 + (k_2 - l_2) \alpha_2 + \dots + (k_s - l_s) \alpha_s = 0$$

$$\alpha_1, \dots, \alpha_s \text{ 无关} \Rightarrow k_1 - l_1 = 0 \dots k_s - l_s = 0$$

表示法唯一.

4. 与子空间的关系: $A = [\alpha_1, \dots, \alpha_s]$

$AX=0$ 有非零解 $\Leftrightarrow \alpha_1, \dots, \alpha_s$ 线性相关.

例: n 个 n 维向量 $\alpha_1, \dots, \alpha_n$ 线性相关 $\Leftrightarrow |\alpha_1 \alpha_2 \dots \alpha_n| = 0$

$[\alpha_1, \dots, \alpha_n]X = 0$ 线性无关 $\Leftrightarrow |\alpha_1 \alpha_2 \dots \alpha_n| \neq 0$

例: $\mathbb{R}^n$ 中 $(n+1)$ 个向量均相关

$$\alpha_1, \alpha_2, \dots, \alpha_{n+1} \in \mathbb{R}^n$$

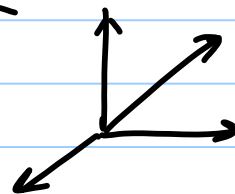
$$k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_{n+1} \alpha_{n+1} = 0$$

$$\underbrace{[\alpha_1 \ \alpha_2 \ \dots \ \alpha_{n+1}]}_{A_{n \times (n+1)}} \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_{n+1} \end{bmatrix} = 0$$

$$r(A) \leq n < n+1 \quad \text{无多余0}$$

从而 $\alpha_1, \dots, \alpha_{n+1}$ 线性相关

例: $\alpha_1 = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 2 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} 1 \\ -1 \\ a \\ -2 \end{bmatrix}$



线性无关, 求 a

$$k_1 \alpha_1 + k_2 \alpha_2 + k_3 \alpha_3 = 0 \quad *$$

$$[\alpha_1 \ \alpha_2 \ \alpha_3] \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = 0 \quad \text{只有零解}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \\ 6 & 2 & a \\ 2 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -4 \\ 0 & -10 & a-6 \\ 0 & -5 & -4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & -4 \\ 0 & 0 & a+2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a \neq -2.$$

$$r(A) = 3$$

定理: $\alpha_1 \dots \alpha_s \in \mathbb{R}^m$
 $\beta_1 \dots \beta_s \in \mathbb{R}^n$

令 $\gamma_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$ 则 $\gamma_i \in \mathbb{R}^{m+n}$

① $\alpha_1 \dots \alpha_s$ 线性无关 则 $\gamma_1 \dots \gamma_s$ 线性无关.

$$k_1 \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} + k_2 \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} + \dots + k_s \begin{bmatrix} \alpha_s \\ \beta_s \end{bmatrix} = 0$$

$$\Rightarrow k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_s \alpha_s = 0 \quad \text{则}$$

$$k_1 = k_2 = \dots = k_s = 0$$

② $\gamma_1 \dots \gamma_s$ 线性相关 $\Rightarrow \alpha_1 \dots \alpha_s$ 线性相关

- 增加向量维数个数不改变相关.
- 增加向量组维数不改变无关性

P104 13. 习题 32

证: P129 3(3)(4) . 5-12.

考试时间 12月5日 2015 8:30-10:30

闭卷

便笺标题

2015/12/3

计 16, 17, 18 N105

不允许计算器

计 19. 数媒 1.2 N106

不带草稿纸



• 1. 向量组的等价

定义: ① $\alpha_1, \dots, \alpha_s$. ② $\beta_1, \beta_2, \dots, \beta_t$ 为 2 个 n 维向量组

$$AX=0.$$

$$AX=b$$

$$AB=C$$

$$A=BX$$

若 ① 中每个向量均可由 ② 中向量表出

则向量组 ① 可被向量组 ② 线性表出.

$$A = [\alpha_1, \alpha_2, \dots, \alpha_s] \quad B = [\beta_1, \beta_2, \dots, \beta_t]$$

$$[\alpha_1, \dots, \alpha_s] = [\beta_1, \dots, \beta_t] C$$

$$\text{即 } A = BC$$

若 ① 可由 ② 表出, 且 ② 也可由 ① 表出, 则称两个向量组

等价

Note:

若 ①② 等价

$$\exists C$$

$$A = B C \quad \begin{matrix} n \times s & n \times t & t \times s \end{matrix}$$

$$\exists D:$$

$$B = A D \quad \begin{matrix} n \times t & n \times s & s \times t \end{matrix}$$

$$\Rightarrow A = ADC$$

D, C 一定有关系?

③ 向量组 α_i 满足:

① 自反性: $\alpha_1 \dots \alpha_s$ 与 $\alpha_1 \dots \alpha_s$ 线性无关

② 对称性: $\alpha_1 \dots \alpha_s$ 与 $\beta_1 \dots \beta_t$ 线性无关

则 $\beta_1 \dots \beta_t$ 与 $\alpha_1 \dots \alpha_s$ 线性无关

⑤ 传递性: $\frac{\alpha_1 \dots \alpha_s \text{ 与 } \beta_1 \dots \beta_t \text{ 线性无关}}{\beta_1 \dots \beta_t \text{ 与 } \gamma_1 \dots \gamma_k \text{ 线性无关}}$

则 $\alpha_1 \dots \alpha_s$ 与 $\gamma_1 \dots \gamma_k$ 线性无关

$\exists D_1, E_1$

$$A = BD_1$$

$$B = AE_1$$

$\exists D_2, E_2$

$$B = CD_2$$

$$C = BE_2$$

$\Downarrow$

$$A = CD_2 D_1 \quad C = AE_1 E_2$$

• 定理: $\alpha_1 \dots \alpha_s$ 可由 $\beta_1 \dots \beta_t$ 表出 $s > t$

则 $\alpha_1 \dots \alpha_s$ 线性相关.

证明: $[\alpha_1 \dots \alpha_s] = [\beta_1 \dots \beta_t] C_{t \times s}$

($[\alpha_1 \dots \alpha_s] X = 0$ 是否有非零解)

$$[\beta_1 \beta_2 \dots \beta_t] C_{\substack{t \times s \\ \underline{t \times s}}} X_{\substack{s \times 1 \\ \underline{s \times 1}}} = 0$$

注意: 则 $C X = 0: r(C) \leq t < s$

有非零解 即 $[\beta_1 \dots \beta_t] C X = 0$ 有非零解

即 $\alpha_1 \dots \alpha_s$ 线性相关

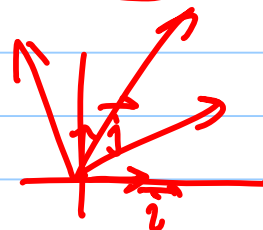
相抵

相合

相抵

α_1, α_2

$$\begin{pmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 - \alpha_2 \\ \alpha_1 + 2\alpha_2 \end{pmatrix}$$



推论: $\alpha_1 \dots \alpha_s$ 可由 $\beta_1 \dots \beta_t$ 表出 且

$\alpha_1 \dots \alpha_s$ 线性无关 $\Rightarrow$ $s \leq t$

二. 极大无关组 (选代表).

1. 定义: 在 $\alpha_1 \dots \alpha_s$ 中, 若存在 r 个向量 $\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_r}$ 线性无关. 且再加上任何一个 α_j 就线性相关.

则称 $\alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_r}$ 是 $\alpha_1 \dots \alpha_s$ 的一个极大无关组

Note 1: ① $\alpha_{i_1} \dots \alpha_{i_r} \rightarrow \alpha_1 \dots \alpha_s$ 子集

② $\alpha_{i_1} \dots \alpha_{i_r}$ 是 $\alpha_1 \dots \alpha_s$ 的极大无关组

$$\Leftrightarrow \begin{cases} \{\alpha_{i_1} \dots \alpha_{i_r}\} \subseteq \{\alpha_1 \dots \alpha_s\} \\ \alpha_{i_1} \dots \alpha_{i_r} \text{ 线性无关} \\ \alpha_1 \dots \alpha_s \text{ 每个向量均可由 } \alpha_{i_1} \dots \alpha_{i_r} \text{ 表出} \end{cases}$$

③ 线性无关的向量组极大无关组是本身.

④ 向量组存在极大无关组 $\Leftrightarrow$ 存在非零向量

2. 定理: $\alpha_{i_1} \dots \alpha_{i_r}$ 与 $\alpha_{j_1} \dots \alpha_{j_t}$ 均为 $\alpha_1 \dots \alpha_s$ 的极大无关组. $\Rightarrow r = t$

证明: $\alpha_{i_1} \dots \alpha_{i_r}$ 为极大无关组

$\alpha_{i_1} \dots \alpha_{i_r}$ 可表出 $\alpha_{j_1} \dots \alpha_{j_t}$

而 $\alpha_{j_1} \dots \alpha_{j_t}$ 线性无关. 则 $t \leq r$

同理 $r \leq t$. 从而 $r = t$

Note: 极大无关组可以不唯一. 但个数唯一

即 极大无关组个数是一个向量组的不变量.

例: $\alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $\alpha_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ $\alpha_3 = \alpha_1 + \alpha_2$

$\alpha_1, \alpha_2, \alpha_3$ 线性相关

α_1, α_2 无关.

α_1, α_3 无关, 故为 $\alpha_1, \alpha_2, \alpha_3$ 之极大线性

α_2, α_3 无关

无关组.

:

定义! $\alpha_1, \dots, \alpha_s$ 中极大线性无关的向量个数 称为 向量组 $\alpha_1, \dots, \alpha_s$ 之秩

3. 极大无关组求法:

① $A_{m \times n} \rightarrow$ 化为行阶梯形后 n 个主元. 列向量 (组之间) 线性无关.

② $AX=0 \Leftrightarrow [\alpha_1, \dots, \alpha_s] X=0$ 是否为非零解.

P 可逆阵

$$PA = B = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1j_2} & \dots & c_{1j_r} & \dots & c_{1s} \\ 0 & 0 & \dots & c_{2j_2} & \dots & c_{2j_r} & \dots & c_{2s} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & c_{rj_r} & \dots & c_{rs} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{bmatrix}$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
 $\beta_1 \quad \beta_2 \quad \beta_{j_2} \quad \beta_{j_r} \quad \beta_s$

$c_{11} \neq 0$
 $c_{2j_2} \neq 0$
 $\vdots$
 $c_{rj_r} \neq 0$

例: $\beta_1, \beta_{j_2}, \dots, \beta_{j_r}$ 线性无关.

$$\begin{bmatrix} \beta_1, \beta_{j_2}, \dots, \beta_{j_r} \end{bmatrix} = \begin{bmatrix} c_{1j_2} & c_{1j_3} & \dots & c_{1j_r} \\ 0 & c_{2j_2} & \dots & c_{2j_r} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & c_{rj_r} \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

r 个向量.

由 $\beta_1, \beta_2, \dots, \beta_s$ 线性组合 $\beta_1, \beta_{j_2}, \dots, \beta_{j_r}$ 表示

即 $\beta_1, \beta_{j_2}, \dots, \beta_{j_r}$ 为 $\beta_1, \dots, \beta_s$ 的一个极大线性无关组

类似 $\alpha_1, \alpha_{j_2}, \alpha_{j_3}, \dots, \alpha_{j_r}$ 是 $\alpha_1, \dots, \alpha_s$ 的一个极大线性无关组

$$PA = B$$

$$\underbrace{P[\alpha_1, \alpha_2, \dots, \alpha_s]}_{n \times n} = \underbrace{[\beta_1, \beta_2, \dots, \beta_s]}_{n \times s}$$

$$P\alpha_i = \beta_i$$

$$① [\alpha_1, \alpha_{j_2}, \dots, \alpha_{j_r}] X = 0$$

$$\rightarrow P[\alpha_1, \alpha_{j_2}, \dots, \alpha_{j_r}] X = 0 \quad \text{同解}$$

$$\rightarrow [\beta_1, \beta_{j_2}, \dots, \beta_{j_r}] X = 0 \quad \text{同解}$$

即只有零解 $\alpha_1, \alpha_{j_2}, \dots, \alpha_{j_r}$ 线性无关

$$② \forall \alpha_i \in \{\alpha_1, \alpha_2, \dots, \alpha_s\} \quad P\alpha_i = \beta_i$$

由 $\beta_1, \beta_{j_2}, \dots, \beta_{j_r}$ 为 $\beta_1, \dots, \beta_s$ 的一个极大

无关组 1.1.20 $\forall \beta_i$ 可由 $\beta_1, \beta_{j_2}, \dots, \beta_{j_r}$ 表出

$$\beta_i = k_1 \beta_1 + k_2 \beta_{j_2} + \dots + k_r \beta_{j_r}$$

左乘 P^{-1} $\alpha_i = k_1 \alpha_1 + k_2 \alpha_{j_2} + \dots + k_r \alpha_{j_r}$

$\alpha_1, \alpha_{j_2}, \dots, \alpha_{j_r}$ 为 $\alpha_1, \dots, \alpha_s$ 中极大无关组

Note: β_i 同样能被 $\beta_1, \beta_{j_2}, \dots, \beta_{j_r}$ 表出.

α_j 就同样能被 $\alpha_1, \alpha_{j_2}, \dots, \alpha_{j_r}$ 表出

例: 求 $\alpha_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} 0 \\ 2 \\ 5 \\ 8 \end{bmatrix}$ $\alpha_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ -1 \end{bmatrix}$ $\alpha_4 = \begin{bmatrix} -1 \\ 7 \\ -1 \\ -2 \end{bmatrix}$

极大线性无关组. 将其余向量表出.

$$[\alpha_1, \alpha_2, \alpha_3, \alpha_4] = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 2 & 2 & 7 \\ 2 & 5 & 0 & -1 \\ 3 & 8 & -1 & -2 \end{bmatrix} \xrightarrow{\text{行变换}}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 4 & 6 \\ 0 & 5 & -4 & 1 \\ 0 & 8 & -7 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -14 & -14 \\ 0 & 0 & -23 & -23 \end{bmatrix}$$

$\alpha_1, \alpha_2, \alpha_3$ 极大无关组.
 $\alpha_1, \alpha_2, \alpha_4$

$$\rightarrow \begin{bmatrix} \textcircled{1} & 0 & 2 & -1 \\ 0 & \textcircled{1} & 2 & 3 \\ 0 & 0 & \textcircled{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \alpha_1, \alpha_2, \alpha_3 \text{ 为极大无关组}$$

$$\alpha_4 = -3\alpha_1 + \alpha_2 + \alpha_3$$

4. 向量组: 秩.

定义: 极大线性无关组中向量的个数. 记为 $r(\alpha_1, \dots, \alpha_s)$

Note: $r(\alpha_1, \dots, \alpha_s) \leq s$.

② $\alpha_1, \dots, \alpha_s$ 线性相关 $\Leftrightarrow r(\alpha_1, \dots, \alpha_s) < s$

③ $\alpha_1, \dots, \alpha_s$ 线性无关 $\Leftrightarrow r(\alpha_1, \dots, \alpha_s) = s$

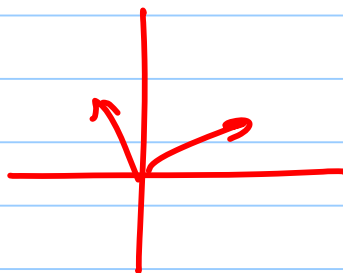
$e_1, e_2, \dots, e_n$ 线性无关. $r(e_1, e_2, \dots, e_n) = n$. $AX=0$

④ $r(\alpha_1, \alpha_2, \dots, \alpha_s) = r$ 则 $\alpha_1, \alpha_2, \dots, \alpha_s$ 中任何 r 个线性无关的向量均为它的极大无关组.

证: 设 $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}$ 线性无关

则 $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}, \alpha_j$ 一定线性相关

否则 $r(\alpha_1, \dots, \alpha_s) = r+1$ 矛盾



证毕 P_{130} . 14-18

一.

$\alpha_1, \alpha_2, \dots, \alpha_s$ 是一个极大无关组. $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}$.

便笺标题

2015/12/8

$$r(\alpha_1, \alpha_2, \dots, \alpha_s) = r.$$

$$\alpha_1, \dots, \alpha_s \text{ 线性相关} \Leftrightarrow r(\alpha_1, \alpha_2, \dots, \alpha_s) < s$$

$$\text{无关} \Leftrightarrow r(\alpha_1, \alpha_2, \dots, \alpha_s) = s$$

定理: $\beta_1, \beta_2, \dots, \beta_t$ 可由 $\alpha_1, \alpha_2, \dots, \alpha_s$ 表出.

$$\begin{aligned} r(\alpha_1, \alpha_2, \dots, \alpha_s) &= r \\ r(\beta_1, \beta_2, \dots, \beta_t) &= p \end{aligned} \quad \text{则} \quad p \leq r$$

证: 设 $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}$ 为 $\alpha_1, \dots, \alpha_s$ 的一个极大无关组

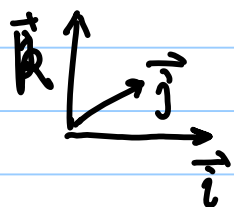
$\beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_p}$ 为 $\beta_1, \dots, \beta_t$ 的一个极大无关组

则 $\beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_p}$ 可由 $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}$ 表出.

$\beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_p}$ 无关 则 $p \leq r$.

推论: 等价向量组具有相同的秩

Note: 反之不成立 α_1, α_2 无关. α_2, α_3 无关.



$\alpha_1, \alpha_2, \alpha_3$ 无关.

但 (α_1, α_2) 与 (α_2, α_3) 不等价

$$\textcircled{2} A = [\alpha_1, \alpha_2, \dots, \alpha_s] \longrightarrow B = [\beta_1, \beta_2, \dots, \beta_s]$$

$$\exists P \text{ 可逆} \quad PA = B \quad \beta_j = P\alpha_j$$

③ 初等行变换不改变向量组的秩.

定义: $r(\alpha_1, \alpha_2, \dots, \alpha_s) = r > 0$. 则 $\alpha_1, \alpha_2, \dots, \alpha_s$ 中任取 m 个线性无关的向量 ($m < r$) 均可扩充为 $\alpha_1, \dots, \alpha_s$ 的一个极大无关组

证: 不妨设 $\alpha_1, \alpha_2, \dots, \alpha_m$ 线性无关. 由定义, 一定存在 $\alpha_{m+1}, \dots, \alpha_s$ 中的一个向量与 $\alpha_1, \dots, \alpha_m$ 线性无关. 不妨设 α_{m+1} 与 $\alpha_1, \dots, \alpha_m$ 无关. 继续扩充下去, 可扩充为 $\alpha_1, \dots, \alpha_s$ 的一个极大无关组

例: $r(\alpha_1, \dots, \alpha_s) = r$ $\alpha_1, \dots, \alpha_r$ 线性无关. 则 $\alpha_1, \dots, \alpha_r$ 为极大无关组.

二. 矩阵秩与向量组秩之间的关系

$$A \in \mathbb{R}^{m \times n} \quad A = [\alpha_1, \alpha_2, \dots, \alpha_n] = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}$$

称 $\alpha_1, \alpha_2, \dots, \alpha_n$ 秩为 A 的列秩.

称 $\beta_1, \beta_2, \dots, \beta_m$ 秩为 A 的行秩

$r(A)$

定义: $r(A) = A$ 列秩 = A 行秩

$$A \xrightarrow{\text{行}} \begin{bmatrix} \boxed{c_{11}} & \dots & \boxed{c_{1j_2}} & \dots & \boxed{c_{1j_r}} & \dots & c_{1n} \\ & & \boxed{c_{2j_2}} & \dots & c_{2j_r} & \dots & c_{2n} \\ & & \vdots & & \vdots & & \vdots \\ & & c_{rj_2} & \dots & c_{rj_r} & \dots & c_{rn} \\ & & - & - & - & - & 0 \end{bmatrix} \quad \begin{matrix} c_{11} \neq 0 \\ c_{2j_2} \sim c_{rj_r} \neq 0 \end{matrix}$$

$$r(A) = r(\alpha_1 \alpha_2 \dots \alpha_n) = r(\beta_1^T \beta_2^T \dots \beta_m^T)$$

2. 线性方程组

存在

唯一

公式: $\rightarrow$ A 为方阵 Cramer 法则

解法问题? $\checkmark$

结构问题? $\checkmark$

(一) 齐次方程组

1. 存在: $A \in M_{m,n}(\mathbb{R})$

$$\textcircled{1} AX=0 \quad A = [\alpha_1 \alpha_2 \dots \alpha_n]$$

$$x_1 \alpha_1 + x_2 \alpha_2 + \dots + x_n \alpha_n = 0$$

有非零解 $\Leftrightarrow \underline{r(A) < n} \Leftrightarrow \alpha_1, \alpha_2, \dots, \alpha_n$ 线性相关.

只有零解 $\Leftrightarrow r(A) = n \Leftrightarrow \alpha_1, \dots, \alpha_n$ 线性无关

2. 解的结构: η_1, η_2 为 $AX=0$ 的解.

$$A\eta_1 = 0 \quad A\eta_2 = 0. \quad A(\eta_1 + \eta_2) = 0$$

$$k \in \mathbb{R} \quad A(k\eta_1) = kA\eta_1 = 0$$

$\therefore \eta_1 + \eta_2, k\eta_1$ 也为 $AX=0$ 的解

Note: 1. 解对线性运算封闭.

null

$$2. N(A) = \{ X \in \mathbb{R}^n \mid AX=0 \}$$

A 的(左)零空间

3. 基础解系: 解空间的基向量.

若 $\textcircled{1} \eta_1, \eta_2, \dots, \eta_t$ 为 $AX=0$ 的解.

$\textcircled{2} \eta_1, \eta_2, \dots, \eta_t$ 线性无关

$\textcircled{3} AX=0$ 的任何解均可由 $\eta_1, \eta_2, \dots, \eta_t$ 线性表示.

证 设 $\eta_1, \eta_2, \dots, \eta_t$ 为 $AX=0$ 的基础解系,

(即 $\eta_1, \eta_2, \dots, \eta_t$ 为 $N(A)$ 的一组基)

证: $A \in M_{m,n}(\mathbb{R})$. $r(A) = r < n$. 证 $AX=0$ 中

基础解系有 $(n-r)$ 个向量

$$A \rightarrow \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1r} & \dots & c_{1n} \\ & c_{22} & \dots & c_{2r} & \dots & c_{2n} \\ & & \ddots & & & \\ & & & c_{rr} & \dots & c_{rn} \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

$$AX=0 \Rightarrow \begin{cases} c_{11}x_1 + c_{12}x_2 + \dots + c_{1r}x_r = -c_{1r+1}x_{r+1} - \dots - c_{1n}x_n \\ c_{22}x_2 + \dots + c_{2r}x_r = -c_{2r+1}x_{r+1} - \dots - c_{2n}x_n \\ \vdots \\ c_{rr}x_r = -c_{rr+1}x_{r+1} - \dots - c_{rn}x_n \end{cases}$$

$$\eta_1 = \begin{bmatrix} d_{11} \\ d_{21} \\ \vdots \\ d_{r1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \eta_2 = \begin{bmatrix} d_{12} \\ d_{22} \\ \vdots \\ d_{r2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad \eta_{n-r} = \begin{bmatrix} d_{1,n-r} \\ d_{2,n-r} \\ \vdots \\ d_{r,n-r} \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$\eta_1, \eta_2, \dots, \eta_{n-r}$ 是 $AX=0$ 的基, $\eta_1, \eta_2, \dots, \eta_{n-r}$ 线性无关.

$$\forall \beta = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_r \\ l_{r+1} \\ \vdots \\ l_n \end{bmatrix} \text{ 为 } AX=0 \text{ 的解}$$

$$\frac{1}{2} \beta = l_{r+1} \eta_1 + l_{r+2} \eta_2 + \dots + l_n \eta_{n-r} - \beta$$

证 $\beta = 0$ $AV=0$.

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1r} & C_{1r+1} & \dots & C_{1n} \\ & C_{22} & \dots & C_{2r} & C_{2r+1} & \dots & C_{2n} \\ & & \ddots & & & & \\ & & & C_{rr} & C_{rr+1} & \dots & C_{rn} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1r} \\ & C_{22} & \dots & C_{2r} \\ & & \ddots & \\ & & & C_{rr} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \underline{C_{11} \cdot C_{22} \cdot \dots \cdot C_{rr} \neq 0}$$

$$\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2p $\beta = l_{r+1} \eta_1 + l_{r+2} \eta_2 + \dots + l_n \eta_{n-r}$

2. $\eta_1, \eta_2, \dots, \eta_{n-r}$ 为 $AX=0$ 的基础解系

3. $X = k_1 \eta_1 + k_2 \eta_2 + \dots + k_{n-r} \eta_{n-r}$

例 1:
$$\begin{cases} x_1 + 3x_2 - 5x_3 - x_4 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 8x_3 + 5x_4 + 3x_5 = 0 \\ x_1 + 3x_2 - 3x_3 + 6x_4 + x_5 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 3 & -5 & -1 & 2 \\ 2 & 6 & -8 & 5 & 3 \\ 1 & 3 & -3 & 6 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} ① & 3 & -5 & -1 & 2 \\ 0 & 0 & ② & 7 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -3k_1 - \frac{33}{2}k_2 + \frac{1}{2}k_3$$

例 2:
$$\begin{cases} x_2 = k_1 \\ x_3 = -\frac{7}{2}k_2 + \frac{1}{2}k_3 \\ x_4 = k_2 \\ x_5 = k_3 \end{cases}$$

例 3:
$$X = k_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -\frac{33}{2} \\ 0 \\ -\frac{7}{2} \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

(二) 非齐次方程组 $AX=b$

$$\Leftrightarrow x_1 \alpha_1 + x_2 \alpha_2 + \dots + x_n \alpha_n = b$$

$$AX=b \text{ 有解} \Leftrightarrow x_1 \alpha_1 + x_2 \alpha_2 + \dots + x_n \alpha_n = b \text{ 有解}$$

$$\Leftrightarrow b \text{ 可由 } \alpha_1, \alpha_2, \dots, \alpha_n \text{ 表出}$$

$$\Leftrightarrow \alpha_1, \alpha_2, \dots, \alpha_n, b \text{ 与 } \alpha_1, \alpha_2, \dots, \alpha_n \text{ 同秩}$$

$$\textcircled{1} \Leftrightarrow \underbrace{r(\alpha_1, \dots, \alpha_n, b)} = \underbrace{r(\alpha_1, \dots, \alpha_n)}$$

$$\Leftrightarrow r(A|b) = r(A)$$

$\textcircled{1} \Leftrightarrow$: 令 $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}$ 为 $\alpha_1, \dots, \alpha_n$ 的一个极大无关组

$$\text{由 } r(\alpha_1, \alpha_2, \dots, \alpha_n) = r, \text{ 从而}$$

$$r(\alpha_1, \alpha_2, \dots, \alpha_n, b) = r$$

从而 $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}$ 也为 $\alpha_1, \dots, \alpha_n, b$ 的一个极大

无关组 从而 b 可由 $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}$ 表出

即 b 可由 $\alpha_1, \dots, \alpha_n$ 表出.

$$\text{证法: } (A|b) \rightarrow \left[\begin{array}{cccc|c} c_{11} & \dots & c_{1j_2} & \dots & c_{1j_r} & \dots & c_{1n} & d_1 \\ & & c_{2j_2} & \dots & c_{2j_r} & \dots & c_{2n} & d_2 \\ & & & & c_{rj_r} & \dots & c_{rn} & d_r \\ 0 & \dots & & & 0 & & & d_{r+1} \\ & & & & & & & \vdots \end{array} \right]$$

即 $AX=b$ 有解 $\Leftrightarrow b$ 可由 $\alpha_1, \dots, \alpha_n$ 表出

$$\Leftrightarrow r(A) = r(A|b)$$

$$Ax=b \text{ 有解} \Leftrightarrow b \text{ 可被 } \alpha_1, \dots, \alpha_n \text{ 表出}$$

$$\Leftrightarrow \begin{cases} r(A) = r(A; b) \\ \alpha_1, \alpha_2, \dots, \alpha_n \text{ 无关} \Rightarrow r(\alpha_1, \dots, \alpha_n) = n \end{cases}$$

$$\Leftrightarrow r(A) = r(A' | b) = n$$

$$AX=b \text{ 有无穷多解} \Leftrightarrow r(A)=r(A|b) < n.$$

解: 结构: $AX = b$ 84 η_1, η_2 .

$$\textcircled{v} \quad A\eta_1 = b, \quad A\eta_2 = b \Rightarrow A(\eta_1 - \eta_2) = 0_{11}$$

$$A\overset{11}{\eta_1} - A\eta_2 = b - b$$

$$\eta_1 - \eta_2 \text{ 为 } AX=0 \text{ 的解.}$$

② η_1, η_2 为 $AX = z$ 的特解。

③ $\eta_0 \neq 0$ $AX=0$ & $A(\eta_0 + \eta_1) = b$

2p $\eta_0 + \eta_1$ 1p $\neq Ax = b$ 83p

• 解法: 当有冗余方程时: $r(A) = r(A|b) < n$

$AX=b$ $[A|b] \rightarrow \begin{bmatrix} \text{ } & \text{ } & \text{ } & \text{ } \\ \text{ } & \text{ } & \text{ } & \text{ } \\ \text{ } & \text{ } & \text{ } & \text{ } \\ \text{ } & \text{ } & \text{ } & \text{ } \end{bmatrix}$ 令自由未知量为 0, 解出之

结论 $AX=b$ 无特解

$$Ax=b \quad A\eta=b$$

$$A(x - \eta) = 0$$

$$X - \eta = k_1 \eta_1 + k_2 \eta_2 + \dots + k_{n-r} \eta_{n-r}$$

$$X = \eta + k_1 \eta_1 + k_2 \eta_2 + \dots + k_{n-r} \eta_{n-r}$$

$$\text{prob: } \begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ 3x_1 + 2x_2 + x_3 + x_4 - 3x_5 = 0 \\ 5x_1 + 4x_2 + 3x_3 + 3x_4 - x_5 = 2 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 & -3 & 0 \\ 5 & 4 & 3 & 3 & -1 & 2 \end{array} \right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -2 & -6 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$X = \begin{bmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

prob: P₁₄₈ 6-12, 14, 16-20.

秩的3种语言 | 矩阵
| 向量组
| 方程组.

Note Title

12/10/2015

例1. $AB=0 \quad r(A)+r(B) \leq n$

证1: $r(A)+r(B)-n \leq r(AB) \leq r(A)$
 $r(B)$

$A_{m \times n} \quad B_{n \times s}$

$\Rightarrow r(A)+r(B) \leq n.$

证2:

$\begin{bmatrix} A_{m \times n} & 0 \\ \textcircled{I} & B_{n \times s} \end{bmatrix} \xrightarrow{\text{行变换}} \begin{bmatrix} 0 & -AB \\ I & B \end{bmatrix}$

$r \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}$

$\geq r(A)+r(B)$

$\parallel$
 $\begin{bmatrix} 0 & 0 \\ \textcircled{I} & \textcircled{B} \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}$

$r \begin{bmatrix} A & 0 \\ I & B \end{bmatrix} = r \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} = n.$
 $\geq r(A)+r(B)$

$\Rightarrow r(A)+r(B) \leq n.$

证2: $AB=0 \quad A[\beta_1 \ \beta_2 \ \dots \ \beta_s] = 0$

$\Rightarrow A\beta_1=0, A\beta_2=0 \dots A\beta_s=0$

$\beta_1, \dots, \beta_s$ 线性无关 $AX=0$ 的解

$\{k_1\beta_1 + k_2\beta_2 + \dots + k_s\beta_s \mid k_1, \dots, k_s \in \mathbb{R}\} \subseteq \{X \mid AX=0\}$

$r(\underbrace{\beta_1, \beta_2, \dots, \beta_s}_{r(B)}) \leq n - r(A).$

例2: $AB=0$ $r(B)=n$ $\Rightarrow A=0$

证1: $r(A) + r(B) - n \leq r(AB) \Rightarrow r(A)=0$
 $\Rightarrow A=0$

证2: $A_{m \times n}, B_{n \times s}$ $r(B)=n$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow \text{零向量}$$

$a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n = 0$ 等式

$r(B)=n$ 从而 B 等式 $\beta_1, \beta_2, \dots, \beta_n$ 线性无关.

$\text{即 } a_{11} = a_{12} = \cdots = a_{1n} = 0$

$A=0$

例3: $\alpha_1, \dots, \alpha_s$ 被 $\beta_1, \dots, \beta_t$ 表出. $s > t$ $\alpha_1, \dots, \alpha_s$ 线性相关.

证1: $\begin{bmatrix} \alpha_1 & \cdots & \alpha_s \end{bmatrix}_{n \times s} = \begin{bmatrix} \beta_1 & \cdots & \beta_t \end{bmatrix}_{n \times t} C_{t \times s}$

$\begin{bmatrix} \alpha_1 & \cdots & \alpha_s \end{bmatrix} X = 0$

$\begin{bmatrix} \beta_1 & \cdots & \beta_t \end{bmatrix} C X = 0$

$CX=0$
 $r(C) \leq t < s$

证2: $\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_s \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_t \end{bmatrix} C$

$r(\alpha_1, \dots, \alpha_s) = r\left(\begin{bmatrix} \beta_1 & \cdots & \beta_t \end{bmatrix} C\right) \leq r(C) \leq t < s$

例4: $r(A+B) \leq r(A) + r(B)$

证1: $r(A+B) \leq r \begin{bmatrix} A+B \\ B \end{bmatrix} = r \begin{bmatrix} A \\ B \end{bmatrix} \leq r(A) + r(B)$

证2: $A = [\alpha_1, \alpha_2, \dots, \alpha_n] \quad B = [\beta_1, \beta_2, \dots, \beta_n]$

$$A+B = [\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n]$$

即 $\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n$ 可由 $\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n$ 表示

即 $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$ 与 $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ 同秩
关系表示。

$$\begin{aligned} r(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n) &\leq r(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \\ &\leq r(\alpha_1, \dots, \alpha_n) + r(\beta_1, \dots, \beta_n) \end{aligned}$$

$$r(A+B) \leq r(A) + r(B)$$

证3: $\begin{bmatrix} A \\ B \end{bmatrix} X = 0 \Rightarrow (A+B)X = 0$

$$\begin{aligned} \{X \mid \begin{bmatrix} A \\ B \end{bmatrix} X = 0\} &\subseteq \{X \mid (A+B)X = 0\} \\ n - r \begin{bmatrix} A \\ B \end{bmatrix} &\leq n - r(A+B) \end{aligned}$$

$$r(A+B) \leq r \begin{bmatrix} A \\ B \end{bmatrix} \leq r(A) + r(B)$$

例: $\overline{A} B \quad r(AB) \leq r(A), r(B)$

$$A_{m \times n} \quad B_{n \times s}$$

证1: 设 $r(A)=r \quad \exists P \in M_m(\mathbb{R}^n) \quad (Q \in M_n(\mathbb{R}))$
 $\overline{Q}$

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} r(AB) &= r(PAB) = r(\underbrace{PAQ}_{\overline{Q}} Q^T B) \\ &= r\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \underline{\underline{Q^T B}}\right) = r\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \begin{matrix} \xrightarrow{r} \\ \xrightarrow{n-r} \end{matrix}\right) \\ &= r\begin{bmatrix} Q_1 \\ 0 \end{bmatrix} = r(Q_1) \leq r = r(A). \end{aligned}$$

$$\begin{aligned} r(B) = r(Q^T B) &= r\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \leq r(Q_1) + r(Q_2) \\ &\leq r(Q_1) + n - r_{r(A)} \end{aligned}$$

$$r(Q_1) \geq r(A) + r(B) - n.$$

$$\text{证 } r(AB) \geq r(A) + r(B) - n.$$

证2: 证 $r(AB) \leq r(A) + r(B).$

$$A = [\alpha_1, \alpha_2, \dots, \alpha_n]_{m \times n}, \quad B = (b_{ij})_{n \times s}$$

$$AB = [\beta_1, \beta_2, \dots, \beta_s]_{m \times s}$$

$$[\beta_1, \beta_2, \dots, \beta_s] = [\alpha_1, \alpha_2, \dots, \alpha_n] \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1s} \\ b_{21} & b_{22} & \dots & b_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{ns} \end{bmatrix}$$

证 $\beta_1, \beta_2, \dots, \beta_s$ 由 $\alpha_1, \alpha_2, \dots, \alpha_n$ 表出.

则 $\beta_1, \beta_2, \dots, \beta_s$ 极大无关组由 $\alpha_1, \alpha_2, \dots, \alpha_n$ 极大无关组表出

$$r(\beta_1, \dots, \beta_s) \leq r(\alpha_1, \dots, \alpha_n)$$

$$\text{证 } r(AB) \leq r(A).$$

$$1.3: BX=0 \Rightarrow ABX=0$$

$$\{X | BX=0\} \subseteq \{X | ABX=0\}$$

$$s - r(B) \leq s - r(AB)$$

$$\Rightarrow r(AB) \leq r(B)$$

线性空间: 空间: 集合加运算.
重点研究结构, 元素与元素关系

$$\mathbb{R} \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^n \rightarrow V$$

$$1. \quad \mathbb{N} \xrightarrow{""} \mathbb{Z} \xrightarrow{""} \mathbb{Q} \xrightarrow{\sqrt{2}} \mathbb{R} \xrightarrow{\sqrt{-1}} \mathbb{C}$$

$$V \text{ 对运算 } \circ \text{ 封闭: } \forall a, b \in V \Rightarrow a \circ b \in V$$

$$2. \text{ 域: } F \subseteq \mathbb{C} \text{ 若 } \textcircled{1} 0, 1 \in F:$$

$$\textcircled{2} F \text{ 对四则运算封闭}$$

则称 F 为一个域.

$$\mathbb{Q}, \mathbb{R}, \mathbb{C}$$

$$3. \mathbb{R}^n.$$

$$M_{m,n}(\mathbb{R})$$

$$\alpha + \beta = \beta + \alpha$$

交换律

$$A + B = B + A$$

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

结合律

$$A + (B + C) = (A + B) + C$$

$$\alpha + 0 = \alpha$$

零元

$$A + O = A$$

$$\alpha + (-\alpha) = 0$$

负元

$$A + (-A) = O$$

$$1 \cdot \alpha = \alpha$$

单位元

$$1 \cdot A = A$$

$$(kl)\alpha = k(l\alpha)$$

结合律

$$k(lA) = (kl)A.$$

$$(k+l)\alpha = k\alpha + l\alpha \quad \text{分配律}$$

$$k(\alpha+\beta) = k\alpha + k\beta$$

$$(k+l)A = kA + lA$$

$$k(A+B) = kA + kB.$$

$(V, +, \cdot, F)$

若满足: ①: V 对 $+$ 封闭 对 $\cdot$ 封闭.

$$\forall \alpha, \beta, \gamma \in V, \quad \forall k, l \in F$$

$$\textcircled{1} \quad \alpha + \beta = \beta + \alpha \quad \textcircled{2} \quad (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

$$\textcircled{3} \quad \alpha + 0 = \alpha \quad \textcircled{4} \quad \exists \theta: \alpha + \theta = 0$$

$$\textcircled{5} \quad 1 \cdot \alpha = \alpha \quad \textcircled{6} \quad k \cdot (l\alpha) = (kl) \cdot \alpha$$

$\downarrow$
标量

$\downarrow$
真正标量.

$$\textcircled{7} \quad (k+l)\alpha = k\alpha + l\alpha \quad \textcircled{8} \quad k(\alpha+\beta) = k\alpha + k\beta$$

$\downarrow$ 真正加法 $\downarrow$ 标量加法.

例1: $C^n[a, b]$. n 次连续可微函数空间

$$f, g \in C^n[a, b]. \quad k \in \mathbb{R}$$

$$(f+g)(x) = f(x) + g(x) \quad \underbrace{(kf)}(x) = kf(x)$$

$$\text{例2: } F_n[x] = \left\{ f(x) \mid f(x) = \sum_{k=0}^{n-1} a_k x^k \quad a_i \in F \right\}$$

数域 F 上 n 次或小于 n 次多项式全体

是一个线性空间

例3: 次数小于 n 的多项式全体. 不构成线性空间

例4: $\underline{AX=0}$ 方程组 求解同解加法. 求解.

构成一个线性空间 $N(A) = \{X \mid AX=0\}$

Bsp 5: $(\mathbb{R}^+, \oplus, \circ, \mathbb{R})$ $\hookrightarrow$ ~~Grp~~ $\mathcal{G} = 1$

$$a \oplus b = ab$$

$$k \circ a = a^k \quad a \in \mathbb{R}^+ \quad k \in \mathbb{R}.$$

交换律结合律 分配律

$a \oplus 0 = a$ 又 $a \oplus b = 0$
 $\therefore \underline{a} = \underline{\frac{1}{a}}$

$$1 \circ a = a' = a$$

$$k \circ (l \circ a) = k \circ (a^l) = (a^l)^k = a^{lk}$$

$$(\kappa l) \circ a = a^{\kappa l} \Rightarrow \kappa \circ (l \circ a) = (\kappa l) \circ a$$

$$k \circ (\underbrace{a \circ b}) = k \circ (\underline{\underline{ab}}) = (ab)^k = a^k \cdot b^k$$

$$K \circ a \underset{\Delta}{\oplus} K \circ b = a^k \underset{\Delta}{\oplus} b^k = a^k b^k$$

$$(k+l) \circ a = a^{k+l}$$

$$k \cdot a \oplus l \cdot a = a^k \oplus a^l = a^k \cdot a^l = a^{k+l}$$

$(\mathbb{R}^+, \oplus, 0, \mathbb{R})$ 是一个线性空间

例: 2阶可逆矩阵加法, 按矩阵加法, 需要 不相成线
4次加法.

$$\begin{bmatrix} 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Note 1: V 中之子集是抽象之符号. 不同空间中子集可能不同.

2. 借助何语言. 仍称 V 中之子为向量..

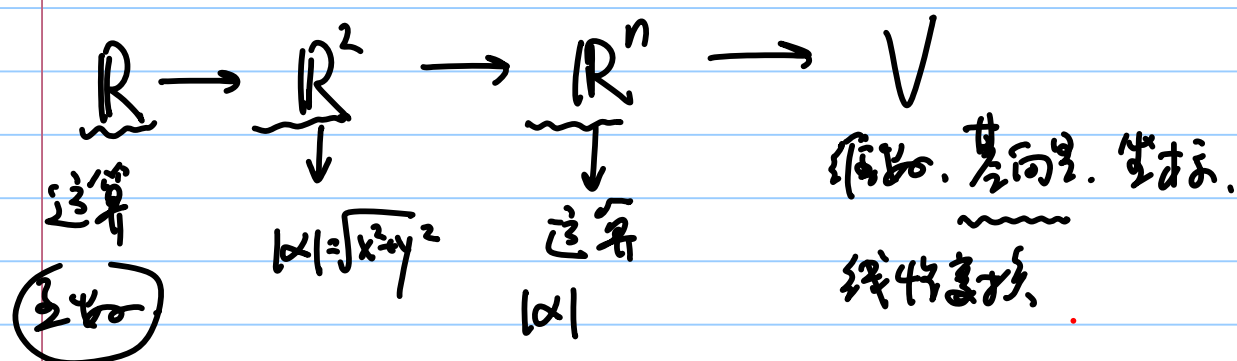
线性空间 也是 向量空间

$M_{n,n}(\mathbb{R})$ $\mathbb{R}^n$ 是 (线性空间) V 之一子集.

3. "加法" "乘法" 借用 $+$, $\cdot$ 之符号 不一定要
加法与乘法.

4. 0 不一定是 0 单位元也不一定是数字 1 .

5. 是否为线性空间也取决于数域 F



例: $\odot$ 零元 0 是唯一的.

$$\alpha + 0 = \alpha$$

证明: 假设 α_1, α_2 均为零元.

$$\begin{aligned} \alpha_1 + \alpha_2 &= \alpha_1 \\ \alpha_2 + \alpha_1 &= \alpha_2 \end{aligned} \Rightarrow \alpha_1 = \alpha_2$$

② 负元也是唯一的: $\forall \alpha \in V$, 设 β_1, β_2 均为 α 之
负元

$$\alpha + \beta_1 = 0 \quad \alpha + \beta_2 = 0$$

$$\begin{aligned} \beta_1 &= \beta_1 + 0 = \beta_1 + (\alpha + \beta_2) = (\beta_1 + \alpha) + \beta_2 \\ &= (\alpha + \beta_1) + \beta_2 \\ &= 0 + \beta_2 \\ &= \beta_2 + 0 = \beta_2 \end{aligned}$$

$$③ \quad 0 \cdot \alpha = \theta.$$

$$\alpha + (0 \cdot \alpha) = [1+0] \cdot \alpha = 1 \cdot \alpha = \alpha$$

由零元性质: $0 \cdot \alpha = \theta$

$$④ \quad (-1)\alpha = -\alpha$$

$\searrow \alpha$ 负元

$$\alpha + (-1)\alpha = [1+(-1)]\alpha = 0 \cdot \alpha = \theta$$

由负元性质: $(-1)\alpha = -\alpha$

定义减法: $\alpha - \beta = \alpha + (-\beta)$

$$⑤ \quad k\alpha = \theta \quad \text{则} \quad k=0 \quad \text{或} \quad \alpha = \theta$$

证: 若 $k \neq 0$

$$k\alpha = \theta$$

$$\frac{1}{k} \cdot k\alpha = \frac{1}{k} \cdot \theta = \theta$$

$$\left[\begin{aligned} l\theta &= l(\alpha + (-\alpha)) \\ &= l\alpha + l(-\alpha) \\ &= l\alpha + \underline{l \cdot (-1)\alpha} = l\alpha + (-l)\alpha \\ &= (l + (-l))\alpha \\ &= 0 \cdot \alpha = \theta \end{aligned} \right]$$

• 类似 $\mathbb{R}^n$ 讨论 $(V, F, +, \cdot)$ 中之子空间之定义.

线性组合: $\alpha_1, \dots, \alpha_s \in V, k_1, \dots, k_s \in F.$

若 $k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s$ 为 $\alpha_1, \dots, \alpha_s$ 的线性组合.

若 $\beta = k_1\alpha_1 + \dots + k_s\alpha_s$ 若 β 可由 $\alpha_1, \dots, \alpha_s$ 表出.

β 可由 $\alpha_1, \dots, \alpha_s$ 表出.

之如何表示?

$$1. \alpha_1 \cdots \alpha_s \in V \quad k_1, k_2, \dots, k_s \in F \quad k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s$$

称为 $\alpha_1 \cdots \alpha_s$ - 线性组合.

若 $\beta = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s$ 则 β 可由 $\alpha_1 \cdots \alpha_s$ 表示

线性相关: $(V, +, \cdot, F)$: 存在不全为 0 的 $k_1, k_2, \dots, k_s \in F$

$$\text{s.t. } k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = \theta \quad \text{则 } \alpha_1 \cdots \alpha_s \text{ 线性相关}$$

线性无关: $k_1\alpha_1 + \cdots + k_s\alpha_s = \theta \Rightarrow k_1 = k_2 = \cdots = k_s = 0$

定义: $(V, +, \cdot, F)$ $\alpha_1 \cdots \alpha_s \in V$ 线性相关 $\Leftrightarrow$ 存在一个向量可由其他向量表示.

定义: $\alpha_1 \cdots \alpha_s \in V$, 线性无关, $\alpha_1, \alpha_2, \dots, \alpha_s, \beta$ 线性相关

则 β 可由 $\alpha_1 \cdots \alpha_s$ 表示. 且表示法唯一.

定义: V 上两个向量组. $\alpha_1 \cdots \alpha_s \in V, \beta_1 \cdots \beta_t \in V$

$\alpha_1 \cdots \alpha_s$ 可由 $\beta_1 \cdots \beta_t$ 表示. $s > t$ 则 $\alpha_1 \cdots \alpha_s$ 相关

$$[\alpha_1 \cdots \alpha_s] = [\beta_1 \cdots \beta_t] C \quad Cx = 0$$

定义: $\alpha_1 \cdots \alpha_s \in V, \beta_1 \cdots \beta_t \in V, \alpha_1 \cdots \alpha_s$ 由 $\beta_1 \cdots \beta_t$ 表示

$\alpha_1 \cdots \alpha_s$ 无关 则 $s \leq t$

例: $F_3[x]$ 次数 ≤ 3 的所有多项式组成 F .

1. $x, x^2 \in F_3[x]$ 线性无关.

$$k_1 + k_2 \cdot x + k_3 x^2 = 0 \rightarrow \text{零多项式} \Rightarrow k_1 = k_2 = k_3 = 0$$

▲ $\alpha_1, \dots, \alpha_n \in \mathbb{R}^n$ 极大无关组 秩.

$(V, +, \cdot, F)$ 无穷个元素. (基) 维数.

2. 基与维数.

定义: 在 V 中, 若存在 n 个线性无关向量 $\alpha_1, \alpha_2, \dots, \alpha_n$

$\forall \beta \in V, \alpha_1, \alpha_2, \dots, \alpha_n, \beta$ 均线性相关. 称 V 是 n 维
 且 记作 $\dim V = n$ $\alpha_1, \dots, \alpha_n$ 称为 V 的一组基

Note 1: 当 V 中有无穷个线性无关向量时. 称 V 是无穷维

(无穷维)

$F[x]$ 多项式环 F 上的无穷维空间.

2 线性代数中只考虑有限维空间

3. 基不唯一. 维数是唯一.

例: $F_n[x]$ $1, x, x^2, \dots, x^{n-1}$

$$f(x) = a_0 \cdot 1 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$$

$a_0, a_1, \dots, a_{n-1} \in F$

例2: $M_{m \times n}(F)$. $A \in M_{m \times n}(F)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

设 $E_{11}, E_{12}, \dots, E_{1n}, E_{21}, E_{22}, \dots, E_{2n}, \dots, E_{m1}, E_{m2}, \dots, E_{mn}$

$$E_{ij} = \begin{bmatrix} & & & \\ & & & \\ & & 1 & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \rightarrow i \quad \dim M_{m \times n}(F) = mn$$

$$A = a_{11}E_{11} + a_{12}E_{12} + \dots + a_{1n}E_{1n} + a_{21}E_{21} + a_{22}E_{22} + \dots + a_{2n}E_{2n} \\ + \dots + a_{m1}E_{m1} + a_{m2}E_{m2} + \dots + a_{mn}E_{mn}.$$

例: $F_n[x] \quad \underline{1, x, x^2, \dots, x^{n-1}}$

$$f(x) = a_0 \cdot 1 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1}$$

$$a_0, a_1, \dots, a_{n-1} \in F$$

$$= [1, x, x^2, \dots, x^{n-1}] \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

故 $\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$ 为 $f(x)$ 在 $1, x, x^2, \dots, x^{n-1}$ 下的坐标

3. $\mathbb{R}^n: \alpha = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1 e_1 + x_2 e_2 + \dots + x_n e_n}_{= [e_1, e_2, \dots, e_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$

I.

3. 类似:

设 $\alpha: \varepsilon_1, \dots, \varepsilon_n \in V$ 为 n 个线性无关的向量, $\alpha \in V$

$$\alpha = x_1 \varepsilon_1 + x_2 \varepsilon_2 + \dots + x_n \varepsilon_n$$

$$= [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

即 $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ 为 α 在 $\varepsilon_1, \dots, \varepsilon_n$ 下的坐标.

10/1: $\alpha \in \mathbb{R}^3$ $\alpha = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 e_1 + x_2 e_2 + x_3 e_3$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\alpha = k_1 e_1 + k_2 e_2 + k_3 e_3$$

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 + k_3 \\ k_2 + k_3 \\ k_3 \end{bmatrix}$$

2/9 $\alpha \in \langle e_1, e_2, e_3 \rangle \Leftrightarrow \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 \end{bmatrix}$

10/2: $M_2(\mathbb{R})$.

$$A \in M_2(\mathbb{R})$$

$$A = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

$$= x E_{11} + y E_{12} + z E_{21} + w E_{22}$$

$$= x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + w \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

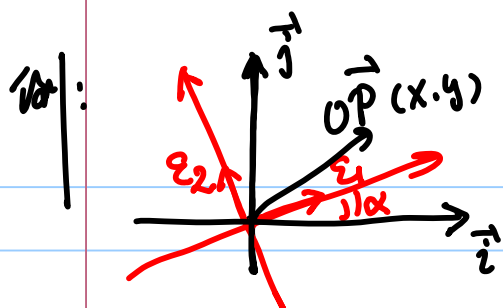
$$A = [E_{11}, E_{12}, E_{21}, E_{22}] \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad e_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

2/9 $k_1 e_1 + k_2 e_2 + k_3 e_3 + k_4 e_4 = 0$

$$\begin{bmatrix} k_1 & k_1 + k_2 \\ k_3 + k_4 & k_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow k_1 = k_2 = k_3 = k_4 = 0$$

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = y_1 e_1 + y_2 e_2 + y_3 e_3 + y_4 e_4 = \begin{bmatrix} y_1 & y_1 + y_2 \\ y_3 + y_4 & y_4 \end{bmatrix} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x \\ y - x \\ z - w \\ w \end{bmatrix}$$



$$\vec{OP} = x \vec{i} + y \vec{j} \\ = [\vec{i}, \vec{j}] \begin{bmatrix} x \\ y \end{bmatrix}$$

例: 坐标变换

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\vec{OP} = [\varepsilon_1, \varepsilon_2] \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

例: $F_{n+1}[x]$ $1, x, x^2, \dots, x^n$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \\ = [1, x, x^2, \dots, x^n] \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$1, (x-x_0), (x-x_0)^2, \dots, (x-x_0)^n$$

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \frac{f''(x_0)}{2!} (x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n \\ = [1, x-x_0, (x-x_0)^2, \dots, (x-x_0)^n] \begin{bmatrix} f(x_0) \\ f'(x_0) \\ \vdots \\ \frac{f^{(n)}(x_0)}{n!} \end{bmatrix}$$

4. 过渡矩阵与坐标变换

定义: $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ 为 n 维空间 V 中一组基.

$\eta_1, \eta_2, \dots, \eta_n \in V$ 可由 $\varepsilon_1, \dots, \varepsilon_n$ 表示

$$\text{即 } (\eta_1, \eta_2, \dots, \eta_n) = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) C$$

则 $\eta_1, \dots, \eta_n$ 线性无关 $\Leftrightarrow C$ 可逆

$\Rightarrow$: 假设 C 不可逆 $CX=0$ 有非零解

$$[\eta_1 \dots \eta_n] X = 0 \Rightarrow [\varepsilon_1 \dots \varepsilon_n] C X = 0$$

与 $\eta_1 \dots \eta_n$ 无关矛盾

$$\Leftrightarrow: (\varepsilon_1 \dots \varepsilon_n) = (\eta_1 \dots \eta_n) C^{-1}$$

定义: $\varepsilon_1 \dots \varepsilon_n, \eta_1 \dots \eta_n$ 为 V 中两组基. 则

$(\varepsilon_1 \dots \varepsilon_n)$ 与 $(\eta_1 \dots \eta_n)$ 可互相表示

$$(\eta_1 \dots \eta_n) = (\varepsilon_1 \dots \varepsilon_n) C$$

称 C 为 $(\varepsilon_1 \dots \varepsilon_n)$ 到 $(\eta_1 \dots \eta_n)$ 的过渡矩阵

Note 1: C 可逆

2: $(\eta_1 \dots \eta_n)$ 到 $(\varepsilon_1 \dots \varepsilon_n)$ 过渡矩阵 C^{-1}

$$[\varepsilon_1 \dots \varepsilon_n] = [\eta_1 \dots \eta_n] C^{-1}$$

坐标变换公式:

$$\alpha = x_1 \varepsilon_1 + x_2 \varepsilon_2 + \dots + x_n \varepsilon_n = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\triangleq [\varepsilon_1 \dots \varepsilon_n] X$$

$$= [\eta_1 \dots \eta_n] C^{-1} X$$

α 在 $\eta_1 \dots \eta_n$ 下的坐标为 $Y = C^{-1} X$

坐标变换公式.

$$\text{例: } \mathbb{R}^3: \alpha_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

$$\beta_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \beta_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \beta_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

求 $\alpha_1, \alpha_2, \alpha_3$ 到 $\beta_1, \beta_2, \beta_3$ 的过渡矩阵

$$\text{Sp: } [\beta_1, \beta_2, \beta_3] = [\alpha_1, \alpha_2, \alpha_3] C$$

$$\begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 2 \\ -1 & -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -1 & | & 1 & 0 & 0 \\ 1 & 1 & 2 & | & 1 & 1 & 0 \\ -1 & -1 & -1 & | & 1 & 1 & 1 \end{bmatrix}$$

$$\downarrow \quad \downarrow$$

$$I \quad C = \begin{bmatrix} -\frac{9}{2} & -4 & -\frac{5}{2} \\ \frac{3}{2} & +1 & \frac{1}{2} \\ 2 & 2 & 1 \end{bmatrix}$$

例: γ 在 $\alpha_1, \alpha_2, \alpha_3$ 下之坐标为 $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ 求 γ 在 $\beta_1, \beta_2, \beta_3$ 下

之坐标

$$\begin{aligned} \gamma &= [\alpha_1, \alpha_2, \alpha_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \\ &= [\beta_1, \beta_2, \beta_3] \underbrace{C^{-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}} \end{aligned}$$

$$\gamma = C^{-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

5. 线性子空间.

① 定义: $(V, +, \cdot, F)$ 是一个线性空间. $W \subseteq V$. 若 $(W, +, \cdot, F)$ 亦构成一个线性空间 则 W 为 V 的子空间



封闭

$$\textcircled{1} \alpha, \beta \in W, \quad \underline{\alpha + \beta \in W}, \quad \underline{k\alpha \in W}$$

$$\textcircled{2} \alpha + \beta = \beta + \alpha \quad \textcircled{3} (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \quad \textcircled{4} \alpha + 0 = \alpha$$

$$\textcircled{5} \alpha + \beta = 0 \quad \textcircled{6} 1 \cdot \alpha = \alpha \quad \textcircled{7} k(l\alpha) = (kl)\alpha$$

$$\textcircled{8} \textcircled{9} k(\alpha + \beta) = k\alpha + k\beta \quad \textcircled{10} (k+l)\alpha = k\alpha + l\alpha$$

定义: $W \subseteq V$ $(W, +, \cdot, F)$ 构成 V 的子空间

$\Leftrightarrow W$ 对 $+$, $\cdot$ 均封闭.

例1: $\{0\}$ 零子空间 平凡子空间

2: $AX=0$ 的解空间: $N(A)$ 是 $\mathbb{R}^n$ 的一个子空间

3: $\{X | AX=b\}$ 不是 $\mathbb{R}^n$ 子空间 (如 $b \neq 0$ 时)

4 $V_1 = \{ (x, -x, 0)^T | x \in \mathbb{R} \} \subseteq \mathbb{R}^3$ 是子空间

$$\forall \alpha, \beta \in V_1. \alpha = \begin{bmatrix} x \\ -x \\ 0 \end{bmatrix} \beta = \begin{bmatrix} y \\ -y \\ 0 \end{bmatrix} \alpha + \beta = \begin{bmatrix} x+y \\ -x-y \\ 0 \end{bmatrix} \in V_1$$

$$k\alpha = \begin{bmatrix} kx \\ -kx \\ 0 \end{bmatrix} \in V_1$$

$V_2 = \{ (1, 0, z)^T | z \in \mathbb{R} \}$ 不是 $\mathbb{R}^3$ 子空间

例: $S = \{ A | A^2=A, A \in M_2(\mathbb{R}) \}$ 不是 $M_2(\mathbb{R})$ 子空间

$$A^2=A, B^2=B \quad (A+B)^2 = A^2 + AB + BA + B^2 \\ = A + B + \boxed{AB + BA}$$

例: W_1, W_2 是 V 的子空间: 知 $W_1 \cap W_2$ 也是 V 的子空间.

证: $W_1 \cap W_2$ 非空. $0 \in W_1, 0 \in W_2 \quad 0 \in W_1 \cap W_2$

$$\forall \alpha, \beta \in W_1 \cap W_2. \alpha, \beta \in W_1 \quad \alpha + \beta \in W_1 \\ \text{同样 } \alpha + \beta \in W_2 \\ \alpha + \beta \in W_1 \cap W_2$$

$$K\alpha \in W_1, \text{ 即 } K\alpha \in W_2 \quad K\alpha \in W_1 \cap W_2$$

$$W_1 + W_2 = \{ \alpha + \beta \mid \alpha \in W_1, \beta \in W_2 \}$$

$$\forall \gamma_1, \gamma_2 \in W_1 + W_2. \quad \gamma_1 = \alpha_1 + \beta_1, \alpha_1 \in W_1, \beta_1 \in W_2$$

$$\gamma_2 = \alpha_2 + \beta_2, \alpha_2 \in W_1, \beta_2 \in W_2$$

$$\gamma_1 + \gamma_2 = \alpha_1 + \beta_1 + \alpha_2 + \beta_2 = \underbrace{\alpha_1 + \alpha_2}_{\in W_1} + \underbrace{\beta_1 + \beta_2}_{\in W_2}$$

$$\in W_1 + W_2$$

线性变换封闭。

证明:

- W_1, W_2 为有限维线性空间 V 的子空间

$$\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$$

$$\text{例: } P_{13} \mid 21-32$$

$$P_{233} \mid 2-13$$

1. 子空间基的判定.

便笺标题

2015/12/17

$$(V, +, \cdot, F) \quad W \subseteq V. \quad (W, +, \cdot, F)$$

例: $N(A) = \{ X \in \mathbb{R}^n \mid \underline{AX=0} \}$

取 $\eta_1, \eta_2, \dots, \eta_{n-r}$ 为 $N(A)$ 的一组基.

$$\dim N(A) = n - r$$

例: 证明: $SM_2(\mathbb{R})$ 是 $M_2(\mathbb{R})$ 的子空间.

① $A \in M_2(\mathbb{R}), B \in SM_2(\mathbb{R})$

$$A^T = A \quad B^T = B$$

$$(A+B)^T = A^T + B^T = A + B$$

$$A+B \in SM_2(\mathbb{R}).$$

$$(kA)^T = kA^T = kA \quad kA \in SM_2(\mathbb{R})$$

$$k \in \mathbb{R}.$$

②: 求出 $SM_2(\mathbb{R})$ 的一组基

$$\forall \begin{bmatrix} x & z \\ z & y \end{bmatrix} \in SM_2(\mathbb{R}) \quad \begin{bmatrix} x & z \\ z & y \end{bmatrix}$$

$$\begin{bmatrix} x & z \\ z & y \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + z \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{线性无关}$$

$$(k_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix})$$

$$\dim SM_2(\mathbb{R}) = 3.$$

例: W_1, W_2 均为 V 的子空间 $W_1 \subseteq W_2$.
证: $\dim W_1 \leq \dim W_2$

证: 设 $\dim W_1 = r_1$

$$\dim W_2 = r_2$$

设 $\alpha_1, \alpha_2, \dots, \alpha_{r_1}$ 为 W_1 的基

$\beta_1, \beta_2, \dots, \beta_{r_2}$ 为 W_2 的基

$W_1 \subseteq W_2$ $\alpha_1, \dots, \alpha_{r_1}$ 也在 W_2 中

$\alpha_1, \alpha_2, \dots, \alpha_{r_1}$ 能被 $\beta_1, \beta_2, \dots, \beta_{r_2}$ 表出.
 $\alpha_1, \dots, \alpha_{r_1}$ 线性无关 则 $r_1 \leq r_2$.
即 $\dim W_1 \leq \dim W_2$

维数公式:

① V 有限维空间 $W \subseteq V$ 子空间 则 W 的任一基
可扩充为 V 的基.

证: 设 $\dim V = n$. $\dim W = m. \Rightarrow n \geq m$.

设 $\alpha_1, \dots, \alpha_m$ 为 W 的基.

若 $m = n$ 也是 V 的基

若 $m < n$. 则 V 中一定存在一个向量 $\alpha_{m+1} \in V$

$\alpha_1, \alpha_2, \dots, \alpha_m, \alpha_{m+1}$ 线性无关 :

若 $m+1 = n$.

$m+1 < n$ 继续找

引理 2.1: W_1, W_2 为有限维 (线性) 空间 V 的子空间

$$2.1 \quad \dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$

$$(\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2))$$

设 $\dim W_1 = r, \dim W_2 = s, \dim(W_1 \cap W_2) = t$

$$2.2 \quad t \leq r, t \leq s.$$

设 $\alpha_1, \alpha_2, \dots, \alpha_t$ 为 $W_1 \cap W_2$ 的一组基

$\alpha_1, \alpha_2, \dots, \alpha_t, \beta_{t+1}, \dots, \beta_r$ 为 W_1 的一组基

$\alpha_1, \alpha_2, \dots, \alpha_t, \gamma_{t+1}, \dots, \gamma_s$ 为 W_2 的一组基

$$W_1 = \left\{ \alpha \mid \alpha = \sum_{i=1}^t k_i \alpha_i + \sum_{i=t+1}^r k_i \beta_i \right\}$$

$$W_2 = \left\{ \beta \mid \beta = \sum_{i=1}^t l_i \alpha_i + \sum_{i=t+1}^s l_i \gamma_i \right\}$$

$$W_1 + W_2 = \left\{ \alpha + \beta \mid \alpha + \beta = \sum_{i=1}^t (k_i + l_i) \alpha_i + \sum_{i=t+1}^r k_i \beta_i + \sum_{i=t+1}^s l_i \gamma_i \right\}$$

即 $W_1 + W_2$ 中任一向量可被 $\alpha_1, \alpha_2, \dots, \alpha_t, \beta_{t+1}, \dots, \beta_r, \gamma_{t+1}, \dots, \gamma_s$ 线性表示, $(r+s-t)$ 个向量线性无关.

证明 2.1 结束

$$\sum_{i=1}^t x_i \alpha_i + \sum_{i=t+1}^r x_i \beta_i + \sum_{i=t+1}^s y_i \gamma_i = 0$$

$$\delta \in W_1, \quad \delta = - \sum_{i=t+1}^r y_i \beta_i \in W_2$$

$$\delta \in W_1 \cap W_2, \quad \left(\delta = \sum_{i=1}^t c_i \alpha_i \right)$$

$$\sum_{i=1}^t c_i \alpha_i + \sum_{i=t+1}^s y_i \beta_i = 0 \dots$$

由 $\alpha_1 \dots \alpha_t, \beta_{t+1} \dots \beta_s$ 无关 $\Rightarrow c_1 = c_2 = \dots = c_t = y_{t+1} = \dots = y_s = 0$

从而 $\delta = 0$

$$\sum_{i=1}^t x_i \alpha_i + \sum_{i=t+1}^r x_i \beta_i = 0$$

$$x_1 = x_2 = \dots = x_t = x_{t+1} = \dots = x_r = 0$$

从而 $\dim(W_1 + W_2) = r + s - t = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

2. 生成子空间

V 为一个线性空间. $\alpha_1, \alpha_2, \dots, \alpha_m \in V$

$$W = \{ k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_m \alpha_m \mid k_i \in F \}$$

对 V 中 $+$, $\cdot$ 均封闭. W 构成了 V 的一个子空间

W 为由 $\alpha_1, \alpha_2, \dots, \alpha_m$ 生成的子空间. 记为

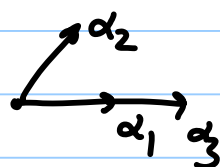
$$L(\alpha_1, \alpha_2, \dots, \alpha_m).$$

Note: $L(\alpha_1, \alpha_2, \dots, \alpha_m)$ 即为 $\alpha_1, \alpha_2, \dots, \alpha_m$ 在 V 中极大无关组

$$\dim L(\alpha_1, \alpha_2, \dots, \alpha_m) = r(\alpha_1, \alpha_2, \dots, \alpha_m)$$

例: $\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\alpha_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\alpha_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$$L(\alpha_1, \alpha_2, \alpha_3) = L(\alpha_1, \alpha_2)$$



例: $A \in M_{m,n}(\mathbb{R})$ $A = [\alpha_1, \alpha_2, \dots, \alpha_n]$

$N(A) = \{ X \in \mathbb{R}^n \mid AX = 0 \}$ A 零空间

$R(A) = \mathcal{L}(\alpha_1, \alpha_2, \dots, \alpha_n)$ A 列空间.

$$= \{ k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_n \alpha_n \mid k_i \in \mathbb{R} \}$$

$$= \{ [\alpha_1, \alpha_2, \dots, \alpha_n] \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \mid k_i \in \mathbb{R} \}$$

$$= \{ AX \mid X \in \mathbb{R}^n \}$$

$AX=b$ 有解 $\iff b \in R(A)$

例: $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 5 \\ 0 & 2 & 2 & 6 \end{bmatrix}$ 求 $R(A)$ 基与维数. 将其余向量用基表出.

$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$

$$A \rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R(A) = \mathcal{L}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \mathcal{L}(\alpha_1, \alpha_2) = \mathcal{L}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right)$$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\dim R(A) = 2.$$

例: $M_2(\mathbb{R})$ $A = \begin{bmatrix} x & x \\ y & 0 \end{bmatrix}$ $B = \begin{bmatrix} x & y \\ x & z \end{bmatrix}$

求 $\mathcal{L}(A)$, $\mathcal{L}(B)$, 并求基与维数.

$$\mathcal{L}(A) = \left\{ x \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}_{\text{元}}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \dim L(A) = 2$$

$$L(B) = \left\{ x \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \dim L(B) = 3$$

② 求 $L(A) + L(B)$ 的基与维数. $L(A) \cap L(B)$ 的基与维数.

$$L(A) + L(B) = L \left(\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{A_0}, \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_3}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{A_4} \right)$$

$$A_0 = -A_1 + A_2 + A_3$$

所以 A_1, A_2, A_3, A_4 无关.

从而 $L(A) + L(B)$ 的基是 4, 维数为 A_1, A_2, A_3, A_4

$$\textcircled{3} \quad \underline{L(A) \cap L(B)} = \left\{ k_1 \begin{bmatrix} x_1 & x_1 \\ y_1 & 0 \end{bmatrix} = k_2 \begin{bmatrix} x_2 & y_2 \\ x_2 & z_2 \end{bmatrix} \right\}$$

$$x_2 = y_2 \quad x_1 = y_1$$

$$= \left\{ k \begin{bmatrix} x & x \\ x & 0 \end{bmatrix} \right\}$$

$$\dim(L(A) \cap L(B)) = 1$$

$$\text{基: } \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

• 线性空间的基与维数 同构 $\dim V = n$
 $V \sim \mathbb{R}^n$

3. 欧氏空间. Euclid 空间.

$$\text{集合 } V \xrightarrow[\text{(8.8)}]{+, \cdot, F} \underbrace{(V, +, \cdot, F)}_{\text{线性空间}} \xrightarrow[\text{范数}]{(\cdot, \cdot) \text{ 内积}} \text{Euclid 空间}$$

↓
欧氏范数空间

1. 内积: $\mathbb{R}^3: \alpha, \beta \in \mathbb{R}^3. \alpha \cdot \beta = |\alpha||\beta| \cos \langle \alpha, \beta \rangle.$

V 中内积: $\varphi: V \times V \rightarrow \mathbb{R}$ 满足:

- ① $\varphi(\alpha, \beta) = \varphi(\beta, \alpha).$
- ② $\varphi(\alpha + \beta, \gamma) = \varphi(\alpha, \gamma) + \varphi(\beta, \gamma)$
- ③ $\varphi(k\alpha, \beta) = k\varphi(\alpha, \beta)$
- ④ $\varphi(\alpha, \alpha) \geq 0, \varphi(\alpha, \alpha) = 0 \Leftrightarrow \alpha = 0.$

则 φ 为 V 上之一个内积.

②. Euclid Space: $(V, +, \cdot, F, \varphi)$

定义了内积之线性空间

例1: $\mathbb{R}^n$ 中. $\varphi(\alpha, \beta) = \alpha^T \beta = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

$$\alpha = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \beta = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$\varphi(\alpha, \beta) = \alpha^T \beta.$$

Note: 内积不唯一. 不同之内积. 定义了不同之线性空间

例: $\mathbb{R}^2$ 中. $\alpha = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \beta = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\text{设 } \varphi(\alpha, \beta) = a_1 b_1 - a_2 b_1 - a_1 b_2 + 3 a_2 b_2$$

$$= [a_1, a_2] \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

作业: 验证 φ 为内积.

例: $M_n(\mathbb{R})$ 中, $\langle A, B \rangle = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij}$
 $= \text{tr}(AB^T)$

$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ $\text{tr} A = a_{11} + a_{22} + \dots + a_{nn}$

于是 $\langle \cdot, \cdot \rangle$ 也是一个内积.

例: $C[0, 1]$. $(f, g) = \int_0^1 f(x)g(x) dx$.

$(f, f) = \int_0^1 \underline{f^2(x)} dx \geq 0$

$(f, f) = 0 \Leftrightarrow \int_0^1 f^2(x) dx = 0 \xrightarrow{\text{连续}} f(x) \equiv 0$

$C[0, 1]$ 构成一个内积空间

例: $\downarrow$
 $\underline{(f+xg, f+xg)} \geq 0$

关于 x 是一个二次式

$\underline{(f, f) + x^2(g, g) + 2x(f, g)} \geq 0$

$\Delta \leq 0 \quad 4(f, g)^2 - 4(f, f)(g, g) \leq 0$

$\underline{(f, g) \leq \sqrt{(f, f)(g, g)}}$ Cauchy-Schwarz 不等式

$\left(\int_0^1 f(x)g(x) dx \right)^2 \leq \left(\int_0^1 f^2(x) dx \right) \left(\int_0^1 g^2(x) dx \right)$

定义: $\cos \theta = \frac{(\alpha, \beta)}{\sqrt{(\alpha, \alpha)} \sqrt{(\beta, \beta)}}$ α, β 非零.

若 $(\alpha, \beta) = 0$ 称 α, β 正交.

$$\text{Def } \forall \alpha \in V, \alpha \perp 0$$

$$(V, +, \cdot, F) \longrightarrow (V, +, \cdot, (\cdot, \cdot)) \text{ 内积空间}$$

Note Title

12/24/2015

$$\begin{aligned} \varphi: V \times V &\rightarrow \mathbb{R} & \varphi(\alpha, \beta) &= \varphi(\beta, \alpha) \\ \varphi(\alpha + \beta, \gamma) &= \varphi(\alpha, \gamma) + \varphi(\beta, \gamma) \\ \varphi(k\alpha) &= k \varphi(\alpha) \\ \varphi(\alpha, \alpha) &\geq 0, \quad \varphi(\alpha, \alpha) = 0 \Leftrightarrow \alpha = \theta \end{aligned}$$

$$C[a, b] \quad (f, g) = \int_a^b f(x) g(x) dx$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

一. 标准正交基.

1. 正交向量组. Euclid Space 中, $\alpha_1, \dots, \alpha_s$ 两两正交:

$$(\forall i \neq j) (\alpha_i, \alpha_j) = 0) \text{ 称 } \alpha_1, \dots, \alpha_s \text{ 为正交向量组}$$

定义: $\alpha_1, \dots, \alpha_s$ 为正交向量组. $\forall \alpha_1, \dots, \alpha_s$ 线性无关

$$k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_s \alpha_s = \theta$$

$$(k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_s \alpha_s, \alpha_1) = (\theta, \alpha_1) = 0.$$

$$\begin{aligned} (\varphi(\theta, \alpha) &= \varphi(\underline{\beta + (-\beta)}, \alpha) \\ &= \varphi(\beta, \alpha) + \varphi(\underline{-\beta}, \alpha) \\ &= \varphi(\beta, \alpha) - \varphi(\underline{\beta}, \alpha) = 0) \end{aligned}$$

$$k_1 (\alpha_1, \alpha_1) = 0. \quad k_1 = 0 \text{ 类似 } k_2 = k_3 = \dots = k_s = 0$$

定义: n 维 Euclid Space 中两两正交的非零向量最多不超过 n 个

② n 维欧氏空间中. 由 n 个两两正交的 nonzero 向量构成的向量组称为正交基. 由单位向量组成的正交基称为标准正交基. (唯一)

例: $\mathbb{R}^2$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$

$$(\alpha_i, \alpha_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

③ 设 $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ 为 V 中一组标准正交基

$$\forall \alpha \in V. \quad \alpha = x_1 \varepsilon_1 + x_2 \varepsilon_2 + \dots + x_n \varepsilon_n$$

$$\begin{aligned} (\alpha, \varepsilon_1) &= (x_1 \varepsilon_1 + x_2 \varepsilon_2 + \dots + x_n \varepsilon_n, \varepsilon_1) \\ &= x_1 (\varepsilon_1, \varepsilon_1) = x_1 \|\varepsilon_1\|^2 = x_1 \end{aligned}$$

$$\alpha \text{ 在 } \varepsilon_1, \dots, \varepsilon_n \text{ 下坐标 } \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad x_i = (\alpha, \varepsilon_i)$$

④ $\varepsilon_1, \dots, \varepsilon_n$ 为 V 中一组标准正交基.

$$\alpha = [\varepsilon_1 \dots \varepsilon_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \beta = [\varepsilon_1 \dots \varepsilon_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\begin{aligned} (\alpha, \beta) &= (x_1 \varepsilon_1 + x_2 \varepsilon_2 + \dots + x_n \varepsilon_n, y_1 \varepsilon_1 + y_2 \varepsilon_2 + \dots + y_n \varepsilon_n) \\ &= x_1 y_1 + x_2 y_2 + \dots + x_n y_n = [x_1 \dots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \end{aligned}$$

⑤*: 施密特正交化方法.

$\alpha_1, \dots, \alpha_n$ 为 n 维欧氏空间 V 的一组基 改造为 $\beta_1, \dots, \beta_n$ (标准正交基).

① 正交化: 取 $\beta_1 = \alpha_1, \quad \beta_2 = \alpha_2 + k\alpha_1$

$$\text{令 } (\beta_1, \beta_2) = 0 \quad (\alpha_1, \alpha_2 + k\alpha_1) = 0$$

$$(\alpha_1, \alpha_2) + k(\alpha_1, \alpha_1) = 0$$

$$k = -\frac{(\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)}$$

$$\beta_1 = \alpha_1, \quad \beta_2 = \alpha_2 - \frac{(\alpha_2, \alpha_1)}{(\alpha_1, \alpha_1)} \alpha_1, \quad \beta_3 = \alpha_3 + k_1 \beta_1 + k_2 \beta_2$$

$$= \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1$$

$$(\beta_3, \beta_1) = 0 \quad (\alpha_3, \beta_1) + k_1(\beta_1, \beta_1) = 0 \quad k_1 = -\frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}$$

$$(\beta_3, \beta_2) = 0 \quad (\alpha_3, \beta_2) + k_2(\beta_2, \beta_2) = 0 \quad k_2 = -\frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}$$

$$\left\{ \begin{array}{l} \beta_1 = \alpha_1 \\ \beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1 \\ \beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} \beta_2 \\ \vdots \\ \beta_n = \alpha_n - \frac{(\alpha_n, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_n, \beta_2)}{(\beta_2, \beta_2)} \beta_2 - \dots - \frac{(\alpha_n, \beta_{n-1})}{(\beta_{n-1}, \beta_{n-1})} \beta_{n-1} \end{array} \right.$$

$$\underline{\text{II}} \quad L(\alpha_1, \alpha_2, \dots, \alpha_n) = L(\beta_1, \beta_2, \dots, \beta_n)$$

$\beta_1, \dots, \beta_n$ 为正交向量组

$$\textcircled{2} \text{ 单位化: } \text{令 } \gamma_1 = \frac{\beta_1}{\|\beta_1\|} \quad \dots \quad \gamma_n = \frac{\beta_n}{\|\beta_n\|}$$

$$\text{例: } \mathbb{R}^3 \text{ 中的 } \alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \alpha_3 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

求为正交向量组

$$\beta_1 = \alpha_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} \beta_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$V_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad V_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad V_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

二. 正交矩阵

1. $\mathbb{R}^n$ 取 $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ 为一组标准正交基.

令 $A = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n]$ 则

$$A^T A = \begin{bmatrix} \varepsilon_1^T \\ \varepsilon_2^T \\ \vdots \\ \varepsilon_n^T \end{bmatrix} [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n] = \begin{bmatrix} \varepsilon_1^T \varepsilon_1 & \varepsilon_1^T \varepsilon_2 \cdots & \varepsilon_1^T \varepsilon_n \\ \varepsilon_2^T \varepsilon_1 & \varepsilon_2^T \varepsilon_2 \cdots & \varepsilon_2^T \varepsilon_n \\ \vdots & \vdots & \vdots \\ \varepsilon_n^T \varepsilon_1 & \varepsilon_n^T \varepsilon_2 \cdots & \varepsilon_n^T \varepsilon_n \end{bmatrix}$$

$$= I$$

定义: $A \in M_n(\mathbb{R})$ 若 $A^T A = I$ 则 A 为正交矩阵

性质: ① 正交矩阵 = 行 (列) 向量构成 $\mathbb{R}^n$ 中一组标准正交基

② $|A| = \pm 1$. $A^{-1} = A^T$.

③ A^T, A^{-1} 均为正交阵.

$$(A^T)^T A^T = A A^T = I$$

④ A, B 均为正交矩阵. 则 AB 也是.

$$A^{-1} = A^T, \quad B^{-1} = B^T$$

$$(AB)^T (AB) = B^T \underline{A^T A} B$$

$$= B^T B = I$$

2. V 为欧氏空间. $\dim V = n$, $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$.
 $\eta_1, \eta_2, \dots, \eta_n$

为两组标准正交基.

$$(\eta_1, \dots, \eta_n) = (\varepsilon_1, \dots, \varepsilon_n) \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}$$

$$(\eta_i, \eta_j) = c_{1i} c_{1j} + c_{2i} c_{2j} + \dots + c_{ni} c_{nj} = 0$$

$i \neq j$.

$$(\eta_i, \eta_i) = 1 = c_{1i}^2 + c_{2i}^2 + \dots + c_{ni}^2$$

证: 由标准正交基到标准正交基的过渡矩阵是
 正交矩阵. 若过渡矩阵为正交矩阵, 其中一列基为
 标准正交基, 则另一组基也是标准正交基.

例: $A \in M_n(\mathbb{R})$. 上三角阵. $a_{ii} > 0$, A 正交

证: $A = I$.

$$\text{证: } A^T A = I. \quad \begin{bmatrix} a_{11} & & & \\ a_{12} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

例: 正交变换保内积, 保距离

$$\mathbb{R}^3: A \in M_3(\mathbb{R}) \quad \alpha \in \mathbb{R}^3$$

$$\underline{A\alpha} \in \mathbb{R}^3$$

$$(\alpha, \beta) = \alpha^T \beta$$

$$(A\alpha, A\beta) = (A\alpha)^T A\beta = \alpha^T \underline{A^T A} \beta = \alpha^T \beta = (\alpha, \beta)$$

$$(A\alpha, A\alpha) = (\alpha, \alpha)$$

$$|A\alpha| = |\alpha|$$

例: 可逆矩阵在 QR 分解.

$A = [\alpha_1, \alpha_2 \dots \alpha_n] \in M_n(\mathbb{R})$ 可逆.

$\alpha_1, \dots, \alpha_n$ 线性无关. 将 $\alpha_1, \dots, \alpha_n$ 化为标准正交基

$$\beta_1 = \alpha_1$$

$$\alpha_1 = \beta_1$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1$$

$$\alpha_2 = \beta_2 + \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1$$

$\vdots$

$$\alpha_n = \beta_n + \frac{(\alpha_n, \beta_1)}{(\beta_1, \beta_1)} \beta_1 + \dots + \frac{(\alpha_n, \beta_{n-1})}{(\beta_{n-1}, \beta_{n-1})} \beta_{n-1}$$

$$\beta_n = \alpha_n - \frac{(\alpha_n, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \dots - \frac{(\alpha_n, \beta_{n-1})}{(\beta_{n-1}, \beta_{n-1})} \beta_{n-1}$$

$$[\alpha_1, \alpha_2 \dots \alpha_n] = [\beta_1, \beta_2 \dots \beta_n] \begin{bmatrix} 1 & \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} & & * \\ 0 & 1 & & \\ 0 & & \ddots & \\ \vdots & & & \ddots \\ 0 & & & & 1 \end{bmatrix}$$

$$= [V_1|\beta_1|, V_2|\beta_2|, \dots, V_n|\beta_n|] \begin{bmatrix} 1 & & * \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$$

$$= [V_1, V_2 \dots V_n] \begin{bmatrix} |\beta_1| & & \\ & |\beta_2| & \\ & & \ddots \\ & & & |\beta_n| \end{bmatrix} \begin{bmatrix} 1 & & * \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$$

$$A = \underbrace{[V_1 \dots V_n]}_{\text{正交阵}} \underbrace{\begin{bmatrix} |\beta_1| & & * \\ & |\beta_2| & \\ & & \ddots \\ & & & |\beta_n| \end{bmatrix}}_{\text{上三角阵}}$$

$$V_1 = \frac{\beta_1}{|\beta_1|} \quad \beta_1 = V_1 |\beta_1|$$

$$A \vec{x} = b \quad \underline{[v_1 \ v_2 \ \dots \ v_n]} \begin{bmatrix} |p_1| \\ \vdots \\ * \\ |p_n| \end{bmatrix} \vec{x} = b$$

$$\begin{bmatrix} |p_1| & \dots & * \\ & & |p_n| \end{bmatrix} \vec{x} = \begin{bmatrix} r_1^T \\ r_2^T \\ \vdots \\ r_n^T \end{bmatrix} \cdot b$$

$$\text{其多} \quad A = LU \quad AX = b$$

$$LUX = b$$

$$\begin{cases} LY = b \\ UX = Y \end{cases}$$

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$f(\alpha) = |\alpha|$$

$$\text{线性变换: } g(\alpha, \beta) = \alpha^T \beta. \quad g: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$\underline{\mathbb{R}} \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^n \rightarrow V \rightarrow \text{Euclid Space.}$$

函数

$$f: A \rightarrow \mathbb{R} \quad (A \subseteq \mathbb{R})$$

$$A \neq \emptyset.$$

定义域, 值域.

零点 不动点 $f(x) = x$.

$$f: A \rightarrow \mathbb{R} \quad (A \subseteq \mathbb{R}) \quad \text{函数.}$$

$$\Delta f: \mathbb{R} \rightarrow \mathbb{R}^3 \quad \text{向量值函数.} \quad f(t) = \begin{bmatrix} t \\ 2t \\ 3t \end{bmatrix}$$

$$f: A \rightarrow B \quad A, B \text{ (线性空间) 线性子.}$$

$$A = B \quad (f: A \rightarrow A \text{ 变换})$$

$f: A \rightarrow \mathbb{R}$ A 为一个线性空间 f : 线性

一. 定义:

1. 设 $\sigma: V \rightarrow V$ 是一个映射. V 是一个线性空间 称 σ 为 V 上的一个变换.

$$\left\{ \begin{array}{l} \sigma \text{ 的值域 } \text{Im } \sigma = \{ \sigma \alpha \mid \alpha \in V \}, \\ \sigma \text{ 的核 } \text{Ker } \sigma = \{ \alpha \mid \sigma \alpha = 0, \alpha \in V \}. \end{array} \right.$$

σ 的不变子空间 W : $\forall \alpha \in W, \sigma \alpha \in W$,

若 σ 满足 ①: $\forall \alpha, \beta \in V \quad \sigma(\alpha + \beta) = \sigma \alpha + \sigma \beta$

② $\forall k \in F, \quad \sigma(k\alpha) = k\sigma\alpha$.

称 σ 为线性变换

例: P_9 1, 2, 4, 5(2) 6, 8, 10(2)(4)
11.

$$\mathbb{R} \rightarrow \underline{\mathbb{R}^2} \rightarrow \underline{\mathbb{R}^n} \rightarrow \underline{(V, +, \cdot, F)} \rightarrow (V, +, \cdot, f(\cdot, \cdot))$$

$$f(A \subseteq \mathbb{R}) \rightarrow \mathbb{R} \text{ 常数.}$$

$$f: A \rightarrow B \quad A, B \text{ 线性空间}$$

算子.

$$f(\alpha) = |\alpha|$$

$$f(x, y) = \sqrt{x^2 + y^2}.$$

$$A=B. \text{ 变换}$$

$$f: A \rightarrow \mathbb{R} \quad A \text{ 线性空间, 泛函.}$$

$$\sigma: V \rightarrow V \quad \textcircled{1} \quad \forall \alpha, \beta \in V \quad \sigma(\alpha + \beta) = \sigma\alpha + \sigma\beta.$$

$$k \in F$$

$$\sigma(k\alpha) = k \sigma\alpha.$$

称 $\sigma\alpha$ 为 α 的像. 用 $\mathcal{L}(V)$ 表示线性空间 V 上的线性变换的全体.

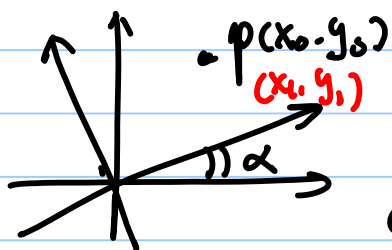
$$C[a, b]$$

$$\text{例: } \sigma: V \rightarrow V \quad \forall \alpha \in V \quad \sigma\alpha = c\alpha \quad c \text{ 常数.}$$

$$c=0 \text{ 零变换}$$

$$c=1 \text{ 恒同 id, } \varepsilon$$

例:



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\sigma: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad A$$

$$\sigma\alpha = A\alpha$$

旋转变换.

$$\text{例: } \sigma: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad \sigma \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix} \text{ 投影变换.}$$

$$\sigma: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad \sigma \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix} \text{ 镜面变换.}$$

$$\text{例: } C[a, b] \text{ 中: } \sigma: C[a, b] \rightarrow C[a, b]$$

$$\sigma(f) = \int_a^x f(t) dt$$

$$C^\infty[a, b] : \sigma : \underbrace{C^\infty[a, b]} \rightarrow C^\infty[a, b]$$

$$\forall f \in C^\infty[a, b] \quad \sigma f = f' \quad \text{微分变换.}$$

$$\sigma : \underline{C^1[a, b]} \rightarrow \underline{C[a, b]} \quad \text{微分算子}$$

$$\sigma f = f'$$

1. 线性变换 = 线性:

$$\textcircled{1} \quad \underline{\sigma(0) = 0.} \quad \sigma(0) = \sigma(0 \cdot \alpha) = 0 \cdot \sigma \alpha = 0.$$

$$\textcircled{2} \quad \sigma(-\alpha) = -\sigma \alpha.$$

③ 线性变换保持向量的线性关系不变

$$\begin{aligned} \beta &= k_1 \alpha_1 + \dots + k_s \alpha_s \\ \Rightarrow \sigma \beta &= \sigma(k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_s \alpha_s) \\ &= k_1 \sigma \alpha_1 + k_2 \sigma \alpha_2 + \dots + k_s \sigma \alpha_s. \end{aligned}$$

A: $\alpha_1, \dots, \alpha_n$ 线性相关 $\Rightarrow \sigma \alpha_1, \sigma \alpha_2, \dots, \sigma \alpha_n$ 线性相关

$$\exists \text{不全为 } 0 \text{ 的 } k_1, k_2, \dots, k_n$$

$$k_1 \alpha_1 + k_2 \alpha_2 + \dots + k_n \alpha_n = 0$$

$$k_1 \sigma \alpha_1 + k_2 \sigma \alpha_2 + \dots + k_n \sigma \alpha_n = 0$$

$\alpha_1, \dots, \alpha_n$ 无关 $\nRightarrow \sigma \alpha_1, \dots, \sigma \alpha_n$ 无关.

2. 线性变换的运算.

$$\begin{array}{ll} kf & (kf)(x) = k f(x) \\ f+g & (f+g)(x) = f(x) + g(x) \end{array}$$

f, g 构造出一个新函数

类似. 两个线性变换也可类似定义: $\sigma, \tau \in L(V)$

① $\sigma = \tau \Leftrightarrow \forall \alpha \in V, \sigma\alpha = \tau\alpha.$

② $\sigma + \tau: \forall \alpha \in V (\sigma + \tau)\alpha = \sigma\alpha + \tau\alpha$

证明 $\sigma + \tau$ 仍为线性变换 $\sigma + \tau \in L(V)$.

$$\left[\begin{aligned} \forall \alpha, \beta \in V, & \quad \underbrace{(\sigma + \tau)}_{\text{}} \underbrace{(k\alpha + l\beta)}_{\text{}} = \underbrace{\sigma(k\alpha + l\beta)}_{\text{}} + \underbrace{\tau(k\alpha + l\beta)}_{\text{}} \\ & = \underbrace{k\sigma\alpha + l\sigma\beta}_{\text{}} + \underbrace{k\tau\alpha + l\tau\beta}_{\text{}} \\ & = k(\sigma\alpha + \tau\alpha) + l(\sigma\beta + \tau\beta) \\ & = k(\sigma + \tau)\alpha + l(\sigma + \tau)\beta \end{aligned} \right]$$

类似定义 $k\sigma$. $\forall \alpha \in V (k\sigma)\alpha = k\sigma\alpha$

易证: $\forall \sigma, \tau, \rho \in L(V)$

① $(\sigma + \tau) + \rho = \sigma + (\tau + \rho)$

② $\boxed{\sigma + \tau} = \boxed{\tau + \sigma}$



$\forall \alpha \in V \quad \begin{aligned} (\sigma + \tau)\alpha &= \sigma\alpha + \tau\alpha \\ (\tau + \sigma)\alpha &= \tau\alpha + \sigma\alpha \end{aligned}$

③ $\sigma + 0 = \sigma \leftarrow 0 \text{ 是恒等变换}$

④ $\sigma + (-\sigma) = 0$

⑤ $1 \cdot \sigma = \sigma$

⑥ $k(l\sigma) = (kl)\sigma$

⑦ $(k+l)\sigma = k\sigma + l\sigma$

⑧ $k(\sigma + \tau) = k\sigma + k\tau$

这样 $L(V)$ 关于 $+$, $\cdot$ 构成一个线性空间.

定义: $\sigma \cdot \tau \quad \forall \sigma, \tau \in L(V)$

$\forall \alpha \in V$

$\sigma \cdot \tau \alpha = \sigma(\tau\alpha)$

类似定义乘法

易证: $\sigma\tau$ 仍是线性变换 证①

其他: ① $\sigma(\tau\rho) = (\sigma\tau)\rho$ ② $\sigma(\tau+\rho) = \sigma\tau + \sigma\rho$

③ $(\sigma+\tau)\rho = \sigma\rho + \tau\rho$ ④ $k(\sigma\tau) = (k\sigma)\tau$

⑤ $\sigma id = id \sigma = \sigma$ ⑥ $\sigma \cdot 0 = 0 \cdot \sigma = 0$ (0变换)

Note: ① 注意 $\sigma\tau \neq \tau\sigma$

② $\sigma\tau = \sigma\rho \nRightarrow \tau = \rho$

定义: 设 $\sigma \in L(V)$. 若 $\exists \tau \in L(V)$. s.t. (反证).
 $\sigma\tau = \tau\sigma = id$ 称 τ 为 σ 的逆变换

Note: 若 σ 是逆变换. 则 唯一.

证: 设 τ_1, τ_2 均为 σ 的逆变换. $\forall \alpha \in V$

$$\begin{aligned}\tau_1 \alpha &= \tau_1(\underline{id} \alpha) = \tau_1(\sigma \tau_2 \alpha) = \underline{(\tau_1 \sigma)} \tau_2 \alpha \\ &= \underline{id}(\tau_2 \alpha) \\ &= \tau_2 \alpha\end{aligned}$$

$$\tau_1 = \tau_2.$$

变换的性质: σ^0 即为 id $\sigma^{k+1} = \sigma \cdot \sigma^k$

σ^2 $f(x) = ax^2 + bx + c$ 证 $f(\sigma) = a\sigma^2 + b\sigma + c id$

$$A \in M_n(\mathbb{R}) \quad f(A) = aA^2 + bA + cI$$
$$f(A)g(B) \neq g(B)f(A).$$

线性变换的多项式

3. 线性变换的矩阵.

$\forall \alpha \in V$. $\alpha_1, \alpha_2, \dots, \alpha_n$ 为 V 的一组基.

$$\alpha = x_1 \alpha_1 + x_2 \alpha_2 + \dots + x_n \alpha_n$$

$$\sigma \in L(V), \sigma \alpha = x_1 \underset{\uparrow V}{\sigma \alpha_1} + x_2 \underset{\uparrow V}{\sigma \alpha_2} + \dots + x_n \underset{\uparrow V}{\sigma \alpha_n}.$$

即 σ 把 α 映射到哪个向量完全由 $\sigma \alpha_1, \sigma \alpha_2, \dots, \sigma \alpha_n$ 决定

$$\sigma \alpha_1, \sigma \alpha_2, \dots, \sigma \alpha_n \in V \quad \alpha_1, \alpha_2, \dots, \alpha_n \in V$$

$$\sigma \alpha_1 = a_{11} \alpha_1 + a_{21} \alpha_2 + \dots + a_{n1} \alpha_n$$

$$\sigma \alpha_2 = a_{12} \alpha_1 + a_{22} \alpha_2 + \dots + a_{n2} \alpha_n$$

$\vdots$

$$\sigma \alpha_n = a_{1n} \alpha_1 + a_{2n} \alpha_2 + \dots + a_{nn} \alpha_n$$

$$\underline{(\sigma \alpha_1, \sigma \alpha_2, \dots, \sigma \alpha_n) = (\alpha_1, \alpha_2, \dots, \alpha_n) A} \quad A = (a_{ij})_{n \times n}.$$

A 称为线性变换 σ 在 $\alpha_1, \dots, \alpha_n$ 下的矩阵.

$$\forall \alpha \in V, \quad \alpha = x_1 \alpha_1 + x_2 \alpha_2 + \dots + x_n \alpha_n$$

$$= [\alpha_1 \ \alpha_2 \ \dots \ \alpha_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\sigma \alpha = x_1 \sigma \alpha_1 + x_2 \sigma \alpha_2 + \dots + x_n \sigma \alpha_n$$

$$= [\underbrace{\sigma \alpha_1, \sigma \alpha_2, \dots, \sigma \alpha_n}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= [\alpha_1 \ \alpha_2 \ \dots \ \alpha_n] A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

设 $\sigma \alpha$ 在 $\alpha_1, \dots, \alpha_n$ 下坐标为 Y

$$Y = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Note: $L(V)$ 上有一个线性变换就对应一个矩阵.

有一个矩阵也对应一个线性变换.

例1: $R_n[x]$ 上的微分变换 $\sigma f = f'$
(σ 称为 n 次微分算子)

$$\sigma(1, x, x^2, \dots, x^{n-1}) = (0, 1, 2x, \dots, (n-1)x^{n-2})$$

$$= (1, x, x^2, \dots, x^{n-1}) \begin{bmatrix} 0 & 1 & 0 & & \\ & 0 & 2 & & \\ & & 0 & 3 & \\ & & & \ddots & \\ & & & & n-1 \\ & & & & & 0 \end{bmatrix}$$

例: σ 在 $\alpha_1, \alpha_2, \alpha_3$ 下矩阵为 $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

σ 在 $\alpha_3, \alpha_2, \alpha_1$ 下矩阵为 B .

求 B .

$$\sigma(\alpha_3, \alpha_2, \alpha_1) = (\sigma\alpha_3, \sigma\alpha_2, \sigma\alpha_1)$$

$$= (\sigma\alpha_1, \sigma\alpha_2, \sigma\alpha_3) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= [\alpha_1, \alpha_2, \alpha_3] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= [\alpha_3, \alpha_2, \alpha_1] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

B

4. 线性变换在不同基下的矩阵之关系.

$\sigma \in L(V)$ $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ 为两组基.

σ 在 $\alpha_1, \dots, \alpha_n$ 下矩阵为 A , 在 $\beta_1, \dots, \beta_n$ 下矩阵为 B

$$(\beta_1 \dots \beta_n) = (\alpha_1 \dots \alpha_n) P \quad (\alpha_1 \dots \alpha_n \text{ 为过渡矩阵})$$

$$(\beta_1 \dots \beta_n) = (\alpha_1 \dots \alpha_n) P$$

$$\underline{\sigma(\beta_1 \dots \beta_n)} = \underline{\sigma(\alpha_1 \dots \alpha_n)} P$$

$$\underbrace{(\beta_1 \dots \beta_n)}_P B = (\alpha_1 \dots \alpha_n) A P$$

$$(\alpha_1 \dots \alpha_n) P B = (\alpha_1 \dots \alpha_n) A P$$

$$(\alpha_1 \dots \alpha_n) (P B - A P) = (0, 0, \dots, 0) \Rightarrow P B = A P$$

过渡可逆从而 $B = P^{-1} A P$. 相似关系

例: $\alpha_1, \alpha_2, \alpha_3$ 为 $\mathbb{R}^3$ 的一组基. $\sigma \alpha_1 = \alpha_3, \sigma \alpha_2 = \alpha_2,$

$$\sigma \alpha_3 = \alpha_1$$

若 α 在 $\alpha_1, \alpha_2, \alpha_3$ 下坐标为 $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ 则 $\sigma \alpha$ 在 $\underline{\alpha_1, \alpha_2, \alpha_3}$

下坐标为?

$$\begin{aligned} \sigma \alpha &= \sigma(\alpha_1, \alpha_2, \alpha_3) \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = [\alpha_3, \alpha_2, \alpha_1] \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \\ &= [\alpha_1, \alpha_2, \alpha_3] \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

5. 线性变换: 基与基的对应关系——对应.

$\alpha_1 \dots \alpha_n$ 为 V 的一组基 $\sigma, \tau \in L(V)$

σ 在该基下坐标为 A .

则 $\sigma \tau$ 在该基下坐标为 AB .

τ ————— B

$$\begin{aligned} \sigma \tau(\alpha_1, \alpha_2, \dots, \alpha_n) &= \sigma(\tau \alpha_1, \tau \alpha_2, \dots, \tau \alpha_n) \\ &= \sigma(\alpha_1, \alpha_2, \dots, \alpha_n) B \\ &= (\alpha_1, \alpha_2, \dots, \alpha_n) AB \end{aligned}$$

$$\text{Im } \sigma = \{ \sigma \alpha \mid \alpha \in V \}$$

线性变换的像核

$$\text{Ker } \sigma = \{ \alpha \mid \alpha \in V, \sigma \alpha = 0 \}$$

不变子空间: W

W 是 V 子空间.

$$\forall \alpha \in W, \sigma \alpha \in W$$

证 4 p₂₃₄ 14-20.

$$\text{tr } A = \sum_{i=1}^n a_{ii} \quad A \in M_n(\mathbb{R})$$

迹

V . $\alpha_1, \alpha_2, \dots, \alpha_n$ 为 V 的一组基

便笺标题

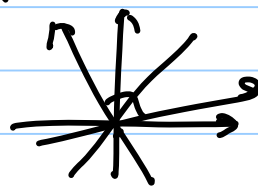
2015/12/29

$$\sigma(\alpha_1, \alpha_2, \dots, \alpha_n) = (\alpha_1, \alpha_2, \dots, \alpha_n) A.$$

$\varepsilon_1, \dots, \varepsilon_n$ 为 V 的另一组基.

$$(\varepsilon_1, \dots, \varepsilon_n) = (\alpha_1, \dots, \alpha_n) P$$

$$\begin{aligned} \sigma(\varepsilon_1, \dots, \varepsilon_n) &= \sigma(\alpha_1, \dots, \alpha_n) P \\ &= (\alpha_1, \dots, \alpha_n) A P \\ &= (\varepsilon_1, \dots, \varepsilon_n) P^{-1} A P \end{aligned}$$

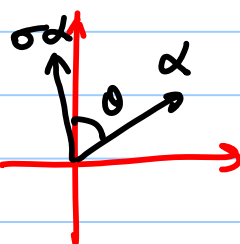


$$B = P^{-1} A P$$

第六章 特征值与特征向量.

$$\sigma: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$\alpha \in \mathbb{R}^2 \quad \sigma \alpha = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \alpha$$



$$\sigma \alpha = k \alpha$$

是否存在只伸缩不旋转变换.

1. 定义: V 为一线性空间 $\sigma \in L(V)$ 若 $\exists \lambda \in \mathbb{C}$.

ξ 为非零向量. s.t. $\sigma \xi = \lambda \xi$. 则 λ 为 σ 的特征值

ξ 称为 σ 属于 λ 的特征向量

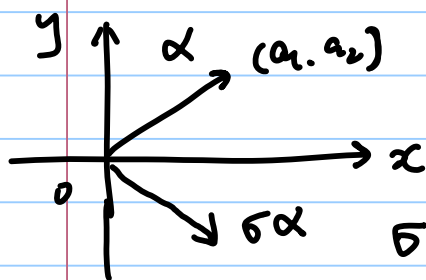
几何意义: $\sigma \xi$ 与 ξ 共线

Note: 特征向量是非零向量.

例: $\sigma \alpha = c \alpha$ ($c \neq 0$). 数乘变换. 所有非零向量均为 c 的特征向量. c 为特征值.

例: 镜面反射 $\sigma \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$

$$\sigma: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$



$$\begin{aligned} \sigma^2 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \sigma \left(\underbrace{\sigma \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}_{\sigma^2 = id} \right) \\ &= \sigma \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \end{aligned}$$

$\lambda = 1$ 为特征值. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 为对应于 $\lambda = 1$ 的特征向量

$$\sigma \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\lambda = -1$ 为特征值 $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ 为对应于 $\lambda = -1$ 的特征向量

例: $\sigma \in L(\mathbb{R}^2)$. $\sigma \alpha = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \alpha$

$\theta = 2k\pi$ 或 $2k\pi + \pi$ 时 σ 有实特征值. 所有非零向量均为其特征向量.

2. 特征向量性质.

① ξ 为 σ 的属于特征值 λ 的特征向量.

$$\sigma \xi = \lambda \xi.$$

$$k \in F, k \neq 0. \quad \sigma(k\xi) = k \underbrace{\sigma \xi}_{= \lambda \xi} = k\lambda \xi = \lambda(k\xi)$$

$$\sigma \xi_1 = \lambda \xi_1, \quad \sigma \xi_2 = \lambda \xi_2$$

$$\begin{aligned} \sigma(\underline{\xi_1 + \xi_2}) &= \sigma \xi_1 + \sigma \xi_2 = \lambda \xi_1 + \lambda \xi_2 \\ &= \lambda(\underline{\xi_1 + \xi_2}). \end{aligned}$$

即 V 中属于线性变换 σ 对应于特征值 λ 的特征向量

全体 + 零元 构成了 V 的一个子空间 称为特征

子空间. V_λ

定义: $V_\lambda = \{ \xi \in V \mid \sigma \xi = \lambda \xi \}$. 称为 σ 的特征子空间

λ 的特征子空间. 特征子空间维数 $\dim V_\lambda$ 称为 λ 的代数重数.

Note: V_λ 为 V 的不变子空间

$$\forall \alpha \in V_\lambda \Rightarrow \sigma \alpha = \lambda \alpha \in V_\lambda$$

3. 求 σ 的特征值.

$\alpha_1, \dots, \alpha_n \in V$ 为 V 的一组基.

$$\sigma(\alpha_1, \dots, \alpha_n) = (\alpha_1, \dots, \alpha_n) A$$

$$\boxed{\sigma \xi = \lambda \xi} \quad \xi \text{ 在 } \alpha_1, \dots, \alpha_n \text{ 下表示为 } X \text{ 即}$$

$$\xi = (\alpha_1, \dots, \alpha_n) X$$

$\Downarrow$

$\Leftarrow$

$$\sigma(\alpha_1, \dots, \alpha_n) X = (\alpha_1, \dots, \alpha_n) A X$$

$$\stackrel{||}{=} \lambda (\alpha_1, \dots, \alpha_n) X$$

$$\Rightarrow (\alpha_1, \dots, \alpha_n) A X = (\alpha_1, \dots, \alpha_n) \lambda X$$

$$(\alpha_1, \dots, \alpha_n) (A X - \lambda X) = (0, 0, \dots, 0)$$

$$\Rightarrow \underline{A X = \lambda X}$$

定义: $A \in M_n(\mathbb{R}) \quad \exists \lambda \in \mathbb{C}. X \in \mathbb{R}^n, X \neq 0$

$AX = \lambda X$. 称 λ 为 A 的特征值. X 为 A 的属于特征值 λ 的特征向量.

Note: 对应于 λ 的特征向量 $+ \vec{0}$ 构成 $\mathbb{R}^n$ 的一个子空间

4. 特征值、特征向量求法.

$$AX = \lambda X. X \neq 0$$

$$=$$

$$(\lambda I - A)X = 0. X \neq 0$$

即 $(\lambda I - A)X = 0$ 有非零解.

$$|\lambda I - A| = 0$$

定义: 称 $f_A(\lambda) = |\lambda I - A|$ 为 A 的特征多项式

Note: ① A 与 A^T 特征值相同, 但特征向量一般不同.

$$|\lambda I - A| = |\lambda I - A^T|$$

② 若 A 特征值为 $\lambda_1, \lambda_2, \dots, \lambda_n$ $|\lambda I - A| = 0$

A 不可逆 $\Leftrightarrow |A| = 0 \Leftrightarrow A$ 有零特征值.

$$|0 \cdot I - A| = |-A| = (-1)^n |A| = 0$$

A 可逆 $\Leftrightarrow |A| \neq 0 \Leftrightarrow AX = 0$ 只有零解 $\Leftrightarrow r(A) = n$

$\Leftrightarrow A$ 列向量线性无关 $\Leftrightarrow A$ 没有零特征值.

$$\Leftrightarrow \dim R(A) = n \Leftrightarrow \dim N(A) = 0.$$

若 A 可逆时: A^{-1} 特征值为 $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}$
对应特征向量不变.

$$A\xi = \lambda\xi. \quad \text{左乘 } A^{-1}$$

$$\xi = \lambda A^{-1}\xi$$

$$A^{-1}\xi = \frac{1}{\lambda}\xi.$$

③ $A, B \in M_n(\mathbb{R})$ AB 特征值与 BA 特征值相同

证法: $A \in M_{m,n}(\mathbb{R}), B \in M_{n,m}(\mathbb{R}).$

$$|I_m - AB| = |I_n - BA|.$$

$$\begin{bmatrix} I_n & B_{n,m} \\ A_{m,n} & I_m \end{bmatrix} \rightarrow \begin{bmatrix} I_n & B_{n,m} \\ 0 & I_m - AB \end{bmatrix}$$

$$\begin{bmatrix} I_n - BA & 0 \\ A & I_m \end{bmatrix}.$$

$$\begin{vmatrix} I & B \\ A & I \end{vmatrix} = \begin{vmatrix} I & B \\ 0 & I - AB \end{vmatrix} = \begin{vmatrix} I - BA & 0 \\ A & I \end{vmatrix}$$

$$|I - AB| = |I - BA|$$

证法: $A, B \in M_n(\mathbb{R})$

$$|\lambda I - AB|$$

若 A, B 中有一个不可逆. $|AB| = |A||B| = 0$

$$|BA| = |B||A| = 0.$$

则 AB, BA 均有零特征值.

$$\begin{aligned}\lambda \neq 0: \quad |\lambda I - AB| &= \lambda^n \left| I - \frac{1}{\lambda} AB \right| \\ &= \lambda^n \left| I - B \cdot \left(\frac{1}{\lambda} A\right) \right| \\ &= |\lambda I - BA|\end{aligned}$$

特征向量关系:

$$\text{若 } B \text{ 可逆时, } AB\xi = \lambda\xi, \quad \xi \neq 0$$

$$\begin{aligned}BA(B\xi) &= B(AB\xi) \\ &= B(\lambda\xi) \\ &= \lambda(B\xi).\end{aligned}$$

$$(B \text{ 可逆 } \xi \neq 0 \Rightarrow B\xi \neq 0)$$

例: 设 0 为 $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & a \end{bmatrix}$ 特征值, 求 A 所有特征值与特征向量.

$$|A| = 0 \quad a = 1. \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda-2 & 0 \\ -1 & 0 & \lambda-1 \end{vmatrix} = 0$$

$$(\lambda-2) \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix} = 0 \quad \text{即 } (\lambda-2) [\lambda^2 - 2\lambda + 1 - 1] = 0$$
$$(\lambda-2)^2 \lambda = 0.$$

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 2$$

$$\lambda = 0 \text{ 时 } AX = 0 \cdot X = 0 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

特征向量为 $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

$$\lambda_2 = \lambda_3 = 2 \text{ of: } AX = 2X$$

$$(2I - A)X = 0$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\lambda = 2 \text{ 特征向量}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Note: ① $f_A(\lambda) = 0$ 中 λ_0 为重数 n 的特征值
 代数重数.

② 对于每一个特征值 λ .

几何重数 $\leq$ 代数重数.

例1: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{vmatrix} = 0 \quad (\lambda - 1)^2 = 0$$

$$\lambda_1 = \lambda_2 = 1.$$

$$\lambda = 1 \text{ of: } (I - A)X = 0$$

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$\lambda = 1 \text{ 特征向量:}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

例2: $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ & a_{22} & \dots & a_{2n} \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}$ 上三角阵是 $a_{11}, a_{22}, \dots, a_{nn}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & * & & \\ & \ddots & \ddots & \\ & & \lambda - a_{nn} \end{vmatrix} = 0$$

特征值的性质: ① $\lambda_1 \lambda_2 \cdots \lambda_n = |A|$

② $\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{tr}(A) \equiv \sum_{i=1}^n a_{ii}$

韦达定理: $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$

$a_n \in \mathbb{R}$ $a_n \neq 0$ 有复数域内有 n 个根 (重根重计)

$z_1, z_2, \dots, z_n$

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n (x - z_1)(x - z_2) \cdots (x - z_n)$$

x^{n-1} : $a_{n-1} = a_n \underbrace{(-z_1 - z_2 - \cdots - z_n)}$

x^0 : $a_0 = a_n (-1)^n z_1 \cdots z_n$

$$|\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} = 0 \text{ 根为 } \lambda_1, \dots, \lambda_n.$$

②

$$|\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

x^{n-1} : $\lambda^{n-1} : -(a_{11} + \cdots + a_{nn}) \equiv -(\lambda_1 + \cdots + \lambda_n)$

x^0 : 令 $\lambda = 0$ $|-A| = (-1)^n \lambda_1 \cdots \lambda_n$

$$(-1)^n |A| = (-1)^n \lambda_1 \cdots \lambda_n$$

$$\Rightarrow \lambda_1 \cdots \lambda_n = |A|.$$

例: $A^2 = A$ 求 A 特征值.

$$AX = \lambda X \quad X \neq 0.$$

左乘 A . $\underline{A^2 X = \lambda \underline{AX} = \lambda^2 X}$

$$\underbrace{(A^2 - A)}_0 X = (\lambda^2 - \lambda) X$$

$$\text{即 } (\lambda^2 - \lambda) X = 0 \quad X \neq 0$$

$$\lambda^2 = \lambda \quad \lambda = 0 \text{ 或 } 1.$$

例: A 为三阶矩阵. 特征值为 $1, 2, -1$

求 $A^3 - 5A^2$ 特征值

$2I - A$ 特征值

$$AX = \lambda X \quad A^2 X = AAX = \lambda AX = \lambda^2 X.$$

$$A^3 X = \lambda^3 X$$

$$A^2 X = \lambda^2 X$$

$$(A^3 - 5A^2) X = (\lambda^3 - 5\lambda^2) X$$

$$A^3 - 5A^2 \text{ 特征值: } -4, -12, -6$$

特征向量不变.

$$(2I - A) X = 2X - AX = 2X - \lambda X = (2 - \lambda) X.$$

$2I - A$ 特征值 $1, 0, 3$ 特征向量不变

⑥ $A \in M_n(\mathbb{R})$. 最多有 n 个不同特征值.

Note:

$|\lambda I - A| = 0$ 是一个 λ 的 n 次方程

② 特征值有可能是复数

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} \quad \lambda = \pm i$$

$$\begin{vmatrix} \lambda - 1 & 2 \\ -1 & \lambda + 1 \end{vmatrix} \\ = \lambda^2 + 1 = 0$$

③ A 的特征值为 λ

$f(A)$ 的特征值为 $f(\lambda)$

$\downarrow$ (A 为 $n \times n$ 矩阵)

例 P_{163} . 1. 2 (2) (4) (6). 3-6, 9. 12, 13.

A1. B5.

$$2A_{21} + 3A_{22} + 4A_{23} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{vmatrix}$$

$$= (4-2)(4-3)(3-2)$$

A2. B6

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$A^n = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}}_{3^{n-1}}$$

$$= 3^{n-1} A = \begin{bmatrix} 3^{n-1} & 3^{n-1} \\ 2 \cdot 3^{n-1} & 2 \cdot 3^{n-1} \end{bmatrix}$$

A3. B7

$$\begin{vmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{vmatrix} = 7 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{vmatrix}$$

$$= 7 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 7 \times 3^3 = 189$$

A4. B8

$$2^3 |A|^2 \cdot |B| = 8 \times 4 \times 3 = 96$$

A5 B9

$$A^2 + A - 4I = 0$$

$$(A-I)(A+2I) = A^2 + A - 2I = 2I$$

$$(A-I) \frac{A+2I}{2} = I$$

$$(A-I)^{-1} = \frac{A+2I}{2}$$

$$A6. B1. |\alpha + \beta|^2 + |\alpha - \beta|^2 = 2(|\alpha|^2 + |\beta|^2)$$

A7. B2. $AE_1E_2 = B$ $B^T = E_2^T E_1^T A^T = E_2 E_1 A^T$
 $A E_2 \widetilde{E_1} = B$ $B^T = E_1^T E_2^T A^T = E_1 E_2 A^T$

A8. B3. $\alpha_{3 \times 1}$ $\alpha_{1 \times 3}^T \alpha_{3 \times 1}$

A9. B4. $\sqrt{29}$

 (x_0, y_0, z_0)
 $ax + by + cz = d \Rightarrow$

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

10. $A[\alpha_1, \alpha_2, \alpha_3] = [\alpha_1, \alpha_2, \alpha_3] \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

$|A| \cdot |\alpha_1, \alpha_2, \alpha_3| = |\alpha_1, \alpha_2, \alpha_3| \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix}$
 $|\alpha_1, \alpha_2, \alpha_3| \neq 0 \quad |A| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = 9$

~~$A\alpha_1 + A\alpha_2 + A\alpha_3 = 3(\alpha_1 + \alpha_2 + \alpha_3)$~~

~~$(A - 3I)\alpha_1 + (A - 3I)\alpha_2 + (A - 3I)\alpha_3 = 0$~~

~~$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0$~~

$A(\alpha_1 + \alpha_2 + \alpha_3) = 3(\alpha_1 + \alpha_2 + \alpha_3)$

~~$|A| \cdot |\alpha_1 + \alpha_2 + \alpha_3| = 3|\alpha_1 + \alpha_2 + \alpha_3|$~~

11. $\frac{x+4}{2} = \frac{y-4}{-1} = \frac{z+1}{-2} = t$

~~$\begin{cases} x = -4 + 2t \\ y = 4 - t \\ z = -1 - 2t \end{cases}$~~ ~~$\begin{cases} x = -5 + 4t \\ y = -5 - 3t \\ z = 5 + 5t \end{cases}$~~

12,

$$\frac{k_1 k_2 \cdot k_3}{k_1 \alpha_1 + k_2 \alpha_2 + k_3 \alpha_3} = 0$$

$$\text{若 } k_1 = 0 \quad k_2 \alpha_2 + k_3 \alpha_3 = 0$$

$$\alpha_2, \alpha_3 \text{ 无关. } k_2 = k_3 = 0$$

矛盾.

$$k_1 \neq 0 \quad \alpha_1 = -\frac{k_2}{k_1} \alpha_2 - \frac{k_3}{k_1} \alpha_3$$

$$\text{若 } \alpha_4 \text{ 由 } \alpha_1, \alpha_2, \alpha_3 \text{ 表示}$$

$$\alpha_4 = l_1 \alpha_1 + l_2 \alpha_2 + l_3 \alpha_3$$

$$= l_1 \left(-\frac{k_2}{k_1} \alpha_2 - \frac{k_3}{k_1} \alpha_3 \right) + l_2 \alpha_2 + l_3 \alpha_3$$

$$13. \quad A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \quad A \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = 0 \quad A \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 0$$

$$A \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

右逆

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{\text{初等变换}} \begin{bmatrix} \checkmark \\ \checkmark \\ \checkmark \\ I \end{bmatrix}$$

14.

$$15. \begin{bmatrix} -1 & -4 & 1 & 1 & 1 \\ 0 & a & -3 & 1 & 3 \\ 1 & 3 & a+1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -4 & 1 & 1 & 1 \\ 0 & a & -3 & 1 & 3 \\ 0 & -1 & a+2 & 1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -1 & -4 & 1 & 1 & 1 \\ 0 & -1 & a+2 & 1 & 1 \\ 0 & \textcircled{a} & -3 & 1 & 3 \end{bmatrix} \rightarrow \dots$$

特征值: $f_A(\lambda) = |\lambda I - A| = 0$

$$(\lambda I - A)X = 0.$$

$$\sum_{i=1}^n \lambda_i = \text{tr}(A) \quad \prod_{i=1}^n \lambda_i = |A|.$$

例: A 是 $n \times n$ 矩阵, $\lambda_1, \dots, \lambda_n$ 是 $2I - A$ 的特征值.

$$\text{证 1: } A\xi_i = \lambda_i \xi_i$$

$$(2I - A)\xi_i = 2\xi_i - A\xi_i = 2\xi_i - \lambda_i \xi_i = (2 - \lambda_i)\xi_i.$$

$2I - A$ 的特征值是 $2 - \lambda_1, 2 - \lambda_2, \dots, 2 - \lambda_n$ 特征向量不变.

$$\text{证 2: } |\mu I - (2I - A)| = 0.$$

$$|(\mu - 2)I + A| = 0$$

$$|(2 - \mu)I - A| = 0 \quad \text{即 } 2 - \mu \text{ 是 } A \text{ 的特征值}$$

$$2 - \mu = \lambda_1, \lambda_2, \dots, \lambda_n \quad \mu = 2 - \lambda_1, 2 - \lambda_2, \dots, 2 - \lambda_n.$$

$$|2I - A| = (2 - \lambda_1)(2 - \lambda_2) \cdots (2 - \lambda_n)$$

例: X 为 A 属于 λ 的特征向量 $\Rightarrow X$ 不可能是 () 特征向量

(1) $A - 2I$ (2) $3A$ (3) A^T (4) A^2 .

$$AX = \lambda X$$

$$3AX = 3\lambda X$$

$$A^2X = A(AX)$$

$$= A\lambda X$$

$$= \lambda AX = \lambda^2 X$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = 1$$

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

矩阵相似:

$\sigma \in L(V)$. $\alpha_1 \dots \alpha_n$ 为 V 的一组基

$\beta_1 \dots \beta_n$ 为 V 的另一组基.

$$(\beta_1 \dots \beta_n) = (\alpha_1 \dots \alpha_n) P$$

P 可逆.

$$\sigma(\alpha_1 \dots \alpha_n) = (\alpha_1 \dots \alpha_n) A$$

$$\sigma(\beta_1 \dots \beta_n) = (\beta_1 \dots \beta_n) B$$

$$\Rightarrow B = P^{-1}AP$$

$$\underbrace{\sigma(\alpha_1 \dots \alpha_n)}_{\ll (\alpha_1 \dots \alpha_n) A} P \quad (\beta_1 \dots \beta_n) P^{-1} A P \quad \Downarrow$$

定义

若 $A, B \in M_n(\mathbb{R})$ $\exists P$ 可逆 $P^{-1}AP = B$ 称

A, B 相似

4.2.1: ① 自反. $I^{-1}AI = A$

② 对称. $P^{-1}AP = B$. $A = PBP^{-1}$
 $= (P^{-1})^{-1}B P^{-1}$

③ 传递 $P^{-1}AP = B$ $Q^{-1}BQ = C$

$$Q^{-1}P^{-1}APQ = C$$

$$(PQ)^{-1}APQ = C$$

4.2.2: A, B 相似 $\Leftrightarrow P^{-1}AP = B$

① $f_A(\lambda) = f_B(\lambda)$.

$$\begin{aligned} |\lambda I - B| &= |\lambda I - P^{-1}AP| \\ &= |\lambda P^{-1}IP - P^{-1}AP| \\ &= |P^{-1}(\lambda I - A)P| \\ &= |P^{-1}| \cdot |\lambda I - A| \cdot |P| = |\lambda I - A| \end{aligned}$$

② A, B 相似 $\Leftrightarrow$ 特征值相等

$\Leftrightarrow$ 有相同之特征多项式. 相同之迹.

例: $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ A, B 相似 $\Leftrightarrow \text{tr} B = 6$

例: $A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$ $P = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}$ $B = P^{-1}AP$.

求 A^{100} .

$$\begin{aligned} B = P^{-1}AP &= \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$A = P \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} P^{-1}$$

$$\begin{aligned} A^2 &= P \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} P^{-1} P \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} P^{-1} \\ &= P \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} P^{-1} \end{aligned}$$

$$A^{100} = P \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}} P^{-1}$$

2. 矩阵可对角化:

$$|\lambda I - A| = 0, \lambda_1, \lambda_2, \dots, \lambda_n. (\lambda I - A)X = 0$$

若存在 n 个线性无关特征向量 $\xi_1, \xi_2, \dots, \xi_n$

$$A\xi_1 = \lambda_1 \xi_1, \quad A\xi_2 = \lambda_2 \xi_2, \quad \dots, \quad A\xi_n = \lambda_n \xi_n$$

$$A \underbrace{[\xi_1 \ \xi_2 \ \dots \ \xi_n]}_P = \underbrace{[\xi_1 \ \xi_2 \ \dots \ \xi_n]}_P \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

$$AP = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

若 A 存在 n 个线性无关特征向量, $\therefore P$ 可逆

$$A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}$$

什么情况成立?

定理: 属于不同特征值的特征向量是线性无关.

证: 设 $\lambda_1, \lambda_2, \dots, \lambda_k$ 是 A 的 k 个不同特征值
 $\xi_1, \dots, \xi_k$ 是对应的特征向量.

$k=1$. $\xi_1 \neq 0$ 线性无关. $0 \cdot \xi_1 = 0$

假设 $k=m$ 时成立. 当 $k=m+1$ 时.

$$k_1 \xi_1 + k_2 \xi_2 + \dots + k_m \xi_m + k_{m+1} \xi_{m+1} = 0 \quad (1)$$

$$\text{用 } A \text{ 左乘 } \lambda_1 k_1 \xi_1 + \lambda_2 k_2 \xi_2 + \dots + \lambda_m k_m \xi_m + \lambda_{m+1} k_{m+1} \xi_{m+1} = 0$$

$$\Rightarrow k_1 (\lambda_{m+1} - \lambda_1) \xi_1 + k_2 (\lambda_{m+1} - \lambda_2) \xi_2 + \dots + k_m (\lambda_{m+1} - \lambda_m) \xi_m = 0$$

由归纳假设 $\xi_1, \dots, \xi_m$ 无关

$$k_1 (\lambda_{m+1} - \lambda_1) = 0 \dots \quad k_m (\lambda_{m+1} - \lambda_m) = 0$$

$$\lambda_i \neq \lambda_j \quad (i \neq j) \quad \Rightarrow \quad k_1 = \dots = k_m = 0$$

$$\text{由 (1)} \quad k_{m+1} \xi_{m+1} = 0 \quad \Rightarrow \quad k_{m+1} = 0.$$

$\xi_1, \dots, \xi_{m+1}$ 无关.

推论1: 若 A 有 n 个不同特征值: $\forall A$ 可对角化.

推论2: 若 A 有 k 个不同特征值 $\lambda_1, \lambda_2, \dots, \lambda_k$
重数为 $d_1, d_2, \dots, d_k$.

$$A \text{ 可对角化} \Leftrightarrow \begin{cases} \lambda_1 \text{ 可提供 } d_1 \text{ 个线性无关特征向量} \\ \vdots \\ \lambda_k \text{ 可提供 } d_k \text{ 个线性无关特征向量} \end{cases}$$

例: $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ 求 A^n .

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 \\ -2 & \lambda - 1 \end{vmatrix} = 0$$

$$(\lambda - 1)^2 + 2 = 0 \quad \lambda = 1 \pm \sqrt{2}i.$$

$\lambda = 1 + \sqrt{2}i$ 时 $(\lambda I - A)X = 0$

$$\begin{bmatrix} \sqrt{2}i & 1 \\ -2 & \sqrt{2}i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\xi_1 = \begin{bmatrix} 1 \\ -\sqrt{2}i \end{bmatrix}$$

$\lambda = 1 - \sqrt{2}i$ 时 $(\lambda I - A)X = 0$

$$\begin{bmatrix} -\sqrt{2}i & 1 \\ -2 & -\sqrt{2}i \end{bmatrix} X = 0 \quad \xi_2 = \begin{bmatrix} 1 \\ \sqrt{2}i \end{bmatrix}$$

$$A\xi_1 = \lambda_1 \xi_1 \quad A\xi_2 = \lambda_2 \xi_2.$$

$$A \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix} = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 1 \\ -\sqrt{2}i & \sqrt{2}i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\sqrt{2}i & \sqrt{2}i \end{bmatrix} \begin{bmatrix} 1 + \sqrt{2}i & \\ & 1 - \sqrt{2}i \end{bmatrix}$$

$$A = \frac{1}{2\sqrt{2}i} \begin{bmatrix} 1 & 1 \\ -\sqrt{2}i & \sqrt{2}i \end{bmatrix} \begin{bmatrix} 1 + \sqrt{2}i & \\ & 1 - \sqrt{2}i \end{bmatrix} \begin{bmatrix} \sqrt{2}i & -1 \\ \sqrt{2}i & 1 \end{bmatrix}$$

$$A^n = \left(\frac{1}{2\sqrt{2}i} \right)^n \begin{bmatrix} 1 & 1 \\ -\sqrt{2}i & \sqrt{2}i \end{bmatrix} \begin{bmatrix} (1 + \sqrt{2}i)^n & \\ & (1 - \sqrt{2}i)^n \end{bmatrix} \begin{bmatrix} \sqrt{2}i & -1 \\ \sqrt{2}i & 1 \end{bmatrix}$$

例: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ 求 A^n 的表达式

$$(\lambda I - A) = (\lambda - 1)^2 (\lambda - 2) = 0 \quad \lambda_1 = \lambda_2 = 1, \lambda_3 = 2$$

$$\lambda=1 \text{ of } (I-A)X=0 \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

A is not diagonalizable

Ex | $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$ A is not diagonalizable

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda-3 & 1 & 2 \\ -2 & \lambda & 2 \\ -2 & 1 & \lambda+1 \end{vmatrix} = \begin{vmatrix} \lambda-3 & 1 & \underline{\lambda-1} \\ -2 & \lambda & 0 \\ -2 & 1 & \underline{\lambda-1} \end{vmatrix} \\ &= (\lambda-1) \begin{vmatrix} -2 & \lambda \\ -2 & 1 \end{vmatrix} + (\lambda-1) \begin{vmatrix} \lambda-3 & 1 \\ -2 & \lambda \end{vmatrix} \\ &= (\lambda-1)(2\lambda-2) + (\lambda-1)(\lambda^2-3\lambda+2) \\ &= (\lambda-1)(\lambda^2-\lambda) \\ &= (\lambda-1)^2 \lambda \end{aligned}$$

$$(\lambda I - A) = 0 \quad \lambda_1 = 0, \lambda_2 = \lambda_3 = 1 = (\lambda-1)^2 \lambda$$

$$\begin{aligned} \lambda_1 = 0 \text{ of } AX = 0 \quad \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \xi_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} &\rightarrow \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\lambda=1 \text{ of } (\lambda I - A) = \begin{bmatrix} -2 & 1 & 2 \\ -2 & 1 & 2 \\ -2 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(\lambda I - A)X = 0 \quad \xi_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A\xi_1 = 0 \quad A\xi_2 = \xi_2 \quad A\xi_3 = \xi_3$$

$$A[\xi_1 \ \xi_2 \ \xi_3] = [\xi_1 \ \xi_2 \ \xi_3] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

~~$\xi_1 \ \xi_2 \ \xi_3$~~
 $\rho \ \bar{\rho} \ \xi$

$$A = P \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} P^{-1}.$$

证: ① P_{158} 是 2 阶. 自共轭 (实对称 $\leq$ 实数).

② P_{164} . 14. 15 (2)(4)(5)

16. 17. 18. 19 (2). 20. - 25 | {23}.

$$\xi \neq 0 \quad A\xi = \lambda\xi. \quad |\lambda I - A| = 0. \quad \underline{(\lambda I - A)X = 0.}$$

一. 实对称矩阵的对角化.

定义: $A \in M_n(\mathbb{R}). \quad A^T = A. \quad \text{或} \quad A \text{ 为实对称矩阵.}$
 $A \in SM_n(\mathbb{R}).$

证: $\lambda \in \mathbb{C}$ 为 A 的特征值.

$X \in \mathbb{C}$ 为 A 对应于 λ 的特征向量.

$$AX = \lambda X.$$

$$A \in SM_n(\mathbb{R}).$$

$$a, b \in \mathbb{C}$$

$$\overline{ab} = \overline{a} \cdot \overline{b}$$

两边取共轭 $A\overline{X} = \overline{\lambda} \overline{X}$

$$\lambda \overline{\lambda} = |\lambda|^2$$

$$(a+bi)(a-bi) = a^2 + b^2$$

$$\textcircled{1} X^T A \overline{X} = \overline{\lambda} X^T \overline{X}$$

数!!

$$\overline{\lambda} X^T \overline{X} = \lambda (\overline{X})^T X$$

$$\textcircled{2} (\overline{X})^T A X = \lambda (\overline{X})^T X$$

$$(X^T A \overline{X})^T = X^T A \overline{X}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \overline{X} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_n \end{bmatrix}$$

$$(\overline{X})^T A X = X^T A \overline{X}$$

$$X^T \overline{X} = x_1 \dots x_n \begin{bmatrix} \overline{x}_1 \\ \vdots \\ \overline{x}_n \end{bmatrix}$$

$$\Rightarrow \overline{\lambda} = \lambda \quad \lambda \in \mathbb{R}.$$

$$= x_1 \overline{x}_1 + x_2 \overline{x}_2 + \dots + x_n \overline{x}_n$$

$$= |x_1|^2 + \dots + |x_n|^2 > 0$$

定理: A 为 n 阶实对称矩阵. $\exists$ 正交矩阵 Q

使 $Q^T A Q = Q^T A Q = \Lambda$. 其中 Λ 为对角阵

(实对称正交相似于对角阵)

证: $n=1$ 时显然成立. 假设 $n-1$ 阶实对称阵成立.

n 阶实对称阵. λ_1 为 A 的一个实特征值. X_1 为实特征向量.

将 X_1 在 $\mathbb{R}^n$ 找到 $X_2 \dots X_n$. 使 $X_1, X_2 \dots X_n$ 为 $\mathbb{R}^n$ 的一组

标准正交基. 令 $Q = (X_1, X_2 \dots X_n)$ 则 Q 为正交矩阵.

$$AQ = A(X_1, X_2 \dots X_n) = (\lambda_1 X_1, AX_2, AX_3 \dots AX_n),$$

$$= \underbrace{(X_1, X_2 \dots X_n)}_{Q} \begin{pmatrix} \lambda_1 & & \\ & \alpha^T & \\ & & A_1 \end{pmatrix}$$

$$AQ = Q \begin{bmatrix} \lambda_1 & \alpha^T \\ 0 & A_1 \end{bmatrix}$$

$$Q^T A Q = \begin{bmatrix} \lambda_1 & \alpha^T \\ 0 & A_1 \end{bmatrix}$$

$$Q^T A Q$$

$$\underbrace{A^T = A}_{\Delta = 0} \quad (Q^T A Q)^T = Q^T A Q$$

$$\exists \begin{bmatrix} \lambda_1 & \alpha^T \\ 0 & A_1 \end{bmatrix} \text{ 为实对称矩阵}$$

$$\exists \begin{bmatrix} \lambda_1 & 0 \\ \alpha & A_1^T \end{bmatrix} = \begin{bmatrix} \lambda_1 & \alpha^T \\ 0 & A_1 \end{bmatrix}$$

$$\alpha = 0. \quad A_1^T = A_1.$$

$$Q^T A Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{bmatrix}$$

由归纳法 $\exists Q_1 \in M_m(\mathbb{R})$

$$Q_1^T A_1 Q_1 = Q_1^T A_1 Q_1 = \Lambda_1$$

$$Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix}$$

$$\begin{aligned} Q_2^T Q^T A Q Q_2 &= \begin{bmatrix} 1 & \\ & Q_1^T \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & Q_1^T A_1 Q_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \underline{\underline{Q_1^T A_1 Q_1}} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \Lambda_1 \end{bmatrix} \end{aligned}$$

之证: 实对称矩阵属于不同特征值: 特征向量是正交.

$$A^T = A, A \in M_n(\mathbb{R})$$

$$\begin{array}{ll} \underline{AX_1 = \lambda_1 X_1} & X_1 \neq 0 \\ \underline{AX_2 = \lambda_2 X_2} & X_2 \neq 0 \end{array} \quad \lambda_1 \neq \lambda_2$$

$$\begin{array}{ll} \text{数} & X_2^T A X_1 = \lambda_1 X_2^T X_1 \\ & \downarrow \\ \text{也} & \underline{X_1^T A X_2} = \lambda_2 X_1^T X_2 \end{array} \quad \begin{array}{l} (X_2^T A X_1)^T = X_2^T A X_1 \\ \parallel \\ X_1^T A X_2 \end{array}$$

$$\underline{\lambda_1 X_2^T X_1} = \underline{\lambda_2 X_1^T X_2}.$$

$$(\lambda_1 - \lambda_2) X_2^T X_1 = 0$$

$$\Rightarrow X_2^T X_1 = 0 \quad \text{or} \quad X_1, X_2 \perp \xi_i$$

$$\text{Sol: } A = \begin{bmatrix} 0 & -2 & -1 \\ -2 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix} \quad |\lambda I - A| = \begin{vmatrix} \lambda & 2 & 1 \\ 2 & \lambda-3 & -2 \\ 1 & -2 & \lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda & 2 & \lambda+1 \\ 2 & \lambda-3 & 0 \\ 1 & -2 & \lambda+1 \end{vmatrix} = 0 \quad (\lambda+1) \begin{vmatrix} \lambda & 2 & 1 \\ 2 & \lambda-3 & 0 \\ 1 & -2 & 1 \end{vmatrix} = 0$$

$$(\lambda+1) \begin{vmatrix} \lambda-1 & 4 & 0 \\ 2 & \lambda-3 & 0 \\ 1 & -2 & 1 \end{vmatrix} = 0$$

$$(\lambda+1) [\lambda^2 - 4\lambda + 3 - 8] = 0$$

$$(\lambda+1)(\lambda^2 - 4\lambda - 5) = 0$$

$$(\lambda+1)^2(\lambda-5) = 0 \quad \lambda_1 = \lambda_2 = -1, \lambda_3 = 5$$

$$\lambda = 5 \text{ or } (5I - A)X = 0$$

$$\begin{bmatrix} 5 & 2 & 1 \\ 2 & 2 & -2 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\rightarrow \begin{bmatrix} 0 & 12 & -24 \\ 0 & 6 & -12 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda = -1 \text{ of } (-I - A)X = 0 \quad (I + A)X = 0$$

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xi_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [\xi_1, \xi_2, \xi_3]$$

$$AP = P \begin{bmatrix} 5 & & \\ & -1 & \\ & & -1 \end{bmatrix}$$

What a pity!

P 不是正交阵

$$\eta_1 = \xi_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\eta_2 = \xi_2 - \frac{(\xi_2, \eta_1)}{(\eta_1, \eta_1)} \eta_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\eta_3 = \xi_3 - \frac{(\xi_3, \eta_1)}{(\eta_1, \eta_1)} \eta_1 - \frac{(\xi_3, \eta_2)}{(\eta_2, \eta_2)} \eta_2$$

η_2, η_3 正交

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 1 \end{bmatrix}$$

$$\varphi_1 = \frac{\eta_1}{|\eta_1|}, \quad \varphi_2 = \frac{\eta_2}{|\eta_2|}, \quad \varphi_3 = \frac{\eta_3}{|\eta_3|}$$

$$Q = [\varphi_1, \varphi_2, \varphi_3]$$

$$Q^T A Q = Q^{-1} A Q = \begin{bmatrix} 5 & & \\ & -1 & \\ & & -1 \end{bmatrix}$$

$$\text{例: } A \in M_3(\mathbb{R}), \quad \lambda_1 = -1, \quad \lambda_2 = \lambda_3 = 1.$$

$$\text{属于 } \lambda_1 = -1 \text{ 的特征向量为 } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{求 } A$$

解: 属于 $\lambda_2 = \lambda_3 = 1$ 的特征向量 $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ 正交

$$0 \cdot x + 1 \cdot y + 1 \cdot z = 0 \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ & -1 \end{bmatrix} \quad P^T A P = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \quad A P = P \begin{bmatrix} & \\ & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

• 二次型

$$z = ax^2 + bxy + cy^2 \text{ 二次齐次式} \cdot a, b, c \text{ 常数.}$$

$$= [x, y] \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

① 判断符号: $ax^2 + bx + c = [x, 1] \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$

$$a > 0, \quad b^2 - 4ac < 0 \Rightarrow \forall x \in \mathbb{R}, \quad ax^2 + bx + c > 0,$$

② 二次曲线在形化:

$$x^2 + xy + y^2 = 2.$$

$$[x, y] \underbrace{\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}}_A \begin{bmatrix} x \\ y \end{bmatrix} = 2$$

$\exists Q \text{ 正交. } Q^T A Q = Q^{-1} A Q$
 $= \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$

$$A = Q \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} Q^T$$

$$[x, y] Q \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} Q^T \begin{bmatrix} x \\ y \end{bmatrix} = 2$$

令 $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = Q^T \begin{bmatrix} x \\ y \end{bmatrix}$ $[x_1, y_1] = [x, y] Q$

$$\text{即 } [x_1, y_1] \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 2$$

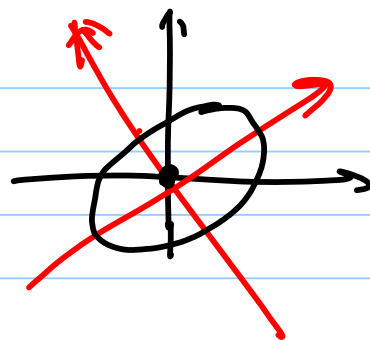
$$\begin{vmatrix} \lambda - 1 & \frac{1}{2} \\ -\frac{1}{2} & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - \frac{1}{4} = 0$$

$$\lambda = 1 \pm \frac{1}{2}$$

$$= \frac{3}{2} \text{ 或 } \frac{1}{2}.$$

$$\frac{3}{2}x_1^2 + \frac{1}{2}y_1^2 = 2$$

$$x^2 + xy + y^2 = 2$$



1. 二次型定义

$$Q(x_1, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \text{ 称为 } n\text{-元二次型}$$

其中 $a_{ij} = a_{ji}$. 称 $A = [a_{ij}]$ 为二次型矩阵.

$$Q(x_1, x_2, x_3) = 3x_1^2 + x_2^2 - x_3^2 + 2x_1x_2 - x_1x_3 + 4x_2x_3$$

$$= \underline{\underline{[x_1, x_2, x_3]}} \underbrace{\begin{bmatrix} 3 & 1 & -\frac{1}{2} \\ 1 & 1 & 2 \\ -\frac{1}{2} & 2 & -1 \end{bmatrix}}_{\text{对称矩阵}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{列向量}}.$$

对称矩阵.

$$Q(x_1, x_2, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j \quad (a_{ij} = a_{ji})$$

$$= X^T A X \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad A = [a_{ij}] \in SM_n(\mathbb{R})$$

2. 相合: $A, B \in M_n(\mathbb{R})$. 若 $\exists$ 可逆矩阵 P

使 $P^T A P = B$. 则称 A, B 相合关系.

① 自反. ② 对称. $P^T A P = B$.

$$\begin{aligned} A &= (P^T)^{-1} B P^{-1} \\ &= (P^{-1})^T B P^{-1} \end{aligned}$$

③ 传递性: $P^T A P = B$ $\xrightarrow{\Delta}$ $Q^T B Q = C$

$$\underbrace{Q^T P^T A P Q}_{(PQ)^T A (PQ)} = C$$

$$(PQ)^T A (PQ) = C$$

3. 化二次型为标准型. (无交叉项)

$$f(x_1, x_2, \dots, x_n) = X^T \underset{\Delta}{A} X \quad A \text{ 为对称阵}$$

$$\exists \text{ 正交阵 } Q \quad Q^T A Q = \underset{\Delta}{Q^T A Q} = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

$$A = Q \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^T$$

$$f(x_1, \dots, x_n) = \underbrace{X^T Q}_{Y^T} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Q^T X$$

$$\text{令 } Y = \underline{\underline{Q^T X}} = Q^T X \quad (\text{可逆线性变换})$$

$$Y^T = X^T Q$$

$$f(x_1, \dots, x_n) = g(y_1, \dots, y_n) = Y^T \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Y$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

P194. 13.(2)(3): 14-17 18-20. 21(4).

23. (1)(2).

$$121: Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$$

Note Title

1/7/2016

$$311: Q = [x_1, x_2, x_3] \underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} |\lambda I - A| &= 0 \\ |A - \lambda I| &= 0 \end{aligned} \quad \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} = 0.$$

$$\begin{vmatrix} \lambda - 1 & -2 & 0 \\ -2 & \lambda - 1 & -\lambda - 1 \\ -2 & -2 & \lambda + 1 \end{vmatrix} = 0$$

$$(\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & 0 \\ -2 & \lambda - 1 & -1 \\ -2 & -2 & 1 \end{vmatrix} = 0$$

$$(\lambda + 1) \begin{vmatrix} \lambda - 1 & -2 & 0 \\ -4 & \lambda - 3 & 0 \\ -2 & -2 & 1 \end{vmatrix} = 0$$

$$(\lambda + 1) [\lambda^2 - 4\lambda + 3 - 8] = 0$$

$$(\lambda + 1) (\lambda^2 - 4\lambda - 5) = 0$$

$$(\lambda + 1) (\lambda + 1) (\lambda - 5) = 0$$

$$\lambda_1 = 5, \quad \lambda_2 = \lambda_3 = -1$$

$$\lambda = 5 \text{ for } \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 & -6 & 6 \\ 0 & 6 & -6 \\ -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = \lambda_3 = -1 \quad \text{of} \quad \begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix} X = 0$$

$$\xi_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\eta_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\eta_2 = \xi_2 - \frac{(\xi_2, \eta_1)}{(\eta_1, \eta_1)} \eta_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\eta_3 = \xi_3 - \frac{(\xi_3, \eta_1)}{(\eta_1, \eta_1)} \eta_1 - \frac{(\xi_3, \eta_2)}{(\eta_2, \eta_2)} \eta_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$s_1 = \frac{\eta_1}{|\eta_1|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$s_3 = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{bmatrix}$$

$$s_2 = \frac{\eta_2}{|\eta_2|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\text{令 } Q = [s_1, s_2, s_3] \quad \text{则} \quad Q^T A Q = Q^{-1} A Q = \begin{bmatrix} 5 & & \\ & -1 & \\ & & -1 \end{bmatrix}$$

正交变换为 $X = Q Y$

$$\text{则} \quad Q(x_1, x_2, x_3) = X^T A X$$

$$= Y^T \underbrace{Q^T A Q}_{\begin{bmatrix} 5 & & \\ & -1 & \\ & & -1 \end{bmatrix}} Y$$

$$= Y^T \begin{bmatrix} 5 & & \\ & -1 & \\ & & -1 \end{bmatrix} Y$$

$$= 5y_1^2 - y_2^2 - y_3^2$$

另2:

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + \underline{\underline{4x_1x_2 + 4x_1x_3 + 4x_2x_3}}$$

$$= x_1^2 + 4x_1(x_2 + x_3) + x_2^2 + x_3^2 + 4x_2x_3$$

$$\begin{aligned}
 &= (x_1 + 2(x_2 + x_3))^2 - 4(x_2 + x_3)^2 + x_2^2 + x_3^2 + 4x_2x_3 \\
 &= (x_1 + 2(x_2 + x_3))^2 - 3x_2^2 - 3x_3^2 - 4x_2x_3 \\
 &= (x_1 + 2(x_2 + x_3))^2 - 3 \left[x_2^2 + \frac{4}{3}x_2x_3 \right] - 3x_3^2 \\
 &= (x_1 + 2(x_2 + x_3))^2 - 3 \left(x_2 + \frac{2}{3}x_3 \right)^2 - \frac{5}{3}x_3^2
 \end{aligned}$$

$$\begin{cases} y_1 = x_1 + 2x_2 + 2x_3 \\ y_2 = x_2 + \frac{2}{3}x_3 \\ y_3 = x_3 \end{cases} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}}_P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

P 可逆变换

$$Q = y_1^2 - 3y_2^2 - \frac{5}{3}y_3^2$$

配方法: ① 如果 x_1^2 系数不为零, 将 x_1 所有项集中起来配方.

② 将不含 x_1 项整理后, 令配方法为 0.

再从 x_2^2 继续配方. 一直到所有项都配好.

③ 若二次型中没有平方项.

$$Q(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$$

$$\begin{cases} x_1 = z_1 + z_2 \\ x_2 = z_2 \\ x_3 = z_3 \end{cases}$$

$$Q = (z_1 + z_2)z_2 + (z_1 + z_2)z_3 + z_2z_3$$

$$= z_1^2 + z_1z_2 + z_1z_3 + z_2^2 + z_2z_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

Note: 配方法变换矩阵 P 必是可逆

③ 主子变换法:

$$P \text{ 可逆} \quad P^T A P = \Lambda \text{ 对称阵}$$

$P \text{ 可逆} \Leftrightarrow P$ 可以写成 n 个初等矩阵乘积.

$$P = P_1 P_2 \cdots P_s$$

$$P_s^T P_{s-1}^T \cdots P_1^T A P_1 P_2 \cdots P_s = \Lambda$$

$$I P_1 P_2 \cdots P_s = P$$

$$Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$$

$$\left[\begin{array}{c|ccc} A & & & \\ \hline & I & & \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & +2 & +2 & & & \\ +2 & 1 & +2 & & & \\ +2 & 2 & 1 & & & \\ \hline & 1 & 0 & 0 & & \\ & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & & \end{array} \right] \xrightarrow{C_1 \times (-2) + C_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & & & \\ 2 & -3 & 2 & & & \\ 2 & -2 & 1 & & & \\ \hline & 1 & -2 & 0 & & \\ & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & & \end{array} \right]$$

$$\xrightarrow{r_1 \times (-2) + r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & & & \\ 0 & -3 & -2 & & & \\ 2 & -2 & 1 & & & \\ \hline & 1 & -2 & 0 & & \\ & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & & \end{array} \right] \xrightarrow{C_1 \times (-2) + C_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & -3 & -2 & & & \\ 2 & -2 & -3 & & & \\ \hline & 1 & -2 & -2 & & \\ & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & & \end{array} \right]$$

$$\xrightarrow{r_1 \times (-2) + r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & -3 & -2 & & & \\ 0 & -2 & -3 & & & \\ \hline & 1 & -2 & -2 & & \\ & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & & \end{array} \right] \xrightarrow{C_2 \times (-\frac{2}{3}) + C_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & -3 & -2 & & & \\ 0 & -\frac{2}{3} & -\frac{5}{3} & & & \\ \hline & 1 & -2 & -2 & & \\ & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & & \end{array} \right]$$

$$\xrightarrow{r_2 \times (-\frac{2}{3}) + r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & -3 & -2 & & & \\ 0 & 0 & -\frac{5}{3} & & & \\ \hline & 1 & -2 & -2 & & \\ & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & & \end{array} \right] \quad \therefore P = \begin{bmatrix} 1 & -2 & -\frac{2}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} 1 & & \\ & -3 & \\ & & -\frac{5}{3} \end{bmatrix}$$

P 可逆 $X = PY$
可逆变换

$$Q = X^T A X$$

$$= Y^T P^T A P Y$$

$$= Y^T \begin{bmatrix} 1 & -3 & -\frac{5}{3} \end{bmatrix} Y$$

$$= y_1^2 - 3y_2^2 - \frac{5}{3}y_3^2.$$

Note: 二次型标准型不唯一.

零个数, 正个数 与 负个数 个数 确定.

$$Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2$$

$$\begin{cases} z_1 = y_1 \\ z_2 = \sqrt{3}y_2 \\ z_3 = \sqrt{\frac{5}{3}}y_3 \end{cases} \quad Q = z_1^2 - z_2^2 - z_3^2$$

正个数 假为 2, 0 时 称其为 二次型 的 规范型.

Note: 规范型 唯一.

2. 非零个数 个数 $= r(A)$

$$Q^T A Q = Q^{-1} A Q = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$r(A) = r \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

正个数 个数 称为 正惯性 个数 也 为 p

负 ————— 负. ————— $(r-p)$.

符号差: $p - (r-p) = 2p - r$.

Note: (实对称矩阵可相似化 $r(A) = A$ 非零特征值个数)

$$Q^T A Q = Q^{-1} A Q = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$r(A) = r \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

不可相似化矩阵 $r(A) \neq A$ 非零特征值个数

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

不可相似化

二: 实二次型正定性.

1. 定义 $Q(x_1 \dots x_n) = X^T A X$ 若 $\forall X \neq 0$ ($x_1 \dots x_n$ 不同时为 0)

$X^T A X > 0$ 称 $Q(x_1 \dots x_n)$ 为正定二次型.

称 A 为正定实矩阵

$$Q(x_1 \dots x_n) = x_1^2 + x_2^2 + \dots + x_n^2$$

Note 1: 正定矩阵首先是实对称矩阵

2. 正定矩阵性质:

$$Q^T A Q = Q^{-1} A Q = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$X = QY$$

实对称 A 正定矩阵 $\Leftrightarrow A$ 特征值 > 0

$\Leftrightarrow A$ 正惯性指数 $= n$

$\Leftrightarrow A$ 与 I 相合.

$\Leftrightarrow \exists C$ 可逆 $C^T C = A$

即 $A = C^T C$.

$$f = X^T A X = Y^T \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} Y$$

$$= \lambda_1 y_1^2 + \dots + \lambda_n y_n^2$$

$$f(x_1, x_2, x_3) = x_1^2$$

例: $Q(x_1 \cdots x_n) = (x_1 + x_2)^2 + (x_2 + x_3)^2 + \cdots + (x_n + x_1)^2$

是否是正定二次型 答: 不是. 半正定.

~~$y_1 = x_1 + x_2$~~

~~$y_2 = x_2 + x_3$~~

~~$\vdots$~~

~~$y_n = x_n + x_1$~~

不可逆.

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots \\ & & & 1 & 1 \\ 1 & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

D_n

$$\begin{vmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{vmatrix}$$

n 为偶数时

$$= 1 + 1 \cdot (-1)^{n+1}$$

1 交换不可逆

正定二次型的判定:

1. 定义: k 阶主子式

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{vmatrix}$$

2. $x^T A x$ 正定 $\Leftrightarrow A$ 的所有 k 阶主子式 > 0 .

$$\Rightarrow: \forall x \neq 0 \quad x^T A x > 0$$

取 $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x_1 & \cdots & x_k & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{bmatrix} > 0$$

$$\begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} > 0$$

若 $\begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$ 正定. 从而 $\langle j, j \rangle > 0$.

$\Leftrightarrow$: 略

例: $f = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2tx_1x_2 - 2tx_1x_3 - 2tx_2x_3$
正定. 求 t 范围.

$$\begin{pmatrix} 2 & -t & -t \\ -t & 2 & -t \\ -t & -t & 2 \end{pmatrix}$$

$$2 > 0 \quad \begin{vmatrix} 2 & -t \\ -t & 2 \end{vmatrix} = 4 - t^2 > 0 \quad \begin{vmatrix} 2 & -t & -t \\ -t & 2 & -t \\ -t & -t & 2 \end{vmatrix} > 0.$$

• 半正定: $\forall X \neq 0 \quad X^T A X \geq 0$ 称
二次型 $X^T A X$ 为半正定. 称 A 为半正定矩阵

• A 半正定 $\Leftrightarrow p = r < n \quad r = r(A)$.

$\Leftrightarrow A$ 相似于 $\begin{bmatrix} I_r & \\ & 0 \end{bmatrix} \quad r < n$.

$\Leftrightarrow A$ 的所有主子式 ≥ 0

• A 负定 $\Leftrightarrow -A$ 正定

• $\exists X \neq 0 \quad X^T A X > 0, \exists Y \neq 0 \quad Y^T A Y < 0$. 称 A 不定.

例: $f = x_1^2 - x_2^2$

• A, B 相似 $\Leftrightarrow \exists C$ 可逆 $C^T A C = B$.
A, B 对称.
 $\Leftrightarrow r(A) = r(B)$ 且 $P_A = P_B$.

A, B 相似 $\Leftrightarrow \exists P$ 可逆 $P^{-1} A P = B$.

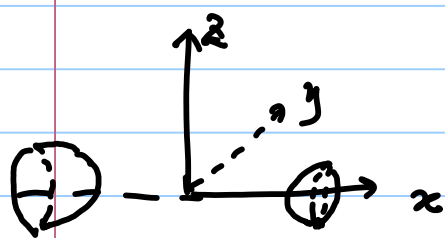
若 A, B 均可对角化 A, B 相似 $\Leftrightarrow \lambda(A) = \lambda(B)$

$\exists P$ 可逆 $P^{-1} A P = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$
 $\exists Q$ 可逆 $Q^{-1} B Q = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$
 $P^{-1} A P = Q^{-1} B Q$ $B = Q P^{-1} A P Q^{-1} = (P Q^{-1})^{-1} A (P Q^{-1})$

A, B 相似 $\Leftrightarrow \exists P, Q$ 可逆 $P A Q = B$.
 $\Leftrightarrow r(A) = r(B)$

第八章 空间曲线曲面

1. 旋转曲面

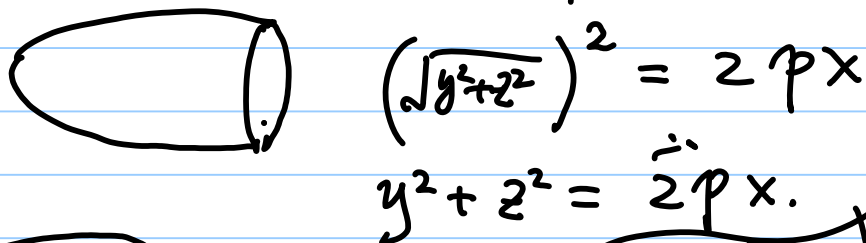
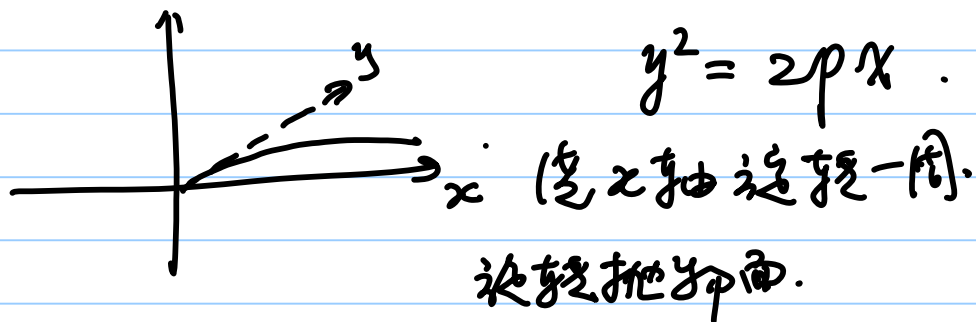
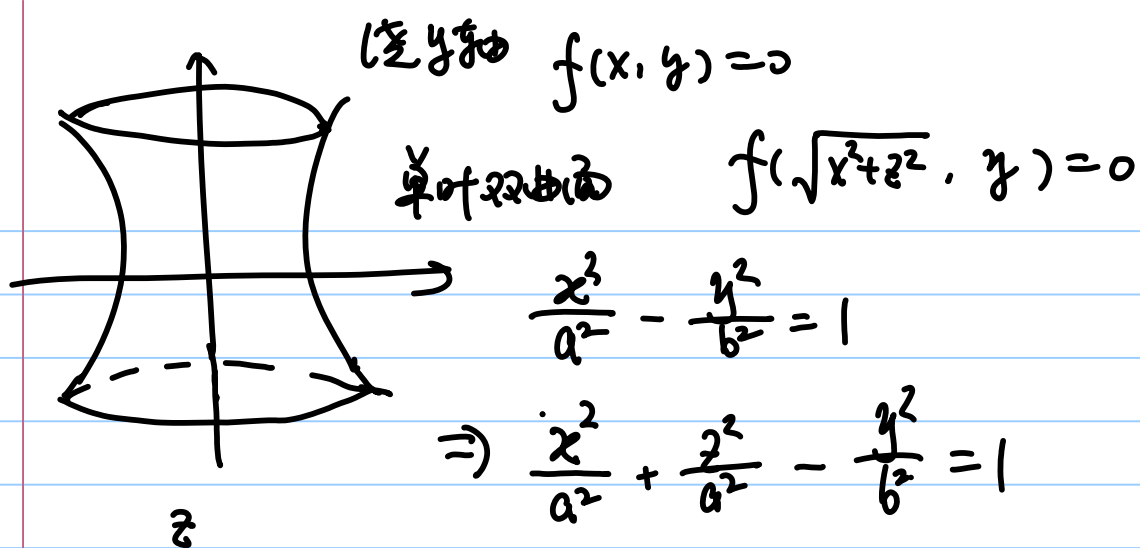


$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
 $f(x, y) = 0$
 $f(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$

绕 x 轴旋转一周所得曲面 $f(x, \sqrt{y^2 + z^2}) = 0$

$$\frac{x^2}{a^2} - \frac{(\sqrt{y^2 + z^2})^2}{b^2} = 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{b^2} = 1$$



柱面.

圆线

