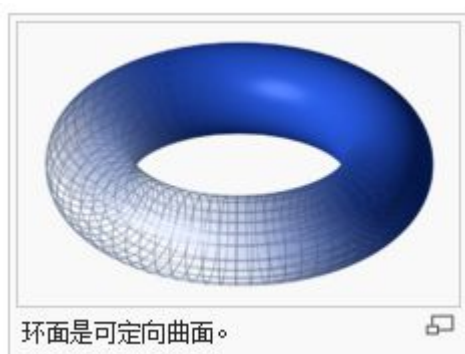
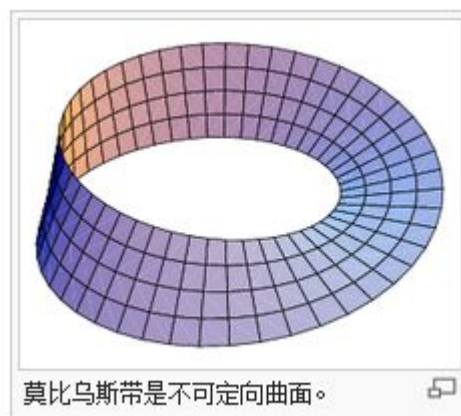




环面是可定向曲面，克莱因瓶是不可定向曲面。

克莱因瓶 (Klein bottle)

是指一种无定向性的平面，比如 2 维平面，就没有“内部”和“外部”之分。克莱因瓶最初的概念是由德国数学家菲利克斯·克莱因提出的。克莱因瓶和莫比乌斯带非常相像。克莱因瓶的结构非常简单，一个瓶子底部有一个洞，现在延长瓶子的颈部，并且扭曲地进入瓶子内部，然后和底部的洞相连接。



$$1. (1) L = L_1 \cup L_2 \cup L_3 \quad \text{其中 } L_1 = \{(x, y) \mid x=0, 0 \leq y \leq 1\}$$

$$L_2 = \{(x, y) \mid y=0, 0 \leq x \leq 1\}$$

$$L_3 = \{(x, y) \mid 0 \leq x \leq 1, y=1-x\}$$

$$\begin{aligned} \text{则 } \int_L (x+y) dl &= \int_{L_1} (x+y) dl + \int_{L_2} (x+y) dl + \int_{L_3} (x+y) dl \\ &= \int_0^1 y dy + \int_0^1 x dx + \int_0^1 \sqrt{1+1} dx = \frac{1}{2} + \frac{1}{2} + \sqrt{2} = \sqrt{2} + 1 \end{aligned}$$

$$(2) L: \begin{cases} x = 1 + \cos \varphi \\ y = \sin \varphi \end{cases} \quad (-\pi \leq \varphi \leq \pi)$$

$$\begin{aligned} \int_L \sqrt{x^2 + y^2} dl &= \int_{-\pi}^{\pi} \sqrt{2 + 2\cos \varphi} \cdot \sqrt{\sin^2 \varphi + \cos^2 \varphi} d\varphi \\ &= 2 \int_{-\pi}^{\pi} \sqrt{\frac{1 + \cos \varphi}{2}} d\varphi = 4 \sin \frac{\varphi}{2} \Big|_{-\pi}^{\pi} = 4 - (-4) = 8 \end{aligned}$$

$$\begin{aligned} (3) \int_L y^2 dl &= \int_0^{2\pi} a^2 (1 - \cos t)^2 \cdot \sqrt{a^2 (1 - \cos t)^2 + a^2 \sin^2 t} dt \\ &= a^3 \int_0^{2\pi} (2 - 2\cos t) \cdot \sqrt{2 - 2\cos t} dt \\ &= 8a^3 \int_0^{2\pi} \sin^2 \frac{t}{2} dt \\ &= 16a^3 \cdot \left(\frac{1}{3} \cos^3 \frac{t}{2} - \cos \frac{t}{2} \right) \Big|_0^{2\pi} = 16a^3 \times \frac{4}{3} = \frac{64}{3} a^3 \end{aligned}$$

$$\begin{aligned} (4) \int_L (x^{\frac{4}{3}} + y^{\frac{4}{3}}) dl &= \int_0^{2\pi} a^{\frac{4}{3}} (\cos^4 t + \sin^4 t) \sqrt{9a^2 (\cos^4 t \sin^4 t + \sin^4 t \cos^4 t)} dt \\ &= 3a^{\frac{7}{3}} \int_0^{2\pi} (\cos^4 t + \sin^4 t) |\sin t \cos t| dt \\ &= 12a^{\frac{7}{3}} \int_0^{\frac{\pi}{2}} [\cos^5 t + \cos t (1 - \cos^2 t)^2] \sin t dt \\ &= 4a^{\frac{7}{3}} \end{aligned}$$

$$2. (1) r^2 = a^2 \cos 2\theta = a^2 \cos^2 \theta - a^2 \sin^2 \theta = x^2 - y^2$$

$$\text{故 } \int_L x \sqrt{x^2 - y^2} dl = r \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a \cos \theta \cdot \sqrt{a^2 \sin^2 \theta + a^2 \cos^2 \theta} d\theta = ar \sin \theta \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \sqrt{2} a^2 r$$

$$\begin{aligned} (2) \int_L (x^2 + y^2 + z^2) dl &= \int_0^{2\pi} (4 + 9t^2) \cdot \sqrt{4 \sin^2 t + 4 \cos^2 t + 9} dt \\ &= \sqrt{13} \int_0^{2\pi} (4 + 9t^2) dt = \sqrt{13} (8\pi + 24\pi^3) \end{aligned}$$

$$\begin{aligned}
 (3) \int_L xyz \, dl &= \int_0^1 t \cdot \frac{2}{3} \sqrt{2} t^{\frac{3}{2}} \cdot \frac{1}{2} t^2 \sqrt{1+2t+t^2} \, dt \\
 &= \frac{\sqrt{2}}{3} \int_0^1 t^{\frac{9}{2}} (t+1) \, dt \\
 &= \frac{\sqrt{2}}{3} \times \left(\frac{2}{11} + \frac{2}{13} \right) = \frac{16\sqrt{2}}{143}
 \end{aligned}$$

$$(4) L = L_1 \cup L_2 \cup L_3 \quad \text{其中} \quad L_1: \begin{cases} x=0 \\ y=2\cos\theta \\ z=2\sin\theta \end{cases} \quad L_2: \begin{cases} x=2\cos\theta \\ y=0 \\ z=2\sin\theta \end{cases} \quad L_3: \begin{cases} x=2\cos\theta \\ y=2\sin\theta \\ z=0 \end{cases} \\
 (0 \leq \theta \leq \frac{\pi}{2}) \quad (0 \leq \theta \leq \frac{\pi}{2}) \quad (0 \leq \theta \leq \frac{\pi}{2})$$

$$\begin{aligned}
 \text{则} \int_L x \, dl &= \int_{L_1} x \, dl + \int_{L_2} x \, dl + \int_{L_3} x \, dl \\
 &= 0 + \int_0^{\frac{\pi}{2}} 2\cos\theta \cdot \sqrt{4\sin^2\theta + 4\cos^2\theta} \, d\theta + \int_0^{\frac{\pi}{2}} 2\cos\theta \cdot \sqrt{4\sin^2\theta + 4\cos^2\theta} \, d\theta \\
 &= 2 \times 4 \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = 2 \times 4 = 8
 \end{aligned}$$

$$3. (1) \int_L dl = \int_0^1 \sqrt{9+36t^2+36t^4} \, dt = 3 \int_0^1 (2t^2+1) \, dt = 5$$

$$\begin{aligned}
 (2) \int_L dl &= \int_0^{+\infty} \sqrt{e^{-2t}(-\cos t - \sin t)^2 + e^{-2t}(-\sin t + \cos t)^2 + e^{-2t}} \, dt \\
 &= \int_0^{+\infty} e^{-t} \sqrt{1+2\cos t \sin t + 1-2\cos t \sin t + 1} \, dt \\
 &= \sqrt{3} (-e^{-t}) \Big|_0^{+\infty} = \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 4. m &= \int_L x^2 \, dl = \int_{\frac{\sqrt{3}}{3}}^{\sqrt{15}} x^2 \cdot \sqrt{1+(\frac{1}{x})^2} \, dx \\
 &= \int_{\frac{\sqrt{3}}{3}}^{\sqrt{15}} x \sqrt{x^2+1} \, dx \\
 &= \frac{1}{3} (x^2+1)^{\frac{3}{2}} \Big|_{\frac{\sqrt{3}}{3}}^{\sqrt{15}} = \frac{64}{3} - \frac{8}{3} = \frac{56}{3}
 \end{aligned}$$

$$5. L: \begin{cases} x=a\cos\theta \\ y=a\sin\theta \end{cases} \quad (0 \leq \theta \leq 2\pi)$$

$$\begin{aligned}
 S &= \int_L (a + \frac{x^2}{a}) \, dl = \int_0^{2\pi} a(1+\cos^2\theta) \sqrt{a^2\sin^2\theta + a^2\cos^2\theta} \, d\theta \\
 &= a^2 \int_0^{2\pi} (1+\cos^2\theta) \, d\theta \\
 &= a^2 \left(\frac{3}{2}\theta + \frac{1}{4}\sin 2\theta \right) \Big|_0^{2\pi} = a^2 \times 3\pi = 3\pi a^2
 \end{aligned}$$

$$6. M = \int_L dl = \int_0^\pi \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t} dt = a \int_0^\pi \sqrt{2-2\cos t} dt = 4a$$

$$M_x = \int_L y dl = \int_0^\pi a(1-\cos t) \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t} dt$$

$$= 4a^2 \int_0^\pi \sin^2 \frac{t}{2} dt = \frac{16}{3} a^2$$

$$M_y = \int_L x dl = \int_0^\pi a(t - \sin t) \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t} dt$$

$$= 2a^2 \int_0^\pi (t \sin \frac{t}{2} - \sin t \sin \frac{t}{2}) dt$$

$$= 2a^2 \left(\int_0^\pi t \sin \frac{t}{2} dt - \int_0^\pi 2 \sin^2 \frac{t}{2} \cos \frac{t}{2} dt \right)$$

$$= 2a^2 \times (4 - \frac{4}{3}) = \frac{16}{3} a^2$$

$$\text{故 } \bar{x} = \frac{M_y}{M} = \frac{4}{3} a \quad \bar{y} = \frac{M_x}{M} = \frac{4}{3} a \quad \text{故重心坐标为 } (\frac{4}{3} a, \frac{4}{3} a)$$

$$7. J_x = \int_L (y^2 + z^2) dl = \int_0^{2\pi} (a^2 \sin^2 t + \frac{b^2}{4\pi^2} t^2) \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + \frac{b^2}{4\pi^2}} dt$$

$$= \sqrt{a^2 + \frac{b^2}{4\pi^2}} \int_0^{2\pi} (a^2 \sin^2 t + \frac{b^2}{4\pi^2} t^2) dt$$

$$= \sqrt{a^2 + \frac{b^2}{4\pi^2}} \times (a^2 \pi + \frac{2b^2}{3} \pi)$$

$$= \sqrt{4\pi^2 a^2 + b^2} \left(\frac{a^2}{2} + \frac{b^2}{3} \right)$$

$$8. L: \begin{cases} x = \cos \theta \\ y = -1 + \sin \theta \end{cases} \quad \left(-\frac{3}{2}\pi \leq \theta \leq \frac{\pi}{2} \right)$$

$$M_x = \int_L y \sqrt{x^2 + y^2} dl = \int_{-\frac{3}{2}\pi}^{\frac{\pi}{2}} (\sin \theta - 1) \sqrt{\sin^2 \theta + \cos^2 \theta} \sqrt{\cos^2 \theta + (\sin \theta - 1)^2} d\theta$$

$$= \sqrt{2} \int_{-\frac{3}{2}\pi}^{\frac{\pi}{2}} (\sin \theta - 1) \left| \sin \frac{\theta}{2} - \cos \frac{\theta}{2} \right| d\theta \quad \left(\begin{array}{l} \cos \frac{\theta}{2} \geq \sin \frac{\theta}{2} \\ \because -\frac{3}{2}\pi \leq \theta \leq \frac{\pi}{2} \end{array} \right)$$

$$= \sqrt{2} \int_{-\frac{3}{2}\pi}^{\frac{\pi}{2}} (2 \cos^2 \frac{\theta}{2} \sin \frac{\theta}{2} - 2 \sin^2 \frac{\theta}{2} \cos \frac{\theta}{2} + \sin \frac{\theta}{2} - \cos \frac{\theta}{2}) d\theta$$

$$= \sqrt{2} \times \left(-\frac{4}{3} \cos^3 \frac{\theta}{2} - \frac{4}{3} \sin^3 \frac{\theta}{2} - 2 \cos \frac{\theta}{2} - 2 \sin \frac{\theta}{2} \right) \Big|_{-\frac{3}{2}\pi}^{\frac{\pi}{2}} = \sqrt{2} \times \left(-\frac{8\sqrt{2}}{3} \right) = -\frac{16}{3}$$

$$M = \int_L \sqrt{x^2 + y^2} dl = \sqrt{2} \int_{-\frac{3}{2}\pi}^{\frac{\pi}{2}} (\cos \frac{\theta}{2} - \sin \frac{\theta}{2}) d\theta = 4\sqrt{2} \cdot \sqrt{2} = 8$$

$$\begin{aligned}
 1. (1) \iint_S (x+y+z) dS &= \iint_{D_{\theta\varphi}} a(\sin\theta\cos\varphi + \sin\theta\sin\varphi + \cos\theta) \cdot a^2 \sin\theta d\theta d\varphi \\
 &= a^3 \int_0^{2\pi} d\varphi \int_0^\pi (\sin^2\theta(\cos\varphi + \sin\varphi) + \sin\theta\cos\theta) d\theta \\
 &= a^3 \int_0^{2\pi} \left[\frac{1}{2} + \frac{\pi}{4}(\cos\varphi + \sin\varphi) \right] d\varphi \\
 &= \pi a^3
 \end{aligned}$$

$$\begin{aligned}
 (2) \iint_S (2x + \frac{4}{3}y + z) dS &= \iint_{D_{xy}} 4 \cdot \sqrt{1+4+\frac{16}{9}} dx dy \quad (z = 4 - 2x - \frac{4}{3}y) \\
 &= \frac{4\sqrt{61}}{3} \iint_{D_{xy}} dx dy \\
 &= 4\sqrt{61}
 \end{aligned}$$

$$\begin{aligned}
 (3) S &= S_1 \cup S_2 \cup S_3 \cup S_4 \quad \text{其中 } S_1 = \{(x, y, z) \mid x=0, 0 \leq y \leq 1, 0 \leq z \leq 1-y\} \\
 S_2 &= \{(x, y, z) \mid y=0, 0 \leq x \leq 1, 0 \leq z \leq 1-x\} \\
 S_3 &= \{(x, y, z) \mid z=0, 0 \leq x \leq 1, 0 \leq y \leq 1-x\} \\
 S_4 &= \{(x, y, z) \mid z=1-x-y, 0 \leq x \leq 1, 0 \leq y \leq 1-x\}
 \end{aligned}$$

$$\begin{aligned}
 \iint_S \frac{dS}{(1+x+y)^2} &= \iint_{S_1} \frac{dS}{(1+x+y)^2} + \iint_{S_2} \frac{dS}{(1+x+y)^2} + \iint_{S_3} \frac{dS}{(1+x+y)^2} + \iint_{S_4} \frac{dS}{(1+x+y)^2} \\
 &= \int_0^1 dy \int_0^{1-y} \frac{1}{(1+y)^2} dz + \int_0^1 dx \int_0^{1-x} \frac{1}{(1+x)^2} dz + \int_0^1 dx \int_0^{1-x} \frac{1}{(1+x+y)^2} dy \\
 &\quad + \int_0^1 dx \int_0^{1-x} \frac{1}{(1+x+y)^2} \cdot \sqrt{1+1+1} dy \\
 &= (1 - \ln 2) + (1 - \ln 2) + (\ln 2 - \frac{1}{2}) + \sqrt{3}(\ln 2 - \frac{1}{2}) \\
 &= (\sqrt{3} - 1)(\ln 2 + \frac{\sqrt{3}}{2})
 \end{aligned}$$

$$\begin{aligned}
 (4) \iint_S (xy + yz + xz) dS &= \iint_{D_{xy}} [xy + (x+y)\sqrt{x^2+y^2}] \cdot \sqrt{2} dx dy \\
 &= \iint_{D_{\rho\theta}} [\rho^2 \sin\theta \cos\theta + \rho^2(\sin\theta + \cos\theta)] \cdot \sqrt{2} \rho d\rho d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \sqrt{2} \rho^3 (\cos\theta \sin\theta + \cos\theta + \sin\theta) d\rho \\
 &= 4\sqrt{2} a^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^5\theta \sin\theta + \cos^4\theta \sin\theta + \cos^3\theta) d\theta = 4\sqrt{2} a^4 \times 2 \times (1 - \frac{2}{3} + \frac{1}{5}) = \frac{64\sqrt{2}}{15} a^4
 \end{aligned}$$

$$\begin{aligned}
 (5) \iint_S x ds &= \iint_{Duv} u \cos v \cdot \sqrt{a^2 \sin^2 v + a^2 \cos^2 v + u^2} du dv \\
 &= \int_0^{2\pi} \cos v dv \cdot \int_0^r u \sqrt{a^2 + u^2} du = 0
 \end{aligned}$$

2. (此题看成曲线积分)

$$L: \begin{cases} x = \frac{1}{2}a + \frac{1}{2}a \cos \theta \\ y = \frac{1}{2}a \sin \theta \end{cases} \quad (0 \leq \theta \leq 2\pi) \quad \text{联立} \begin{cases} x^2 + y^2 = ax \\ x^2 + y^2 + z^2 = a^2 \end{cases} \quad \text{得 } z = \sqrt{a^2 - ax} \quad (\text{只考虑 } z \geq 0)$$

由对称性得, $z < 0$ 的面积与 $z \geq 0$ 时相同.

$$\begin{aligned}
 \text{故 } S &= 2 \int_L \sqrt{a^2 - ax} dl = 2 \int_0^{2\pi} \sqrt{a^2 - \frac{1}{2}a^2 - \frac{1}{2}a^2 \cos \theta} \cdot \sqrt{\frac{1}{4}a^2 \sin^2 \theta + \frac{1}{4}a^2 \cos^2 \theta} d\theta \\
 &= a^2 \cdot (-2 \cos \frac{\theta}{2}) \Big|_0^{2\pi} = 4a^2
 \end{aligned}$$

$$\begin{aligned}
 3. M &= \iint_S \frac{x^2 + y^2}{2} ds = \iint_{Dxy} \frac{x^2 + y^2}{2} \cdot \sqrt{1 + x^2 + y^2} dx dy \quad (Dxy = \{(x, y) | 0 \leq x^2 + y^2 \leq 2\}) \\
 &= \iint_{D\rho\theta} \frac{\rho^2}{2} \cdot \sqrt{1 + \rho^2} \rho d\rho d\theta \\
 &= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \frac{1}{2} \rho^3 \sqrt{1 + \rho^2} d\rho \\
 &= 2\pi \cdot \frac{1}{4} \cdot \left[\frac{2}{5} (1 + \rho^2)^{\frac{5}{2}} - \frac{2}{3} (1 + \rho^2)^{\frac{3}{2}} \right] \Big|_0^{\sqrt{2}} \\
 &= \frac{12\sqrt{3} + 2}{15} \pi
 \end{aligned}$$

4. 令该直径在 z 轴上.

$$\begin{aligned}
 M &= \iint_S \sqrt{x^2 + y^2} ds = \iint_{D\rho\varphi} a \sin \theta \cdot a^2 \sin \theta d\theta d\varphi \\
 &= \int_0^{2\pi} d\varphi \int_0^\pi a^3 \sin^2 \theta d\theta = 2\pi \cdot \frac{\pi}{2} \cdot a^3 = \pi^2 a^3
 \end{aligned}$$

5. 直线为经过点 $(0, 0, b)$, 平行于 x 轴的直线. $z = \frac{b}{a} \sqrt{x^2 + y^2} \quad (0 \leq z \leq b)$

$$\begin{aligned}
 J &= \iint_{Dxy} (y^2 + (z - b)^2) \delta_0 \sqrt{1 + \frac{b^2 x^2}{a^2 (x^2 + y^2)} + \frac{b^2 y^2}{a^2 (x^2 + y^2)}} dx dy \\
 &= \sqrt{1 + \frac{b^2}{a^2}} \delta_0 \int_0^{2\pi} d\theta \int_0^a \left[\rho^2 \sin^2 \theta + b^2 \left(\frac{\rho^2}{a^2} - \frac{2\rho}{a} + 1 \right) \right] \rho d\rho \\
 &= \sqrt{1 + \frac{b^2}{a^2}} \delta_0 \int_0^{2\pi} \left(\frac{\rho^4}{4} \sin^2 \theta + \frac{1}{12} a^2 b^2 \right) d\theta = a \sqrt{a^2 + b^2} \pi \delta_0 \left(\frac{a^2}{4} + \frac{b^2}{6} \right)
 \end{aligned}$$

6. 第一卦限: $M_1 = \frac{1}{2}\pi a^2$ $z = \sqrt{a^2 - x^2 - y^2}$

$$M_{yz} = \iint_S x dS = \iint_{D_{xy}} ax \frac{1}{\sqrt{a^2 - x^2 - y^2}} dx dy = \int_0^{\frac{\pi}{2}} d\theta \int_0^a \frac{a\rho^2 \cos\theta}{\sqrt{a^2 - \rho^2}} d\rho$$

$$= a \cdot \frac{\pi}{4} a^2 = \frac{\pi}{4} a^3$$

故 $\bar{x} = \frac{M_{yz}}{M} = \frac{a}{2}$ 由对称性得 $\bar{y} = \bar{z} = \bar{x}$ 故重心坐标 $(\frac{a}{2}, \frac{a}{2}, \frac{a}{2})$.

上半球面, $M_2 = 2\pi a^2$ 由对称性得, 重心在 z 轴上.

$$M_{xy} = \iint_S z dS = \iint_{D_{xy}} \sqrt{a^2 - x^2 - y^2} \cdot \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx dy = a \cdot \pi a^2 = \pi a^3$$

$$\bar{z} = \frac{M_{xy}}{M_2} = \frac{a}{2} \quad \text{故重心坐标 } (0, 0, \frac{a}{2})$$

7. 由对称性得 $\bar{y} = 0$, 由例 4.3.3 得 $M = \sqrt{2}\pi a^2$ (书上图 4.3.3 是错的!!!)

$$M_{yz} = \iint_S x dS = \iint_{D_{\theta\rho}} \rho \cos\theta \cdot \sqrt{2} \cdot \rho d\rho d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} \sqrt{2} \rho^2 \cos\theta d\rho$$

$$= \frac{8\sqrt{2}a^3}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta d\theta = \frac{8\sqrt{2}}{3} a^3 \cdot \frac{3}{8}\pi = \sqrt{2}\pi a^3$$

$$M_{xy} = \iint_S z dS = \iint_{D_{\theta\rho}} \sqrt{2} \rho^2 d\rho d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} \sqrt{2} \rho^2 d\rho$$

$$= \frac{8\sqrt{2}}{3} a^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta d\theta = \frac{8\sqrt{2}}{3} a^3 \cdot \frac{4}{3} = \frac{32\sqrt{2}}{9} a^3$$

$$\text{故 } \bar{x} = \frac{M_{yz}}{M} = a \quad \bar{z} = \frac{M_{xy}}{M} = \frac{32}{9\pi} a \quad \text{故重心坐标 } (a, 0, \frac{32}{9\pi} a)$$

8. 联立 $\begin{cases} z = \sqrt{x^2 + y^2} \\ z^2 = 2x \end{cases}$ 得 $S: x^2 + y^2 \leq 2x$

$$dS = \sqrt{2} dx dy$$

$$\text{故 } S = \iint_S dS = \iint_{D_{xy}} \sqrt{2} dx dy = \sqrt{2}\pi$$

1. (1) L^+ : $\begin{cases} x = a \cos^3 t \\ y = a \sin^3 t \end{cases} \quad (0 \leq t \leq \frac{\pi}{2})$ 正向的参数值从 $\frac{\pi}{2}$ 到 0.

$$\begin{aligned} \text{故} \int_{L^+} \frac{x^2 dy - y^2 dx}{x^{\frac{5}{3}} + y^{\frac{5}{3}}} &= \int_{\frac{\pi}{2}}^0 \frac{a^2 \cos^6 t \cdot 3a \sin^2 t \cos t - a^2 \sin^6 t \cdot (-3a) \cos^2 t \sin t}{a^{\frac{5}{3}} (\cos^5 t + \sin^5 t)} dt \\ &= 3a^{\frac{4}{3}} \int_{\frac{\pi}{2}}^0 \cos^3 t \sin^3 t dt = -\frac{3\pi}{16} a^{\frac{4}{3}} \end{aligned}$$

(2) L^+ : $\begin{cases} x = 1+t \\ y = 1+2t \\ z = 1+3t \end{cases} \quad (0 \leq t \leq 1)$ 正向对应的参数值为从 0 到 1

$$\begin{aligned} \text{故} \int_{L^+} x dx + y dy + z dz &= \int_0^1 [(1+t) + 2(1+2t) + 3(1+3t)] dt \\ &= \int_0^1 (6+14t) dt = (6t+7t^2) \Big|_0^1 = 13 \end{aligned}$$

(3) L^+ : $\begin{cases} x = a \cos t \\ y = a \sin t \\ z = bt \end{cases} \quad (0 \leq t \leq 2\pi)$ 正向对应的参数值为从 0 到 2π .

$$\begin{aligned} \int_{L^+} \frac{-y dx + x dy}{x^2 + y^2} + b dz &= \int_0^{2\pi} \left(\frac{-a \sin t \cdot (-a \sin t) + a^2 \cos^2 t}{a^2 \cos^2 t + a^2 \sin^2 t} + b^2 \right) dt \\ &= \int_0^{2\pi} (1+b^2) dt = 2\pi(1+b^2) \end{aligned}$$

2. (1) $\int_{L^+} (x^2 - y^2) dx = \int_0^2 (x^2 - x^4) dx = \left(\frac{1}{3}x^3 - \frac{1}{5}x^5 \right) \Big|_0^2 = \frac{8}{3} - \frac{32}{5} = -\frac{56}{15}$

(2) L^+ : $\begin{cases} x = a \cos \varphi \\ y = a \sin \varphi \end{cases} \quad (0 \leq \varphi \leq 2\pi)$ 正向对应的参数值为从 0 到 2π .

$$\begin{aligned} \oint_{L^+} \frac{(x+y)dx + (y-x)dy}{x^2 + y^2} &= \int_0^{2\pi} \frac{-a^2(\cos \varphi + \sin \varphi) \sin \varphi + a^2(\sin \varphi - \cos \varphi) \cos \varphi}{a^2} d\varphi \\ &= -\int_0^{2\pi} d\varphi = -2\pi \end{aligned}$$

(3) L^+ : $|x| + |y| = 1$ 则 $\oint_{L^+} \frac{dx+dy}{|x|+|y|} = \oint_{L^+} (dx+dy)$

由对称性可得 $\oint_{L^+} \frac{dx+dy}{|x|+|y|} = 0$

(4) 在 xOy 平面上, L^+ : $\begin{cases} x = \cos\varphi \\ y = \sin\varphi \\ z = 0 \end{cases} \quad (0 \leq \varphi \leq \frac{\pi}{2})$ 正向对应的参数值为从 0 到 $\frac{\pi}{2}$

$$\begin{aligned} \text{故} \int_{L^+} (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz &= \int_{L^+} y^2 dx - x^2 dy \\ &= \int_0^{\frac{\pi}{2}} (-\sin^2\varphi - \cos^2\varphi) d\varphi \\ &= -\frac{4}{3} \end{aligned}$$

由轮换对称性得, 原式 = $3 \times (-\frac{4}{3}) = -4$

(5) L^+ : $\begin{cases} x = \cos\varphi \\ y = \frac{\sqrt{2}}{2}\sin\varphi \\ z = \frac{\sqrt{2}}{2}\sin\varphi \end{cases} \quad (0 \leq \varphi \leq 2\pi)$ 正向对应的参数值为从 0 到 2π

$$\text{故} \int_{L^+} xyz dz = \int_0^{2\pi} \frac{\sqrt{2}}{4} \sin^2\varphi \cos^2\varphi d\varphi = \frac{\sqrt{2}}{4} \times \frac{\pi}{4} = \frac{\sqrt{2}}{16}\pi$$

3.(1) $\int_{L^+} \vec{F} d\vec{r} = \int_{L^+} -ydx + xdy = \int_0^1 0 \cdot dx + \int_0^1 dy + \int_1^0 (-x+x) dx$
 $= 0 + 1 + 0 = 1$

(注: 由做功角度很容易看出, 第一段和第三段力的方向与位移方向垂直)

(2) L^+ : $\begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \end{cases} \quad (0 \leq \varphi \leq 2\pi)$ 正向对应的参数值为从 0 到 2π .

$$\text{故} \int_{L^+} \vec{F} d\vec{r} = \int_{L^+} \frac{ydx - xdy}{x^2 + y^2} = \int_0^{2\pi} \frac{-a^2\sin^2\varphi - a^2\cos^2\varphi}{a^2} d\varphi = -2\pi$$

(3) $\int_{L^+} \vec{F} d\vec{r} = \int_{L^+} F dx = \int_0^a F dx = aF$

(4) L^+ : $\begin{cases} x = 1 + (x_0 - 1)t \\ y = 1 + (y_0 - 1)t \\ z = 1 + (z_0 - 1)t \end{cases} \quad (0 \leq t \leq 1)$ 正向对应的参数值为从 0 到 1
 因为 $\vec{F}(x, y, z) = Gm \frac{x\vec{e} + y\vec{j} + z\vec{k}}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$

$$\begin{aligned} \text{故} \int_{L^+} \vec{F} d\vec{r} &= \int_0^1 Gm \frac{[(x_0 - 1) + (y_0 - 1) + (z_0 - 1)] + [(x_0 - 1)^2 + (y_0 - 1)^2 + (z_0 - 1)^2]t}{3 + 2[(x_0 - 1) + (y_0 - 1) + (z_0 - 1)]t + [(x_0 - 1)^2 + (y_0 - 1)^2 + (z_0 - 1)^2]t^2} dt \\ &= Gm \cdot \frac{1}{2} \cdot (-2) \cdot \left\{ 3 + 2[(x_0 - 1) + (y_0 - 1) + (z_0 - 1)]t + [(x_0 - 1)^2 + (y_0 - 1)^2 + (z_0 - 1)^2]t^2 \right\}^{-\frac{1}{2}} \Big|_0^1 \\ &= Gm \cdot \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{x_0^2 + y_0^2 + z_0^2}} \right) \end{aligned}$$

$$4. (1) \vec{F} = -(x\vec{i} + y\vec{j}) \quad L^+: \begin{cases} x = a \cos \varphi \\ y = b \sin \varphi \end{cases} \quad (0 \leq \varphi \leq \frac{\pi}{2}) \quad \begin{array}{l} \text{正向对应的参数值} \\ \text{为从 } 0 \text{ 到 } \frac{\pi}{2} \end{array}$$

$$\int_{L^+} \vec{F} d\vec{r} = \int_0^{\frac{\pi}{2}} -(a^2 \cos \varphi \sin \varphi + b^2 \sin \varphi \cos \varphi) d\varphi = \frac{a^2 - b^2}{4} \cdot (-\cos 2\varphi) \Big|_0^{\frac{\pi}{2}} = \frac{a^2 - b^2}{2}$$

(2) 由对称性得, 力F做的功为0.

$$5. (1) \vec{F} = -\frac{k}{\sqrt{x^2+y^2}} \cdot \frac{x\vec{i}+y\vec{j}}{\sqrt{x^2+y^2}} = -\frac{k}{x^2+y^2} (x\vec{i}+y\vec{j})$$

$$\int_{L^+} \vec{F} d\vec{r} = \int_1^0 -\frac{k}{x^2+y^2} \cdot x dx = -k \int_1^0 \frac{x}{x^2+1} dx = \frac{k}{2} \ln 2$$

$$(2) \vec{F} = -\frac{k}{z} \cdot \frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{x^2+y^2+z^2}} \quad L^+: \begin{cases} x = at \\ y = bt \\ z = ct \end{cases} \quad (1 \leq t \leq 2) \quad \begin{array}{l} \text{正向对应的参数值} \\ \text{为从 } 1 \text{ 到 } 2 \end{array}$$

$$\begin{aligned} \text{故 } \int_{L^+} \vec{F} d\vec{r} &= \int_1^2 -\frac{k}{|ct|} \cdot \frac{(a^2+b^2+c^2)t}{\sqrt{a^2+b^2+c^2} \cdot t} dt = -\frac{k\sqrt{a^2+b^2+c^2}}{|c|} \int_1^2 \frac{dt}{t} \\ &= -\frac{k\sqrt{a^2+b^2+c^2}}{|c|} \ln 2 \end{aligned}$$

$$1.1) \oint_{S^+} dx \wedge dy = 0$$

$$1.2) \oint_{S^+} z \, dx \wedge dy$$

$$= \int_{\text{上半球}} z \, dx \, dy - \int_{\text{下半球}} z \, dx \, dy$$

$$= \int_0^R dr \int_0^{2\pi} d\theta r [(R + \sqrt{R^2 - r^2}) - (R - \sqrt{R^2 - r^2})]$$

$$= \frac{4}{3} \pi R^3$$

$$1.3) \oint_{S^+} z^2 \, dx \wedge dy$$

$$= \int_{\text{上半球}} z^2 \, dx \, dy - \int_{\text{下半球}} z^2 \, dx \, dy$$

$$= \int_0^R dr \int_0^{2\pi} d\theta r [(R + \sqrt{R^2 - r^2})^2 - (R - \sqrt{R^2 - r^2})^2]$$

$$= \frac{8}{3} \pi R^4$$

$$2.1) \oint_{S^+} z \, dx \wedge dy$$

$$= \int_0^1 dx \int_0^{1-x} dy (1-x-y)$$

$$= \frac{1}{6}$$

$$2.2) \oint_{S^+} x^2 \, dy \wedge dz$$

$$= \int_0^1 dy \int_0^{1-y} dz (1-y-z)^2$$

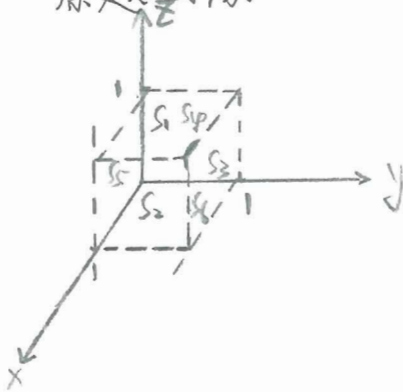
$$= \frac{1}{12}$$

$$2.3) \oint_{S^+} y^2 \, dz \wedge dx$$

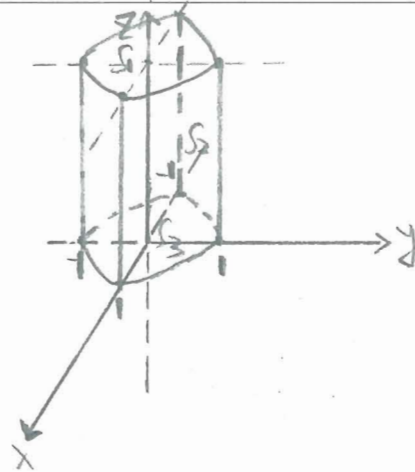
$$= \int_0^1 dx \int_0^{1-x} dz (1-x-z)^2$$

$$= \frac{1}{12}$$

习题 4.5

3 (1)
原题图像为:

(2)



$$\therefore \iint_{S^+} = \iint_{S_1^+} + \iint_{S_2^+} + \iint_{S_3^+} + \iint_{S_4^+} + \iint_{S_5^+} + \iint_{S_6^+}$$

$$\iint_{S_1^+} = \iint_{S_2^+} = 0 + 0 + \iint_{R_{xy}} z dx dy = \iint_{R_{xy}} 1 dx dy = 1$$

$$\iint_{S_2^+} = \iint_{S_3^+} = 0 + 0 + \iint_{R_{yz}} z dy dz = 0$$

$$\iint_{S_3^+} = \iint_{S_4^+} = 0 + 0 + \iint_{R_{xz}} z dx dz = 0$$

$$\iint_{S_4^+} = \iint_{S_5^+} = 0 + 0 + \iint_{R_{xy}} z dx dy = 0$$

$$\iint_{S_5^+} = \iint_{S_6^+} = 0 + 0 + \iint_{R_{yz}} z dy dz = 0$$

$$\iint_{S_6^+} = \iint_{S_1^+} = 0 + 0 + \iint_{R_{xy}} z dx dy = 0$$

$$\therefore \iint_{S^+} = \iint_{S_1^+} + \iint_{S_2^+} + \iint_{S_3^+} + \iint_{S_4^+} + \iint_{S_5^+} + \iint_{S_6^+} = 3$$

$$\therefore \iint_{S^+} = \iint_{S_1^+} + \iint_{S_2^+} + \iint_{S_3^+}$$

① 对于 S_1^+ :

$$\iint_{S_1^+} = \iint_{S_2^+} = 0 + 0 + \iint_{R_{xy}} z dx dy = \iint_{R_{xy}} 1 dx dy = 1$$

② 对于 S_2^+ :

$$\iint_{S_2^+} = \iint_{S_3^+} = 0 + 0 + \iint_{R_{yz}} z dy dz = 0$$

$$\iint_{S_3^+} = \iint_{S_4^+} = 0 + 0 + \iint_{R_{xz}} z dx dz = 0$$

$$\iint_{S_4^+} = \iint_{S_5^+} = 0 + 0 + \iint_{R_{xy}} z dx dy = 0$$

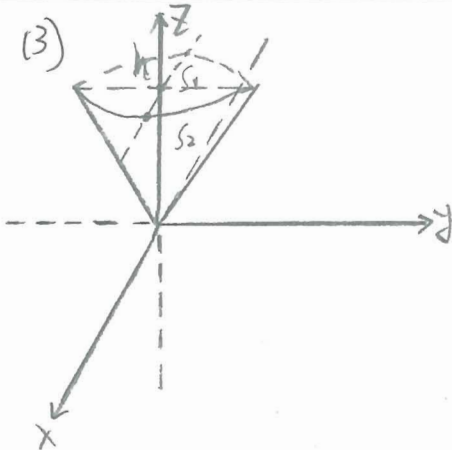
$$\therefore \iint_{S^+} = 0$$

③ 对于 S_3^+ :

$$\iint_{S_3^+} = \iint_{S_4^+} = 0 + 0 + \iint_{R_{xz}} z dx dz = 0$$

$$= - \iint_{R_{xy}} 0 dx dy = 0$$

$$\therefore \text{综上: } \iint_{S^+} (x^2 dy dz + y^2 dz dx + z^2 dx dy) = 0$$



$$\therefore \iint_{S^+} = \iint_{S_1^+} + \iint_{S_2^+}$$

① 对 S_1^+ 使用:

$$\iint_{S_1^+} = \iint_{S_1^+} 0 + 0 + (x-y) dx dy = \iint_{D_{xy}} (x-y) dx dy \quad \text{其中 } D_{xy} = \{(x,y) | x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$$

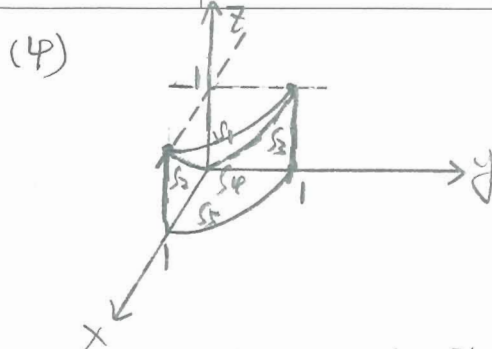
$$\text{由对称性知} \iint_{D_{xy}} (x-y) dx dy = 0 \quad \text{令 } x = r \cos \varphi, y = r \sin \varphi$$

$$\therefore \iint_{S_1^+} (x-y) dx dy = 0 \quad \text{其中 } D_{xy} = \{(r, \varphi) | 0 \leq r \leq 1, 0 \leq \varphi \leq \frac{\pi}{2}\}$$

② 对 S_2^+ 使用:

$$\iint_{S_2^+} = \iint_{S_2^+} (x-y) dx dy = \iint_{D_{xy}} (x-y) dx dy = 0$$

$$\therefore \iint_{S^+} = 0$$



$$\therefore \iint_{S^+} = \iint_{S_1^+} + \iint_{S_2^+} + \iint_{S_3^+} + \iint_{S_4^+} + \iint_{S_5^+}$$

① 对 S_1^+ 使用:

$$\iint_{S_1^+} z^2 dx dy = \iint_{D_{xy}} (1 - x^2 - y^2) dx dy$$

$$\therefore \iint_{S_1^+} z^2 dx dy = \iint_{D_{xy}} (1 - x^2 - y^2) dx dy = \frac{\pi}{4}$$

$$\therefore \iint_{S_1^+} z^2 dx dy = \frac{\pi}{4}$$

$$\therefore \iint_{S_1^+} z^2 dx dy = \frac{\pi}{4}$$

$$\therefore \iint_{S_1^+} z^2 dx dy = \frac{\pi}{4}$$

微积分

章节: 4.5

做题人: 余小豪

$$\therefore \int_1^{\frac{7}{4}} = \frac{7}{24} - \frac{7}{16} - \frac{7}{48} = -\frac{7}{24}$$

$$\therefore \int_4^+ = 0 + \frac{7}{12} + \frac{7}{16} = \frac{71}{48}$$

② 对于 G^+ :

$$\int_1^+ \int_2^+ \int_3^+ z^2 dx dy dz = 0 \quad \int_1^+ \int_2^+ \int_3^+ z^2 x dy dz dx = 0$$

$$\int_1^+ \int_2^+ \int_3^+ x^2 y dz dx dy = 0 \quad \int_1^+ \int_2^+ \int_3^+ x^2 y dz dx dy = 0$$

$$\therefore \int_1^+ = 0$$

③ 对于 G^+ :

$$\int_1^+ \int_2^+ \int_3^+ z^2 dx dy dz = 0$$

$$\int_1^+ \int_2^+ \int_3^+ z^2 x dy dz dx = 0$$

$$\int_1^+ \int_2^+ \int_3^+ x^2 y dz dx dy = 0$$

$$\therefore \int_1^+ = 0$$

③ 对于 G^+ :

$$\int_1^+ \int_2^+ \int_3^+ y^2 z dx dy dz = 0 \quad \int_1^+ \int_2^+ \int_3^+ x^2 y dz dx dy = 0$$

$$\int_1^+ \int_2^+ \int_3^+ z^2 x dy dz dx = 0 \quad \int_1^+ \int_2^+ \int_3^+ z^2 y dx dy dz = 0$$

$$\therefore \int_1^+ = 0$$

$$\begin{aligned} \therefore \int_1^+ \int_2^+ \int_3^+ y^2 z dx dy dz + \int_1^+ \int_2^+ \int_3^+ x^2 y dz dx dy + \int_1^+ \int_2^+ \int_3^+ z^2 x dy dz dx \\ = -\frac{7}{24} + 0 + 0 + \frac{71}{48} + 0 \\ = \frac{57}{48} \end{aligned}$$

④ 对于 G^+ :

$$\int_1^+ \int_2^+ \int_3^+ z^2 dx dy dz = 0$$

$$\int_1^+ \int_2^+ \int_3^+ z^2 x dy dz dx = \int_1^+ \int_2^+ \int_3^+ z^2 \cdot \sqrt{1-x^2} dy dz$$

$$\text{其中 } D_{xz} = \{(x, z) \mid 0 \leq x \leq 1, 0 \leq z \leq 1\}$$

$$\therefore \int_1^+ \int_2^+ \int_3^+ z^2 x dy dz dx = \int_0^1 \int_0^1 z^2 dz \int_0^1 \sqrt{1-x^2} dx = \frac{\pi}{12} = \frac{5\pi}{32}$$

$$\int_1^+ \int_2^+ \int_3^+ z^2 y dy dz dx = \int_1^+ \int_2^+ \int_3^+ x^2 \cdot \sqrt{1-x^2} dx dy dz$$

$$\text{其中 } D_{xz} = \{(x, z) \mid 0 \leq x \leq 1, 0 \leq z \leq 1\}$$

$$\begin{aligned} \therefore \int_1^+ \int_2^+ \int_3^+ z^2 y dy dz dx &= \int_0^1 \int_0^1 \int_0^1 x^2 \sqrt{1-x^2} dx dy dz \\ &= \frac{\pi}{16} \end{aligned}$$

$$(5) \int_1^+ \int_2^+ \int_3^+ z^2 dx dy dz = \int_1^+ \int_2^+ \int_3^+ (x^2 - x^2 - y^2) dx dy dz$$

$$\text{其中 } D_{xz} = \{(x, z) \mid (x - \frac{R}{2})^2 + z^2 \leq (\frac{R}{4})^2\}$$

$$\text{令 } x = \frac{R}{2} + \rho \cos \theta, y = \rho \sin \theta$$

$$\therefore D_{\rho\theta} = \{(\rho, \theta) \mid 0 \leq \rho \leq \frac{R}{2}, 0 \leq \theta \leq 2\pi\}$$

$$\therefore \int_1^+ \int_2^+ \int_3^+ z^2 dx dy dz = \int_0^1 \int_0^{2\pi} \int_0^{\frac{R}{2}} \rho^2 \left(\frac{R}{2} - \rho \cos \theta + \rho^2 \right) \rho d\rho d\theta dz$$

$$4. \int_1^+ \vec{A} \cdot d\vec{s} = \int_1^+ \frac{x}{\sqrt{x^2+y^2+z^2}} \cos \alpha + \frac{y}{\sqrt{x^2+y^2+z^2}} \cos \beta + \frac{z}{\sqrt{x^2+y^2+z^2}} \cos \gamma$$

$$+ \frac{z}{\sqrt{x^2+y^2+z^2}} \cos \gamma$$

$$\int_1^+ \frac{x}{\sqrt{x^2+y^2+z^2}} dx dy dz + \int_1^+ \frac{y}{\sqrt{x^2+y^2+z^2}} dy dz dx + \int_1^+ \frac{z}{\sqrt{x^2+y^2+z^2}} dz dx dy$$

$$\begin{aligned}
 \text{其中 } \iint_S \frac{x}{\sqrt{x^2+y^2+z^2}} dy \wedge dz &= \frac{1}{R} \iint_S x dy \wedge dz = \frac{2\pi}{3} R^3 \\
 &= \frac{1}{R} \left(\iint_{S_1} -\sqrt{R^2-y^2-z^2} \cdot dy \wedge dz + \frac{1}{R} \iint_{S_2} (R^2-y^2-z^2) \cdot dy \wedge dz \right) \therefore \iint_S \frac{z}{\sqrt{x^2+y^2+z^2}} dx \wedge dy \\
 &= -\frac{2\pi}{3} R^3 = -\frac{2\pi}{3} R^3
 \end{aligned}$$

$$\text{其中 } \gamma \wedge z = \{ (x, y, z) \mid x^2+y^2 \leq R^2, z=0 \} \quad \therefore \iint_{\gamma} \vec{A} \cdot d\vec{S} = -\frac{10\pi}{3} R^2$$

$$\text{令 } x = \rho \cos \theta \quad z = \rho \sin \theta$$

$$R \setminus \gamma \cap \rho = \{ (\rho, \theta) \mid 0 \leq \rho \leq R, 0 \leq \theta \leq 2\pi \}$$

$$\begin{aligned}
 \therefore \iint_{\gamma \wedge z} \sqrt{R^2-x^2-z^2} dy \wedge dz &= \int_0^{2\pi} d\theta \int_0^R \sqrt{R^2-\rho^2} \rho d\rho \\
 &= \frac{2\pi}{3} R^3
 \end{aligned}$$

$$\therefore \iint_S \frac{x}{\sqrt{x^2+y^2+z^2}} dy \wedge dz = -\frac{4\pi}{3} R^3$$

$$\text{而 } \iint_S \frac{y}{\sqrt{x^2+y^2+z^2}} dz \wedge dx = \frac{1}{R} \iint_S y dz \wedge dx$$

$$\begin{aligned}
 &= \frac{1}{R} \left(\iint_{S_1} \sqrt{R^2-x^2-z^2} \cdot (-dx \wedge dz) + \frac{1}{R} \iint_{S_2} (-\sqrt{R^2-x^2-z^2}) \cdot dx \wedge dz \right) \\
 &= -\frac{2}{R} \iint_{\gamma \wedge z} \sqrt{R^2-x^2-z^2} dx \wedge dz = -\frac{4\pi}{3} R^3
 \end{aligned}$$

$$\text{而 } \iint_S \frac{z}{\sqrt{x^2+y^2+z^2}} dx \wedge dy = \frac{1}{R} \iint_S z dx \wedge dy$$

$$= \frac{1}{R} \iint_{D_{xy}} \sqrt{R^2-x^2-y^2} \cdot (-dx \wedge dy)$$

$$\text{其中 } D_{xy} = \{ (x, y) \mid x^2+y^2 \leq R^2 \}$$

$$\text{令 } x = \rho \cos \theta \quad y = \rho \sin \theta$$

$$\therefore D_{xy} = \{ (\rho, \theta) \mid 0 \leq \rho \leq R, 0 \leq \theta \leq 2\pi \}$$

$$\therefore \iint_{D_{xy}} \sqrt{R^2-x^2-y^2} dx \wedge dy = \int_0^{2\pi} d\theta \int_0^R \sqrt{R^2-\rho^2} \rho d\rho$$

$$5. \vec{v} = xy\vec{i} + yz\vec{j} + zx\vec{k}$$

$$\begin{aligned} Q &= \iint_{S^+} \vec{v} \cdot d\vec{s} = \iint_{S^+} xy dy \wedge dz + yz dz \wedge dx + zx dx \wedge dy \\ &= \iint_{S^+} xy dy dz + yz dz dx + zx dx dy \\ &= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} [(\sin^2 \theta \cos \varphi)(\sin^2 \theta \cos \varphi \sin \varphi) + (\sin^2 \theta \sin \varphi)(\sin \theta \cos \theta \sin \varphi) \\ &\quad + (\sin \theta \cos \theta)(\sin \theta \cos \theta \cos \varphi)] d\theta d\varphi = \frac{3}{16}\pi \end{aligned}$$

$$6. (1) Q_1 = \iint_{S_1^+} x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$$

$$= \iint_{S_1^+} x dy dz + y dz dx - z dx dy$$

$$= \iint_{S_1^+} \sqrt{b^2+y^2} dx dy - \sqrt{b^2+y^2} dx dy = 0.$$

$$(2) Q_2 = \iint_{S_2^+} dy \wedge dz + y dz \wedge dx + z dx \wedge dy$$

$$= \iint_{S_2^+} 0 + 0 + \sqrt{b^2+y^2} dx dy$$

$$= \iint_{S_2^+} r^2 dr dy = \int_0^{2\pi} \int_0^h r^2 dr dy = \frac{2\pi}{3} h^3$$

$$7. \Omega D_1 = \{(u, v) \mid 0 \leq v \leq \pi, 0 \leq u \leq 1\}, D_2 = \{(u, v) \mid 0 \leq u \leq 1, \pi \leq v \leq 2\pi\}$$

$$\iint_{S^+} (x^2+y^2) dx \wedge dy + y^2 dy \wedge dz + z^2 dz \wedge dx$$

$$= \iint_{D_{uv}} u^3 du dv + [a^2 \sin^2 v a \sin v + (-a^3) v^2 \cos v] du dv$$

$$= \frac{\pi}{2} - 4a^3\pi$$

$$\begin{aligned}
 1. (1) \oint_{L^+} (x+y)^2 dx - (x^2+y^2) dy \\
 = \iint_D (-2x - 2x - 2y) dx dy \\
 = \int_0^1 dx \int_0^{1-x} (-2y - 4x) dy \\
 = \int_0^1 \left[-(1-x)^2 - 4x(1-x) \right] dx \\
 = -1.
 \end{aligned}$$

$$\begin{aligned}
 (2) \oint_{L^+} (x+y) dx + (x-y) dy \\
 = - \iint_D (1-1) dx dy = 0.
 \end{aligned}$$

$$\begin{aligned}
 (3) \oint_{L^+} (x^2+y) dx - (x-y^2) dy \\
 = \iint_D (-1-1) dx dy \\
 = -2S = -2\pi ab.
 \end{aligned}$$

$$\begin{aligned}
 (4) \oint_{L^+} e^x [(1-\cos y) dx - (y-\sin y) dy] \\
 = \iint_D [(y-\sin y)e^x - e^x \sin y] dx dy \\
 = \int_0^\pi dx \int_0^{\sin x} -ye^x dy \\
 = \int_0^\pi -\frac{1}{2} e^x \sin^2 x dx.
 \end{aligned}$$

$$\begin{aligned}
 \int e^x \sin^2 x dx &= e^x \sin^2 x - \int e^x \cdot 2 \sin x \cos x dx \\
 &= e^x \sin^2 x - \int e^x \sin 2x dx.
 \end{aligned}$$

$$\begin{aligned}
 \int e^x \sin 2x dx &= e^x \sin 2x - \int 2e^x \cos 2x dx \\
 &= e^x \sin 2x - 2(e^x \cos 2x + \int 2e^x \sin 2x dx) \\
 &= e^x \sin 2x - 2e^x \cos 2x - 4 \int e^x \sin 2x dx
 \end{aligned}$$

$$\therefore \int e^x \sin 2x dx = \frac{1}{5} e^x \sin 2x - \frac{2}{5} e^x \cos 2x.$$

$$\therefore \int e^x \sin^2 x dx = e^x \sin^2 x - \frac{1}{5} e^x \sin 2x + \frac{2}{5} e^x \cos 2x.$$

$$\therefore \text{原式} = -\frac{1}{2} \int_0^\pi e^x \sin^2 x dx = -\frac{1}{5} (e^\pi - 1)$$

$$2. X(x,y) = \frac{x+y}{x^2+y^2}, Y(x,y) = \frac{y-x}{x^2+y^2}$$

$$\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = \frac{+x^2-y^2-2xy}{(x^2+y^2)^2} - \frac{x^2+y^2-2xy-2y^2}{(x^2+y^2)^2} = 0$$

$V(x,y) = (X(x,y), Y(x,y))$ 在原点不连续.

(2)(3)(4) 问曲线围成的面积包含原点

$\therefore$ 作半径为 ε ($\varepsilon \rightarrow 0$) 的顺时针圆周 L_ε^+

$$\int_{L^+} + \int_{L_\varepsilon^+} = \iint_{D'} \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dx dy = 0.$$

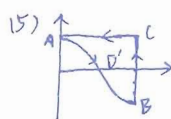
其中 D' 为 L^+ 和 L_ε^+ 围成的区域.

$$\text{对于 } L_\varepsilon^+, \text{ 令 } \begin{cases} x = \varepsilon \cos \theta \\ y = \varepsilon \sin \theta \end{cases}$$

$$\begin{aligned}
 &\int_{L_\varepsilon^+} \frac{(x+y)dx + (y-x)dy}{x^2+y^2} \\
 &= \int_{2\pi}^0 \frac{\varepsilon(\sin \theta + \cos \theta)(-\varepsilon \sin \theta) + \varepsilon(\cos \theta - \sin \theta)(\varepsilon \cos \theta)}{\varepsilon^2} d\theta \\
 &= \int_{2\pi}^0 (-d\theta) = 2\pi
 \end{aligned}$$

$$\therefore \int_{L^+} = -2\pi.$$

$$11) \text{原式} = \iint_D \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) dx dy = 0.$$



由 C 到 B 的曲线积分为

$$\begin{aligned}
 \int_{\pi}^{\frac{\pi}{2}} \frac{y-x}{x^2+y^2} dy &= \frac{1}{2} \ln(x^2+y^2) \Big|_{\pi}^{\frac{\pi}{2}} - \arctan \frac{y}{x} \Big|_{\pi}^{\frac{\pi}{2}} \\
 &= \frac{\pi}{2}
 \end{aligned}$$

由 A 到 C 的曲线积分为

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \frac{x-y}{x^2+y^2} dx &= \arctan \frac{y}{x} \Big|_0^{\frac{\pi}{2}} - \frac{1}{2} \ln(x^2+y^2) \Big|_0^{\frac{\pi}{2}} \\
 &= \frac{\pi}{4} - \frac{1}{2} \ln 2
 \end{aligned}$$

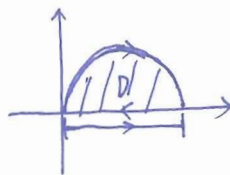
$$\therefore \int_{L^+} = \iint_{D'} 0 dx dy + \frac{\pi}{2} + \frac{\pi}{4} - \frac{1}{2} \ln 2 = \frac{3\pi}{4} - \frac{1}{2} \ln 2.$$

微积分

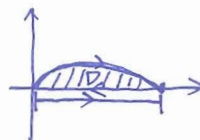
章节:

做题人:

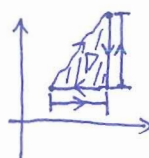
$$\begin{aligned}
 3.1) & \int_{L^+} (1 + xe^{xy}) dx + (x^2 e^{xy} - y^2) dy \\
 &= - \int_0^4 (2xe^{xy} - 2xe^{xy}) dx dy + \int_0^4 (1+x) dx \\
 &= 12
 \end{aligned}$$



$$\begin{aligned}
 12) \text{ 原式} &= - \int_0^{\pi/2} (2x - 2x) dx dy + \int_0^{\pi/2} 3x \sin x dx \\
 &= 3 \left(-x \cos x \Big|_0^{\pi/2} + \int_0^{\pi/2} \cos x dx \right) \\
 &= 3 [-\pi/2 \cos(\pi/2) + \sin(\pi/2)] \\
 &= -3\pi/2 \cos(\pi/2) + 3\sin(\pi/2)
 \end{aligned}$$



$$\begin{aligned}
 13) \text{ 原式} &= - \int_0^1 \left(\frac{1}{y} - \frac{x}{y} \cdot \frac{1}{x} \right) dx dy + \int_1^3 \left(\ln \frac{1}{x} - 1 \right) dx + \int_1^{3e} \frac{3}{y} dy \\
 &= \int_1^3 \ln \frac{1}{x} dx - 2 + 3 [\ln(3e) - \ln 1] \\
 &= x \ln \frac{1}{x} \Big|_1^3 - \int_1^3 x \cdot x \left(-\frac{1}{x^2} \right) dx - 2 + 3 \ln(3e) \\
 &= -3 \ln 3 + 2 - 2 + 3(\ln 3 + 1) \\
 &= 3.
 \end{aligned}$$



$$4.1) \begin{cases} x = a \cos^3 t \\ y = a \sin^3 t \end{cases} \quad (0 \leq t \leq 2\pi).$$

$$\begin{aligned}
 S &= \frac{1}{2} \oint_{\partial D} (a \cos^3 t \cdot 3a \sin^2 t \cos t + a \sin^3 t \cdot 3a \cos^2 t \sin t) dt \\
 &= \frac{1}{2} \int_0^{2\pi} 3a^2 \sin^2 t \cos^2 t dt \\
 &= \frac{3a^2}{8} \int_0^{2\pi} \sin^2 t \cos^2 t dt \\
 &= \frac{3a^2}{8} \int_0^{2\pi} \frac{1 - \cos 4t}{2} dt \\
 &= \frac{3\pi a^2}{8}
 \end{aligned}$$

2) 设 $x = r \cos \theta$, $y = r \sin \theta$. 代入方程得

$$r^4 = a^2 r^2 (\cos^2 \theta - \sin^2 \theta) \Rightarrow r = a \sqrt{\cos 2\theta}$$

$$\therefore \cos 2\theta \geq 0 \Rightarrow \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}]$$

$$\therefore x = a \cos \theta \sqrt{\cos 2\theta}, \quad y = a \sin \theta \sqrt{\cos 2\theta}.$$

$$\begin{aligned}
 \therefore S &= \frac{1}{2} \oint_{\partial D} [a \cos \theta \sqrt{\cos 2\theta} (a \cos \theta \sqrt{\cos 2\theta} - a \sin \theta \frac{\sin 2\theta}{\sqrt{\cos 2\theta}}) - a \sin \theta \sqrt{\cos 2\theta} (-a \sin \theta \sqrt{\cos 2\theta} - a \cos \theta \frac{\sin 2\theta}{\sqrt{\cos 2\theta}})] d\theta \\
 &= \frac{1}{2} \left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2 \cos 2\theta d\theta + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} a^2 \cos 2\theta d\theta \right) \\
 &= a^2.
 \end{aligned}$$

3) 令 $x = \rho \cos \theta$, $y = \rho \sin \theta$ 代入方程得

$$\rho^3 (\sin^3 \theta + \cos^3 \theta) = 3a \rho^2 \sin \theta \cos \theta$$

$$\therefore \rho = \frac{3a \sin \theta \cos \theta}{\sin^3 \theta + \cos^3 \theta} \quad \theta \in [0, \frac{\pi}{2}]$$

$$\begin{aligned} S &= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{3a \sin \theta \cos \theta}{\sin^3 \theta + \cos^3 \theta}} \rho \, d\rho \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} \frac{9a^2 \sin^2 \theta \cos^2 \theta}{(\sin^3 \theta + \cos^3 \theta)^2} d\theta \\ &= \frac{9a^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin^2 \theta \cos^2 \theta}{\sin^4 \theta + 2\sin^2 \theta \cos^2 \theta + \cos^4 \theta} d\theta \\ &= \frac{9a^2}{2} \int_0^{\frac{\pi}{2}} \frac{\tan^2 \theta}{\cos^2 \theta (\tan^3 \theta + 1)^2} d\theta \\ &= \frac{3a^2}{2} \int_0^{\frac{\pi}{2}} \frac{d(\tan^3 \theta + 1)}{(\tan^3 \theta + 1)^2} \\ &= \frac{3a^2}{2} \int_1^{+\infty} \frac{dt}{t^2} \\ &= \frac{3a^2}{2} \end{aligned}$$

$$\begin{aligned} 5. (1) & \oint_{L^+} f(x,y) (y \, dx + x \, dy) \\ &= \iint_D \left(\frac{\partial (x f(x,y))}{\partial x} - \frac{\partial (y f(x,y))}{\partial y} \right) dx \, dy \\ &= \iint_D [f(x,y) + x f'(x,y) \cdot y - f(x,y) - y f'(x,y) \cdot x] dx \, dy \\ &= 0 \end{aligned}$$

$$\begin{aligned} (2) & \oint_{L^+} f(x^2+y^2) (x \, dx + y \, dy) \\ &= \iint_D \left(\frac{\partial (y f(x^2+y^2))}{\partial x} - \frac{\partial (x f(x^2+y^2))}{\partial y} \right) dx \, dy \\ &= \iint_D [y \cdot f'(x^2+y^2) \cdot 2x - x \cdot f'(x^2+y^2) \cdot 2y] dx \, dy \\ &= 0 \end{aligned}$$

$$6. (1) \oint_{\partial D} x^2 \, dy = \iint_D 2x \, dx \, dy = 2 \frac{\iint_D x \, dx \, dy}{\iint_D dx \, dy} \cdot \iint_D dx \, dy = 2\sigma(D) \bar{x}$$

$$(2) \oint_{\partial D} xy \, dx = \iint_D y \, dx \, dy = \frac{\iint_D y \, dx \, dy}{\iint_D dx \, dy} \cdot \iint_D dx \, dy = \sigma(D) \bar{y}$$

$$7. d\vec{l} = (dx, dy)$$

$$\therefore \vec{n} = (dy, -dx)$$

$$\begin{aligned} \oint_{\partial D} \frac{\partial f}{\partial n} dl &= \oint_{\partial D} \left(\frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx \right) \cdot \frac{dl}{\sqrt{(dx)^2 + (dy)^2}} \\ &= \oint_{\partial D} \left(\frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx \right) \\ &= \iint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx dy \end{aligned}$$

$$\begin{aligned} 8. (1) \oint_{\partial D} \frac{\partial u}{\partial n} dl &= \iint_D \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx dy \\ &= \iint_D \Delta u dx dy \end{aligned}$$

$$\begin{aligned} (2) \oint_{\partial D} v \frac{\partial u}{\partial n} dl &= \oint_{\partial D} \left(v \frac{\partial u}{\partial x} dy - v \frac{\partial u}{\partial y} dx \right) \\ &= \iint_D \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + v \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} + v \frac{\partial^2 u}{\partial y^2} \right) dx dy \\ &= \iint_D \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} \right) dx dy + \iint_D v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx dy \\ &= \iint_D \nabla v \nabla u dx dy + \iint_D v \Delta u dx dy \end{aligned}$$

$$\begin{aligned} (3) \oint_{\partial D} \left| \frac{\partial u}{\partial n} \frac{\partial v}{\partial n} \right| dl &= \oint_{\partial D} v \frac{\partial u}{\partial n} dl - \oint_{\partial D} u \frac{\partial v}{\partial n} dl \\ &= \iint_D \nabla v \nabla u dx dy + \iint_D v \Delta u dx dy \\ &\quad - \iint_D \nabla u \nabla v dx dy - \iint_D u \Delta v dx dy \\ &= \iint_D (v \Delta u - u \Delta v) dx dy \\ &= \iint_D \left| \begin{matrix} \Delta u & \Delta v \\ u & v \end{matrix} \right| dx dy \end{aligned}$$

$$9. d\vec{l} = (dx, dy), \quad \vec{n} = \left(\frac{dy}{dl}, -\frac{dx}{dl} \right)$$

$$\therefore \cos \langle \vec{n}, \hat{i} \rangle = \frac{dy}{dl}, \quad \cos \langle \vec{n}, \hat{j} \rangle = -\frac{dx}{dl}$$

$$\begin{aligned} \therefore \oint_{L^+} (x \cos \langle \vec{n}, \hat{i} \rangle + y \cos \langle \vec{n}, \hat{j} \rangle) dl \\ &= \oint_{L^+} x dy - y dx \\ &= \iint_S (1+1) dx dy = 2S. \end{aligned}$$

S 为 L 所围区域面积.

$$10. (1) X = x^2 - y, \quad Y = -(x + \sin^2 y)$$

$$\frac{\partial X}{\partial y} = -1, \quad \frac{\partial Y}{\partial x} = -1 = \frac{\partial X}{\partial y}$$

$$\therefore \exists u(x, y) \text{ 使 } du = X dx + Y dy.$$

$$\frac{\partial u}{\partial x} = x^2 - y \Rightarrow \sqrt{x} u = \frac{1}{3} x^3 - xy + C(y).$$

$$\frac{\partial u}{\partial y} = -x + C'(y) = -x - \sin^2 y$$

$$\therefore C'(y) = -\sin^2 y.$$

$$C(y) = \int -\sin^2 y dy$$

$$= \int \frac{\cos 2y}{2} dy - \int \frac{1}{2} dy$$

$$= \frac{1}{4} \sin 2y - \frac{1}{2} y + C_0.$$

$$\therefore u = \frac{1}{3} x^3 - xy + \frac{1}{4} \sin 2y - \frac{1}{2} y + C_0$$

$\therefore$ 原方程解为

$$\frac{1}{3} x^3 - xy + \frac{1}{4} \sin 2y - \frac{1}{2} y = \text{const.}$$

$$(2) X = e^y, \quad Y = xe^y - 2y.$$

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x} = e^y$$

$$\therefore \exists u(x, y) \text{ 使 } du = X dx + Y dy$$

$$\frac{\partial u}{\partial x} = e^y \Rightarrow \sqrt{x} u = xe^y + C(y)$$

$$\frac{\partial u}{\partial y} = xe^y + C'(y) = xe^y - 2y$$

$$\therefore C'(y) = -y^2 + C_0$$

$$\therefore u = xe^y - y^2 + C_0$$

$$\therefore \text{解为 } xe^y - y^2 = \text{const.}$$

$$(3) \frac{x dx + y dy}{\sqrt{x^2 + y^2}} = d(\sqrt{x^2 + y^2})$$

$$\text{对于 } \frac{y dx - x dy}{x^2}, \quad X = \frac{y}{x^2}, \quad Y = -\frac{1}{x}$$

$$\frac{\partial X}{\partial y} = \frac{1}{x^2} = \frac{\partial Y}{\partial x}$$

$$\therefore \exists u(x, y) \text{ 使 } du = X dx + Y dy.$$

$$\frac{\partial u}{\partial x} = \frac{y}{x^2} \Rightarrow u = -\frac{y}{x} + C(y).$$

$$\frac{\partial u}{\partial y} = -\frac{1}{x} + C'(y) = -\frac{1}{x} \Rightarrow C'(y) = C_0$$

$$\therefore u = -\frac{y}{x} + C_0$$

$$\therefore \text{解为 } \sqrt{x^2 + y^2} = -\frac{y}{x} + C.$$

微积分	章节:	做题人:
4) $X = \cos x + \frac{1}{y}$ $Y = \frac{1}{y} - \frac{x}{y^2}$ $\frac{\partial X}{\partial y} = -\frac{1}{y^2} = \frac{\partial Y}{\partial x}$ $\therefore \exists u(x, y)$ 使 $du = Xdx + Ydy$ $\frac{\partial u}{\partial x} = \cos x + \frac{1}{y} \Rightarrow u = \sin x + \frac{x}{y} + C(y)$ $\frac{\partial u}{\partial y} = -\frac{x}{y^2} + C'(y) = \frac{1}{y} - \frac{x}{y^2}$ $\therefore C(y) = \ln y + C$ $\therefore$ 解为 $\sin x + \frac{x}{y} + \ln y = \text{const.}$ 11.11) $(y \cos x - x \sin x)dx + (y \sin x + x \cos x)dy = 0$ $e^y(y \cos x - x \sin x)dx + e^y(y \sin x + x \cos x)dy = 0$ $X = e^y(y \cos x - x \sin x)$, $\frac{\partial X}{\partial y} = e^y(\cos x + y \cos x - x \sin x)$ $Y = e^y(y \sin x + x \cos x)$, $\frac{\partial Y}{\partial x} = e^y(y \cos x + \cos x - x \sin x) = \frac{\partial X}{\partial y}$ $\therefore \exists u(x, y)$ 使 $du = Xdx + Ydy$ $\frac{\partial u}{\partial x} = e^y(y \cos x - x \sin x)$ $\therefore u = e^y \cdot y \sin x - e^y \int x \sin x dx + C(y)$ $= e^y \cdot y \sin x - e^y(-x \cos x + \int \cos x dx) + C(y)$ $= e^y \cdot y \sin x + x e^y \cos x - e^y \sin x + C(y)$ $\frac{\partial u}{\partial y} = e^y(y \sin x + \sin x + x \cos x - \sin x) + C'(y)$ $= e^y(y \sin x + x \cos x)$ $\therefore C(y) = C$ $\therefore u = e^y(y \sin x + x \cos x - \sin x) + C$ $\therefore$ 通解为 $e^y(y \sin x + x \cos x - \sin x) = \text{const.}$ 12) 课本 P252 例 4, 6, 7. 13) $(3x^3 + y)dx + (2x^2y - x)dy = 0$ $(3x + \frac{y}{x^2})dx + (2y - \frac{1}{x})dy = 0$ $X = 3x + \frac{y}{x^2}$, $\frac{\partial X}{\partial y} = \frac{1}{x^2}$ $Y = 2y - \frac{1}{x}$, $\frac{\partial Y}{\partial x} = \frac{1}{x^2} = \frac{\partial X}{\partial y}$ $\therefore \exists u(x, y)$ 使 $du = Xdx + Ydy$ $\frac{\partial u}{\partial x} = 3x + \frac{y}{x^2}$ $\therefore u = \frac{3}{2}x^2 - \frac{y}{x} + C(y)$ $\frac{\partial u}{\partial y} = -\frac{1}{x} + C'(y) = 2y - \frac{1}{x}$ $\therefore C(y) = y^2 + C$ $\therefore u = \frac{3}{2}x^2 - \frac{y}{x} + y^2 + C$ $\therefore$ 解为 $\frac{3}{2}x^2 - \frac{y}{x} + y^2 = \text{const.}$ 4) $(x+y)(dx - dy) = dx + dy$ $(x+y-1)dx - (x+y+1)dy = 0$ $\therefore (1 - \frac{1}{x+y})dx - (1 + \frac{1}{x+y})dy = 0$ $dx - dy - \frac{dx + dy}{x+y} = 0$ $d(x-y) - d(\ln x+y) = 0$ $\therefore$ 解为 $x-y - \ln x+y = \text{const.}$ 5) $(x^2 - \sin^2 y)dx + x \sin 2y dy = 0$ $\therefore (1 - \frac{\sin^2 y}{x^2})dx + \frac{\sin 2y}{x} dy = 0$ $X = 1 - \frac{\sin^2 y}{x^2}$, $Y = \frac{\sin 2y}{x}$ $\frac{\partial X}{\partial y} = -\frac{1}{x^2} 2 \sin y \cos y = -\frac{\sin 2y}{x^2}$ $\frac{\partial Y}{\partial x} = -\frac{\sin 2y}{x^2} = \frac{\partial X}{\partial y}$ $\therefore \exists u(x, y)$ 使 $du = Xdx + Ydy$ $\therefore \frac{\partial u}{\partial x} = 1 - \frac{\sin^2 y}{x^2}$ $\therefore u = x + \frac{\sin^2 y}{x} + C(y)$ $\frac{\partial u}{\partial y} = \frac{\sin 2y}{x} + C'(y) = \frac{\sin 2y}{x}$ $\therefore C(y) = C$ $\therefore u = x + \frac{\sin^2 y}{x} + C$ $\therefore$ 解为 $x + \frac{\sin^2 y}{x} = \text{const.}$		

微积分

章节: 4.7

做题人: 何乐

1. 证明: 设 V 是 Ω 中连续可微的向量场(1), $\text{rot } V = 0 \Rightarrow$ (2) 对任意闭曲线 L , $\oint_L V \cdot dl = 0$ 的证明:由 Stokes 公式, $\oint_L V \cdot dl = \iint_S \text{rot } V \cdot dS = 0$, S 为以 L 为边界的曲面,故 (1) $\Rightarrow$ (2) 成立(2) $\Rightarrow$ (3) $\int_{L(A)}^{(B)} V \cdot dl$ 与路径无关, 其中 $L \subset \Omega$ 的证明:由 Ω 是单连通域, 存在一条由 B 到 A 的曲线 L_0 , 记 $\int_{L_0} V \cdot dl = t$ 则对任意由 A 到 B 的曲线 L , 由 (2), L 与 L_0 构成闭曲线故 $\int_L V \cdot dl + \int_{L_0} V \cdot dl = 0$, 即 $\int_{L(A)}^{(B)} V \cdot dl = -t$ 为定值, 与路径无关.(3) $\Rightarrow$ (4) 存在三元函数 $u(x, y, z)$ 使 $du = Xdx + Ydy + Zdz$ 的证明:考虑使 A 为定点, $B(x, y, z)$, $\int_{L(A)}^{(B)} V \cdot dl$ 与路径无关, P 与 B 坐标 (x, y, z) 有关从而 $u = \int_{L(A)}^{(B)} V \cdot dl$ 是关于 x, y, z 的函数, 其中 $V \cdot dl = Xdx + Ydy + Zdz$ 从而 $\frac{\partial u}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{u(x+\Delta x, y, z) - u(x, y, z)}{\Delta x} = \int_{L(A)}^{(x+\Delta x, y, z)} \frac{Xdx + Ydy + Zdz}{\Delta x}$, 其中 $dy=dz=0$ 由积分第一中值定理, $= \int_{L(A)}^{(x+\Delta x, y, z)} \frac{Xdx}{\Delta x} = X(x+\theta\Delta x, y, z) \quad (0 < \theta < 1) \quad (\Delta x \rightarrow 0)$ 由 X 的连续性有 $\frac{\partial u}{\partial x} = X$, 同理 $\frac{\partial u}{\partial y} = Y, \frac{\partial u}{\partial z} = Z$ 由 X, Y, Z 的连续性, 结合定理 1.4.5, u 可微 $du = Xdx + Ydy + Zdz$ (4) $\Rightarrow$ (1) 的证明: $\frac{du}{dx} = X, \frac{du}{dy} = Y, \frac{du}{dz} = Z, \frac{dX}{dy} = \frac{d^2u}{dydx}, \frac{dY}{dx} = \frac{d^2u}{dx dy}$, 由偏导数连续 $\frac{d^2u}{dx dy} = \frac{d^2u}{dy dx}, \therefore X_y = Y_x$, 类似, $X_z = Z_x, Y_z = Z_y, \text{rot } V = 0$

综上, 定理 4.7.3 得证

2. $X = \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, Y = \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, Z = \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ $\frac{\partial X}{\partial x} = \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}, \frac{\partial Y}{\partial y} = \frac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}, \frac{\partial Z}{\partial z} = \frac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$, 在 $(0,0,0)$ 以外的点连续

从而对于 (1) 和 (2),

$$\oiint_{S^+} V \cdot dS = \iiint_{\Omega} \text{div } V \, dx dy dz = 0$$

对于 (3), 由对称性

$$\oiint_{S^+} X dy dz + Y dx dz + Z dx dy = \oiint_{S^+} 3x dy dz$$

对于 $x > 0, dy dz = dy dz, x = \sqrt{\varepsilon^2 - y^2 - z^2}$; 对于 $x < 0, dy dz = -dy dz, x = -\sqrt{\varepsilon^2 - y^2 - z^2}$

$$\text{原积分} = \oiint_{S^+} \frac{6\sqrt{\varepsilon^2 - y^2 - z^2}}{\varepsilon^3} dy dz = \frac{1}{\varepsilon^3} \int_0^{2\pi} d\theta \int_0^\varepsilon 6\sqrt{\varepsilon^2 - \rho^2} \cdot \rho \, d\rho = 4\pi$$

对于 (4), 由于 Ω 包括非连续可微点 $(0,0,0)$, 考虑 $\Omega_\varepsilon: x^2+y^2+z^2 \leq \varepsilon^2 \quad (\varepsilon > 0)$ 及其边界 S_ε

$$\text{由 Gauss 公式: } \left(\oiint_{S^+} + \oiint_{S_\varepsilon^-} \right) X dy dz + Y dx dz + Z dx dy = \iiint_{\Omega - \Omega_\varepsilon} (X+Y+Z) dx dy dz = 0$$

 $\therefore \oiint_{S^+} V \cdot dS = \oiint_{S_\varepsilon^-} V \cdot dS$, 归入 (3) 的情况, 为 4π

(2)

微积分

章节: 4.7

做题人: 何乐

3. (1) 由Gauss公式

$$\oint_{\Sigma^+} x^2 dy dz + y^2 dz dx + z^2 dx dy = \iiint_{\Sigma} 3(x^2 + y^2 + z^2) dx dy dz, \Sigma: \{(x, y, z) | x^2 + y^2 + z^2 \leq a^2\}$$

在球坐标下, $= \int_0^{2\pi} d\varphi \int_0^\pi \sin\theta d\theta \int_0^a 3r^4 dr = \frac{12}{5}\pi a^5$

(2) 由Gauss公式

$$\oint_{\Sigma^+} (x-y) dx dy + (x-y-z) dy dz = \iiint_{\Sigma} (y-z) dx dy dz, \Sigma: \{(x, y, z) | 0 \leq z \leq 1, x^2 + y^2 \leq 1\}$$

在柱坐标下, $= \int_0^1 dz \int_0^{2\pi} d\theta \int_0^1 \rho(\rho \sin\theta - z) d\rho = -\frac{1}{2}\pi$

(3) 由Gauss公式

$$\oint_{\Sigma^+} (x+2y+3z) dx dy + (y+2z) dy dz + (z^2-1) dz dx = \iiint_{\Sigma} 3 dx dy dz = 3V_{\Sigma} = \frac{1}{2}$$

(4) 由Gauss公式

$$\oint_{\Sigma^+} a^2 x^2 dy dz + b^2 x^2 y dz dx + a^2 y^2 z dx dy = \iiint_{\Sigma} (a^2 b^2 z^2 + a^2 c^2 y^2 + b^2 c^2 x^2) dx dy dz$$

$\Sigma: \{(x, y, z) | \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}$ 在变换: $\begin{cases} x = r \cos\theta \cos\varphi \\ y = r \cos\theta \sin\varphi \\ z = r \sin\theta \end{cases}$ 下为 $\Sigma: \{(r, \theta, \varphi) | 0 \leq r \leq 1, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi\}$

$$\text{且 } dx dy dz = abc r^2 \sin\theta dr d\theta d\varphi$$

$$\text{故原积分} = \iiint_{\Sigma} r^2 \cdot a^2 b^2 c^2 \cdot abc r^2 \sin\theta dr d\theta d\varphi = \frac{4}{5}\pi a^3 b^3 c^3$$

(5) 由2题中的(4)得, 第二类曲面积分值为 4π 4. (1) 同题3的(1)(2), 为 $\frac{12}{5}\pi R^5$

(2) 由Gauss公式

$$\oint_{\Sigma^+} A \cdot dS = \iiint_{\Sigma} 3 dx dy dz = 3V = 4\pi abc$$

5. (1) 由Stokes公式

$$\oint_{L^+} y dx + z dy + x dz = \iint_D (1, -1, -1) \cdot n^0 ds, D \text{ 为 } x+y+z=0 \text{ 与球的交面}$$

n^0 为其正向法向量 $(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$, 故原积分 $= -\sqrt{3} \iint_D ds = -\sqrt{3}S = -\sqrt{3}\pi R^2$

(2) 由Stokes公式

$$\oint_{L^+} (y-x) dx + (z-y) dy + (x-z) dz = \iint_{\Sigma^+} -dx dy - dy dz - dz dx$$

(这里题意不清应为印刷错误, 此处按 L 为 $\begin{cases} x^2+y^2=1 \\ x+z=1 \end{cases}$ 的交线, 从上方看

逆时针为正, S 为 $x+z=1$ 在 $x^2+y^2 \leq 1$ 的部分, 由右手定则, 向上为正)

则单位正法向量为 $(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2})$, $dx dy = dx dy$, $dy dz = dy dz$, $dz dx = dx dz$

投影: $D_{xy}: x^2+y^2 \leq 1$, $D_{xz}: x+z=1$ (面积为0, 无意义), $D_{yz}: y^2+(1-z)^2 \leq 1$

$$\text{故原积分} = - \iint_{D_{xy}} dx dy - \iint_{D_{xz}} dx dz - \iint_{D_{yz}} dy dz = -2\pi$$

(3) 由Stokes公式

$$\oint_{L^+} y^2 dx + z^2 dy + x^2 dz = \iint_{\Sigma^+} 2z dy dz - 2x dz dx - 2y dx dy, S \text{ 为 } \triangle ABC, \text{ 向上为正}$$

$$\iint_{\Sigma^+} 2z dy dz - 2x dz dx - 2y dx dy = \iint_{D_{yz}} 2z dy dz - \iint_{D_{xz}} 2x dz dx - \iint_{D_{xy}} 2y dx dy = 0$$

(3)

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投影区域: $\{(y, z) | 0 \leq y \leq b \text{ 且 } 0 \leq z \leq c - \frac{c}{b}y\}$

$$\text{则 } \iint_{S^+} -2z dy dz = \iint_{Dyz} -2z dy dz = \int_0^b dy \int_0^{c-\frac{c}{b}y} -2z dz = -\frac{bc^2}{3}$$

由轮换对称性, 原第二类曲线积分值为 $-\frac{ab^2bc^2+ca^2}{3}$

$$6. (1) X = \frac{yz}{x^2}, Y = -\frac{z}{x}, Z = -\frac{y}{x}$$

$$\text{则 } X_y = Y_x = \frac{z}{x^2}, X_z = Z_x = \frac{y}{x^2}, Y_z = Z_y = -\frac{1}{x}, \text{ 故 } \text{rot } V = 0$$

从而存在 u , $du = Xdx + Ydy + Zdz$ 成立

$$\frac{\partial u}{\partial x} = \frac{yz}{x^2}, u = -\frac{yz}{x} + C(y, z), \text{ 验证 } Y, Z \text{ 得 } C(y, z) = C$$

$$\text{故 } u = -\frac{yz}{x} + C$$

$$(2) X = \frac{y}{(x+z)^2+y^2}, Y = -\frac{x+z}{(x+z)^2+y^2}, Z = \frac{y}{(x+z)^2+y^2}$$

$$X_z = Z_x = -\frac{2(x+z)y}{(x+z)^2+y^2}, X_y = Y_x = \frac{(x+z)-y^2}{(x+z)^2+y^2}, \text{ 故 } \text{rot } V = 0$$

从而存在 u , $du = Xdx + Ydy + Zdz$

$$\text{且 } X = \frac{y}{(x+z)^2+y^2} = \frac{1}{y} \cdot \frac{1}{(\frac{x+z}{y})^2+1}, X \text{ 的原函数为 } \arctan(\frac{x+z}{y})$$

且验证 Y, Z 亦成立

$$\text{故 } u = \arctan(\frac{x+z}{y}) + C$$

$$7. (1) X = yxz, Y = xtz, Z = x+y, X_y = Y_x = X_z = Z_x = Y_z = Z_y = 1, \text{rot } V = 0$$

存在 u , $du = Xdx + Ydy + Zdz$, $u = xyz + xz + yz + C$

$$\therefore \int_{(0,0,0)}^{(1,2,1)} Xdx + Ydy + Zdz = u(1,2,1) - u(0,0,0) = 5$$

$$(2) X = y^2z, Y = 2xy^2z, Z = xy^2, X_y = Y_x = 2yz, X_z = Z_x = y^2, Y_z = Z_y = 2xy, \text{rot } V = 0$$

故存在 u , $du = Xdx + Ydy + Zdz$, 得 $u = xy^2z + C$

$$\therefore \int_{(1,1,1)}^{(1,1,-1)} Xdx + Ydy + Zdz = -2$$

8. 由于 z 为定值, $dz = 0$, 可以看成是平面向量场中的第二类曲面积分

$$(1) X = -\frac{y}{x^2+y^2}, Y = \frac{x}{x^2+y^2}, L: x^2+y^2 = \varepsilon^2, \text{ 逆时针为正, 在极坐标系下}$$

$$\oint_L -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy = \int_0^{2\pi} (\sin^2\theta + \cos^2\theta) d\theta = 2\pi$$

(2) X, Y 在 $(0,0)$ 以外的点均连续, 且 $(0,0)$ 在 $(x-1)^2+y^2=r^2$ ($r>2$) 内
增开一个曲线 $L_0: x^2+y^2=\varepsilon^2$, ($\varepsilon \rightarrow 0^+$), 其围成区域为 D_0 .

则由格林公式

$$\oint_{L+L_0} Xdx + Ydy = \iint_{D-D_0} (X_y - Y_x) dx dy = 0, \text{ 其中 } X_y = Y_x = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\text{故 } \oint_L Xdx + Ydy = \oint_{L_0} Xdx + Ydy, \text{ 由 (1), 结果为 } 2\pi$$

微积分

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9. 为方便起见, 默认向量值函数为 (x, y, z) , 多个向量值函数以下标区分

$$\begin{aligned} (1) uV &= uXi + uYj + uZk, \operatorname{div}(uV) = \frac{\partial(uX)}{\partial x} + \frac{\partial(uY)}{\partial y} + \frac{\partial(uZ)}{\partial z} \\ &= u\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) + \frac{\partial u}{\partial x} \cdot X + \frac{\partial u}{\partial y} \cdot Y + \frac{\partial u}{\partial z} \cdot Z \\ &= u \cdot \operatorname{div} V + \operatorname{grad} u \cdot V, \text{ 其中 } \operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) \end{aligned}$$

$$(2) \operatorname{rot}(uA) = u \operatorname{rot} A + \operatorname{grad} u \times A \Leftrightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ uX & uY & uZ \end{vmatrix} = u \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ X & Y & Z \end{vmatrix}$$

$$\begin{aligned} \text{考查 } k \text{ 分量, 左边系数} &= \frac{\partial(uY)}{\partial x} - \frac{\partial(uX)}{\partial y} \\ &= u\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) + \left(Y \cdot \frac{\partial u}{\partial x} - X \cdot \frac{\partial u}{\partial y}\right) = \text{右边系数} \end{aligned}$$

 i, j 类似, 故得证

$$(3) A \times B = (YA ZB - ZA YB)i + (ZA XB - XA ZB)j + (XA YB - YA XB)k$$

则 $\operatorname{div}(A \times B) = B \cdot \operatorname{rot}(A) - A \cdot \operatorname{rot}(B)$ 等价于

$$\begin{aligned} &\frac{\partial(YA ZB - ZA YB)}{\partial x} + \frac{\partial(ZA XB - XA ZB)}{\partial y} + \frac{\partial(XA YB - YA XB)}{\partial z} \\ &= (XB i + YB j + ZB k) \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ XA & YA & ZA \end{vmatrix} - (XA i + YA j + ZA k) \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ XB & YB & ZB \end{vmatrix} \end{aligned}$$

$$\text{其中 } XA \text{ 的系数, 左边为 } \frac{\partial YB}{\partial z} - \frac{\partial ZB}{\partial y} = - \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ YB & ZB \end{vmatrix} = \text{右边}$$

同理, XB, YA, YB, ZA, ZB 系数均相等

故得证

$$(4) \operatorname{rot}(\operatorname{grad} u) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix}, \text{ 其中 } i \text{ 分量为 } \frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 u}{\partial z \partial y}$$

但是, 题目中没有给出 u 二阶导连续的条件, 需要更精确的条件

故本题有误, 如例 1.4.16 就是反例

$$\begin{aligned} (5) \operatorname{div}(\operatorname{rot} A) &= \operatorname{div}\left(\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right)i + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right)j + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)k\right) \\ &= \frac{\partial^2 Z}{\partial x \partial y} - \frac{\partial^2 Z}{\partial y \partial x} + \frac{\partial^2 Y}{\partial z \partial x} - \frac{\partial^2 Y}{\partial x \partial z} + \frac{\partial^2 X}{\partial y \partial z} - \frac{\partial^2 X}{\partial z \partial y} \end{aligned}$$

同样, 由于没有二阶光滑的条件, $\operatorname{div}(\operatorname{rot} A) = 0$ 亦不能得证

$$10. (1) V = x^2 i + y^2 j + z^2 k, \operatorname{div} V = 2x + 2y + 2z, \operatorname{rot} V = 0, P(-1, 1, 2) \text{ 处}$$

散度为 4, 旋度为 0

$$(2) V = x^2 y z i + x y^2 z j + x y z^2 k, \operatorname{div} V = 6xyz, \operatorname{rot} V = (xz^2 - xy^2)i + (xy^2 - yz^2)j + (yz^2 - xz^2)k$$

 $P(-1, 3, 2)$, 散度为 -36, 旋度为 $(5, -9, 16)$

微积分

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$$(3) V = (3x^2z + 2y)i + (y^2 - 3xz)j + 3xyzk$$

$$\operatorname{div} V = 6xz + 2y + 3xy, \operatorname{rot} V = (3xz + 3x)i + (3x^2 - 3yz)j + (3yz + 2)k$$

PC (1, 2, 3), 散度 28, 旋度 (12, -15, -11)

$$11. \vec{r} = (x, y, z), |\vec{r}| = \sqrt{x^2 + y^2 + z^2}, \vec{a} = (a_1, a_2, a_3) \text{ 为常向量}$$

$$\frac{\partial f(|\vec{r}|)}{\partial x} = \frac{\partial f(|\vec{r}|)}{\partial |\vec{r}|} \cdot \frac{\partial |\vec{r}|}{\partial x} = f'(|\vec{r}|) \cdot \frac{x}{\sqrt{x^2 + y^2 + z^2}} = f'(|\vec{r}|) \cdot \frac{x}{|\vec{r}|}$$

$$\text{类似 } \frac{\partial f(|\vec{r}|)}{\partial y} = f'(|\vec{r}|) \cdot \frac{y}{|\vec{r}|}, \frac{\partial f(|\vec{r}|)}{\partial z} = f'(|\vec{r}|) \cdot \frac{z}{|\vec{r}|}$$

$$(1) \operatorname{div}(\vec{r} \cdot \vec{a}) = \operatorname{div}(a_1 |\vec{r}| i + a_2 |\vec{r}| j + a_3 |\vec{r}| k)$$

$$= a_1 \frac{d|\vec{r}|}{dx} + a_2 \frac{d|\vec{r}|}{dy} + a_3 \frac{d|\vec{r}|}{dz} = \frac{\vec{a} \cdot \vec{r}}{|\vec{r}|}$$

$$(2) \operatorname{div}(|\vec{r}|^n \vec{a}) = \operatorname{div}(a_1 |\vec{r}|^n i + a_2 |\vec{r}|^n j + a_3 |\vec{r}|^n k)$$

$$= n \cdot |\vec{r}|^{n-1} (a_1 \frac{d|\vec{r}|}{dx} + a_2 \frac{d|\vec{r}|}{dy} + a_3 \frac{d|\vec{r}|}{dz}) = n \cdot |\vec{r}|^{n-2} \vec{a} \cdot \vec{r}$$

② 恕难以孤陋寡闻, 若将 1, 2 中书中的 ' $\vec{r}$ ' 理解为向量实在不理解, ...
感觉是书上印错

$$(3) \text{由 (2), } n=0$$

$$(4) \operatorname{div}(f(|\vec{r}|) \vec{r}) = \operatorname{div}(f(|\vec{r}|) x i + f(|\vec{r}|) y j + f(|\vec{r}|) z k)$$

$$= 3f(|\vec{r}|) + x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z} = 3f(|\vec{r}|) + f'(|\vec{r}|) \cdot |\vec{r}| = 0$$

$$\text{故 } f(|\vec{r}|) = -\frac{|\vec{r}|}{3} f'(|\vec{r}|), \text{ 得 } f(|\vec{r}|) = \frac{C}{|\vec{r}|^3}$$

$$(5) \operatorname{div}(\operatorname{grad} f(|\vec{r}|)) = \operatorname{div}\left(f'(|\vec{r}|) \left(\frac{x}{|\vec{r}|^2} i + \frac{y}{|\vec{r}|^2} j + \frac{z}{|\vec{r}|^2} k\right)\right)$$

$$\text{即 } \frac{\partial}{\partial x} \left(f'(|\vec{r}|) \cdot \frac{x}{|\vec{r}|^2} \right) = f'(|\vec{r}|) \cdot \frac{x^2}{|\vec{r}|^4} + f'(|\vec{r}|) \cdot \frac{y^2 + z^2}{(|\vec{r}|^2)^2} \cdot \frac{2}{|\vec{r}|^2}$$

$$\therefore \operatorname{div}(\operatorname{grad} f(|\vec{r}|)) = f'(|\vec{r}|) + f'(|\vec{r}|) \cdot \frac{2}{|\vec{r}|} = 0, f'(|\vec{r}|) = \frac{C}{|\vec{r}|^2}$$

$$\therefore f(|\vec{r}|) = \frac{C_1}{|\vec{r}|} + C_2$$

$$(6) \text{显然 } \operatorname{rot} \vec{r} = 0$$

$$(7) \operatorname{rot}(f(|\vec{r}|) \vec{r}) = \operatorname{rot}(f(|\vec{r}|) x i + f(|\vec{r}|) y j + f(|\vec{r}|) z k)$$

$$\text{其中 } \gamma_z = z_y = f'(|\vec{r}|) \cdot \frac{yz}{|\vec{r}|^2}, \text{ 故 } \operatorname{rot}(f(|\vec{r}|) \cdot \vec{r}) = 0$$

$$(8) \operatorname{rot}(f(|\vec{r}|) \vec{a}) = \operatorname{rot}(f(|\vec{r}|) a_1 i + f(|\vec{r}|) a_2 j + f(|\vec{r}|) a_3 k)$$

$$= (a_3 \frac{f'(|\vec{r}|)}{|\vec{r}|} y - a_2 \frac{f'(|\vec{r}|)}{|\vec{r}|} z) i + (a_1 \frac{f'(|\vec{r}|)}{|\vec{r}|} z - a_3 \frac{f'(|\vec{r}|)}{|\vec{r}|} x) k + (a_2 \frac{f'(|\vec{r}|)}{|\vec{r}|} x - a_1 \frac{f'(|\vec{r}|)}{|\vec{r}|} y) k$$

$$= \frac{f'(|\vec{r}|)}{|\vec{r}|} \cdot \vec{r} \times \vec{a}$$

⑥

微积分

章节: 4.7

做题人: 1/可乐

$$(1) \vec{r} \times f(r) \vec{a} = f(r) \vec{r} \times \vec{a}$$

利用题给的(2), $\text{rot}(uA) = u \text{rot} A + \text{grad} u \times A$. 这里 $u = f(r)$, $\text{grad} u = \frac{f'(r)}{r} \cdot \vec{r}$

$$A = \vec{r} \times \vec{a} = (a_3 y - a_2 z) \mathbf{i} + (a_1 z - a_3 x) \mathbf{j} + (a_2 x - a_1 y) \mathbf{k}$$

$$\text{rot} A = -2a_1 \mathbf{i} - 2a_2 \mathbf{j} - 2a_3 \mathbf{k} = -2\vec{a}$$

$$\text{故 } \text{rot}(\vec{r} \times f(r) \vec{a}) = -2f(r) \cdot \vec{a} + \frac{f'(r)}{r} \cdot \vec{r} \times (\vec{r} \times \vec{a}) \quad (\text{注意叉乘没有结合律})$$

$$12. (1) X = y \cos(xy), Y = x \cos(xy); Z = \sin z$$

$$X_y = Y_x = \cos(xy) - xy \sin(xy), X_z = Z_x = Y_z = Z_y = 0$$

存在原函数 u , $du = Xdx + Ydy + Zdz$

$$\text{故 } V \text{ 为保守场, } u = \sin(xy) - \cos z + C$$

$$\int_{(4,0,1)}^{(2,1,-1)} V \cdot dr = \sin 2$$

$$(2) X = 2x \cos y - y^2 \sin x, Y = 2y \cos x - x^2 \sin y$$

$$X_y = -2x \sin y - 2y \sin x, Y_x = -2y \sin x - 2x \sin y, V \text{ 为保守场}$$

$$u = x^2 \cos y + y^2 \cos x + C$$

$$\int_{(4,0,1)}^{(2,1,-1)} V \cdot dr = 4 \cos 1 + \cos 2 - 16$$

$$(3) X = 6xy + z^2, Y = 3x^2 - z, Z = 3xz^2 - y$$

$$X_z \neq Z_x, \text{ 不为保守场}$$

$$(4) X = 2xy^2 + y^2 z + yz^2, Y = x^2 z + 2xy^2 + xz^2, Z = x^2 y + xy^2 + 2xyz$$

$$X_y = Y_x = 2yz + 2xz + z^2, X_z = Z_x = 2xy + y^2 + 2yz, Y_z = Z_y = x^2 + 2xy + 2xz$$

$$V \text{ 为保守场, } u = xyz(x + y + z)$$

$$\int_{(4,0,1)}^{(2,1,-1)} V \cdot dr = -4$$