

科目: 微积分

章节: 3.1

做题人: 钱雨杰

1. 证明过程与定积分类似

2. 略

$$3. \iint_{[0,1] \times [0,1]} xy \, dx \, dy = \lim_{n \rightarrow +\infty} \sum_{i=1}^n \sum_{j=1}^n \frac{i}{n} \cdot \frac{j}{n} \cdot \left(\frac{1}{n^2}\right) = \lim_{n \rightarrow +\infty} \left(\sum_{i=1}^n \frac{i}{n^2}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$4. \iint_{[-a,a] \times [-a,a]} \sin(x+y) \, dx \, dy.$$

用 $x = \frac{i}{n} \cdot a$, $y = \frac{j}{n} \cdot a$ ($-n+1 \leq i, j \leq n$) 将积分域分成 $(2n)^2$ 个小正方形.

在 x 轴上方的小正方形, 选其右上顶点的函数值; x 轴下方的小正方形, 选其左下顶点的函数值.

$$\therefore \iint_{[-a,a]^2} \sin(x+y) \, dx \, dy = \lim_{n \rightarrow +\infty} \left(\sum_{i=1}^n \sum_{j=-n+1}^n \sin\left(\frac{ia}{n}\right) \left(\frac{1}{n^2}\right) + \sum_{i=1}^n \sum_{j=-n+1}^n \sin\left(\frac{-ja}{n}\right) \left(\frac{1}{n^2}\right) \right) = 0$$

5. $\because f(x,y) \in R(I)$ $\therefore f$ 在 I 上的间断点集合是零面积集.

$\because J \subset I$. $\therefore J$ 的间断点集合是 I 的间断点集合的子集, 亦为零面积集

$\therefore f \in R(J)$.

6. 令分割 $T_x: 0=x_0 < x_1 < \dots < x_n=1$. $T_y: 0=y_0 < y_1 < \dots < y_m=1$.

$$\begin{aligned} \iint_{[0,1] \times [0,1]} f(x)g(y) \, dx \, dy &= \lim_{|T| \rightarrow 0} \sum_{i=1}^n \sum_{j=1}^m f(x_i)g(y_j) \Delta x_i \Delta y_j = \lim_{|T| \rightarrow 0} \left(\sum_{i=1}^n f(x_i) \Delta x_i \right) \left(\sum_{j=1}^m g(y_j) \Delta y_j \right) \\ &= \int_0^1 f(x) \, dx \cdot \int_0^1 g(y) \, dy. \end{aligned}$$

$$\iint_{[0,1] \times [0,1]} e^{-(x+y)} \, dx \, dy = \int_0^1 e^{-x} \, dx \int_0^1 e^{-y} \, dy = (1-e^{-1})^2$$

7. $f(x,y)$ 与 $|f(x,y)|$ 在 I 上的间断点集合是相同的.

故若 $f(x,y)$ 在 I 上可积, $|f(x,y)|$ 在 I 上可积.

令 $I_1 = \{(x,y) | (x,y) \in I \text{ 且 } f(x,y) \geq 0\}$. $I_2 = \{(x,y) | (x,y) \in I \text{ 且 } f(x,y) < 0\}$.

$$\iint_I f(x,y) \, dx \, dy = \iint_{I_1} f(x,y) \, dx \, dy + \iint_{I_2} f(x,y) \, dx \, dy$$

$$\iint_I |f(x,y)| \, dx \, dy = \iint_{I_1} f(x,y) \, dx \, dy - \iint_{I_2} f(x,y) \, dx \, dy$$

$\therefore \iint_{I_1} f(x,y) \, dx \, dy > 0$, $\iint_{I_2} f(x,y) \, dx \, dy < 0$. 由绝对值不等式 $|\iint_I f(x,y) \, dx \, dy| \leq \iint_I |f(x,y)| \, dx \, dy$

8. 略. 参阅《数学分析》

科目: 微积分

章节: 3.1

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9. 设点列 $\{x_n\}$ 收敛于 A . 即 $\forall \varepsilon > 0, \exists N, \forall n > N, \|x_n - A\| < \varepsilon$.

记 $A = (x_a, y_a)$. 取矩形 $I = [x_a - \varepsilon, x_a + \varepsilon] \times [y_a - \varepsilon, y_a + \varepsilon]$. $\sigma(I) = 4\varepsilon^2$.

当 $n > N$ 时, x_n 落在矩形 I 内. 当 $n \leq N$ 时, 有限个点, 构成零面积集.

由 $\varepsilon > 0$ 的任意性可知, $\{x_n\}$ 是零面积集.

10. (1) 考虑 $f(x, y)$ 的间断点. 当 $(1-x^2) + (1-y^2) = 0$ 时, $f(x, y)$ 间断.

即有 4 个间断点 $(1, 1), (1, -1), (-1, 1), (-1, -1)$.

$\therefore f(x, y)$ 在 $[-2, 2] \times [-2, 2]$ 上可积.

(2) 考虑 $f(x, y)$ 的间断点. 当 $y = x^2$ 时, $f(x, y)$ 间断.

由 $y = x^2$ 是二维平面上一条光滑曲线, 是零面积集.

$\therefore f(x, y)$ 在 $[0, 1] \times [0, 1]$ 上可积.

11. 12. 易证.

13. $\because f, g \in R(I)$ $\therefore f, g$ 的间断点集合均为零面积集.

$\because f, g$ 的间断点集合 $\subset f$ 的间断点集合 $\cup g$ 的间断点集合 $\therefore$ 也是零面积集

$\therefore fg \in R(I)$. 同理. 若 $g \neq 0, \frac{f}{g} \in R(I)$.

科目: 微积分

章节: 3.2

做题人: 姚炫容

1. 按定义, 将二重化为Riemann和的形式, 则显然.

$$2. \quad \frac{1}{100} > \frac{1}{100 + \cos^2 x + \cos^2 y} > \frac{1}{102}$$

$$\therefore 1.96 < \iint_{|x|+|y| \leq 1.0} \frac{dx dy}{100 + \cos^2 x + \cos^2 y} < 2.$$

3. (1) 在D上. 有 $x+y \geq 1$. $\therefore (x+y)^3 \geq (x+y)^2$ 等号只在一点取到.

$$\therefore \cancel{\iint_D (x+y)^2 dx dy} < \iint_D (x+y)^3 dx dy \quad \therefore \iint_D (x+y)^2 dx dy < \iint_D (x+y)^3 dx dy$$

(2) 在D上. $x+y \leq 1 \quad \therefore \ln(x+y) \leq 0. \quad xy \geq 0.$

$$\therefore xy \geq \ln(x+y) \quad \text{且} \exists S \subset D \text{ 不为零面积集.}$$

等号取到的集合为零面积集.

$$\therefore \iint_D \ln(x+y) dx dy < \iint_D xy dx dy.$$

4. 反证法. 若 $\exists (x_0, y_0) \in D$. s.t. $f(x_0, y_0) \neq 0$. 则由f非负.

得 $f(x_0, y_0) > 0$. 又 $\because f \in C(D)$.

$\therefore \exists B \subset D$. s.t. $(x_0, y_0) \in B$. 且B不为零面积集.

$\forall (x, y) \in B$. 有 $f(x, y) > 0$.

$$\therefore \iint_B f(x, y) dx dy > 0. \quad \text{又} \because \iint_{D-B} f(x, y) dx dy \geq 0$$

$$\therefore \iint_D f(x, y) dx dy > 0 \quad \text{与题设矛盾.}$$

$\triangle$ 得证.

科目: 微积分

章节: 3.2

做题人: 赵煜溶

5. $\exists \xi, \eta$ s.t. $\xi^2 + \eta^2 \leq r^2$ 且

$$\begin{aligned}\frac{1}{r^2} \iint_{x^2+y^2 \leq r^2} f(x,y) dx dy &= \frac{1}{r^2} \cdot f(\xi, \eta) \cdot \iint_{x^2+y^2 \leq r^2} dx dy \\ &= f(\xi, \eta) \cdot \pi\end{aligned}$$

当 $r \rightarrow 0^+$ 时 $f(\xi, \eta) \rightarrow f(0,0)$.

$\therefore$ 原式 $= \pi \cdot f(0,0)$.

科目: 微积分

章节: § 3.3

做题人: 姜天健

1.

(1) 几何意义是半球的体积

$$\iint_D \sqrt{R^2 - x^2 - y^2} dx dy, D = \{(x, y) | x^2 + y^2 \leq R^2\}$$

$$= \frac{2}{3} \pi R^3$$

(2) 几何意义是从圆柱中挖去圆锥

$$\iint_D \sqrt{R^2 - x^2 - y^2} dx dy, D = \{(x, y) | x^2 + y^2 \leq R^2\}$$

$$= \pi R^3 - \frac{1}{3} \pi R^3 = \frac{2}{3} \pi R^3$$

(3)

$$\iint_D 1 dx dy, D = \{(x, y) | |x| + |y| + |z| \leq 1\}$$

注: 本题的 z 的取值未给出, 这里默认为绝对值小于等于 1 的常数.

几何意义为正方形的体积

$$= 2(1 - |z|)^2$$

科目: 微积分

章节: § 3.3

做题人: 姜子健

2.

(1)

$$\iint_D \frac{x^2}{1+y^2} dx dy, D = [0, 1]^2$$

$$= \int_0^1 dx \int_0^1 \frac{x^2}{1+y^2} dy$$

$$= \frac{1}{12} \pi$$

(2)

$$\iint_D x \cos(xy) dx dy, D = [0, \frac{\pi}{2}] \times [0, 1]$$

$$= \int_0^{\frac{\pi}{2}} dx \int_0^1 x \cos(xy) dy$$

$$= 1$$

(3)

$$\iint_D \sin(x+y) dx dy, D = [0, \pi]^2$$

$$= \int_0^{\pi} dx \cdot \int_0^{\pi} \sin(x+y) dy$$

$$= 0$$

3.

$\therefore$ 函数 $f(x, y)$ 在 $D = [a, b] \times [c, d]$ 上有连续的 二阶偏导数

$\therefore f(x, y)$ 在 D 上处处可微

$\therefore f(x, y)$ 在 D 上处处连续

又 $\because D$ 为有限区域

$\therefore f(x, y)$ 在 D 上可积

$\therefore \forall x \in [a, b]$ 有 $f(x, y)$ 在闭区

间上连续

$\therefore$ 关于 y 在 $[c, d]$ 上可积

$\therefore \forall x, y$ 在闭区间上连续

$\therefore \frac{\partial^2 f}{\partial x \partial y}$ 可积

$$\therefore \iint_D f(x, y) d\sigma = \int_a^b dx \int_c^d \frac{\partial^2 f(x, y)}{\partial x \partial y} dy$$

$$= \int_a^b dx \left(\frac{\partial f(x, d)}{\partial x} - \frac{\partial f(x, c)}{\partial x} \right)$$

$$= f(b, d) - f(b, c)$$

$$+ f(a, c) - f(a, d)$$

科目: 微积分

章节: § 3.3

做题人: 李亚建

4.

(1)

$$D = \{(x, y) \mid x+y \leq 1, y-x \leq 1, y \geq 0\}$$

$$\therefore D = \{(x, y) \mid y-1 \leq x \leq 1-y, 0 \leq y \leq 1\}$$

$$\therefore \iint_D f(x, y) d\sigma = \int_0^1 dy \int_{y-1}^{1-y} f(x, y) dx$$

(2)

$$D = \{(x, y) \mid y \geq x-2, x \geq y^2\} = D_1 \cup D_2$$

$$\text{其中 } D_1 = \{(x, y) \mid x \geq y^2, 0 \leq x \leq 1\}$$

$$D_2 = \{(x, y) \mid y^2 \leq x \leq y+2, 1 \leq x \leq 4\}$$

$$\begin{aligned} \iint_D f(x, y) d\sigma &= \iint_{D_1} f(x, y) d\sigma + \iint_{D_2} f(x, y) d\sigma \\ &= \int_0^1 dx \int_{\sqrt{x}}^{\sqrt{x}} f(x, y) dy + \int_1^4 dx \int_{x-2}^{\sqrt{x}} f(x, y) dy \end{aligned}$$

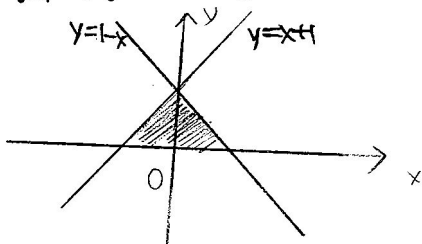
(3)

$$D = \{(x, y) \mid \frac{2}{x} \leq y \leq 2x, 1 \leq x \leq 2\}$$

$$\iint_D f(x, y) d\sigma = \int_1^2 dx \int_{\frac{2}{x}}^{2x} f(x, y) dy$$

5.

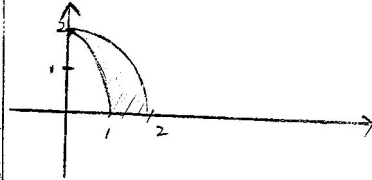
$$(1) \int_{-1}^0 dx \int_0^{1-x} f(x, y) dy + \int_0^1 dx \int_0^{1-x} f(x, y) dy$$



$$\text{原式} = \int_0^1 dy \int_{y-1}^{1-y} f(x, y) dx$$

(2)

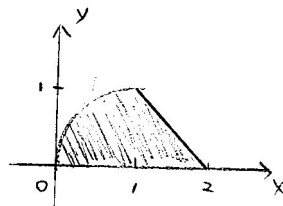
$$\int_0^1 dx \int_{\frac{\sqrt{4-x^2}}{2\sqrt{x}}}^{\sqrt{4-x^2}} f(x, y) dy + \int_1^2 dx \int_0^{\sqrt{4-x^2}} f(x, y) dy$$



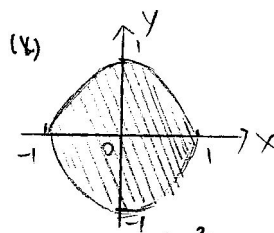
$$\Rightarrow \int_0^2 dy \int_{1-\frac{y^2}{2}}^{\sqrt{4-y^2}} f(x, y) dx$$

(3)

$$\int_0^1 dx \int_0^{\sqrt{2x-x^2}} f(x, y) dy + \int_1^2 dx \int_0^{2-x} f(x, y) dy$$



$$\Rightarrow \int_0^1 dy \int_{1-\sqrt{1-y^2}}^{2y} f(x, y) dx$$



$$\int_{-1}^1 dx \int_{x^2-1}^{1-x^2} f(x, y) dy$$

$$\begin{aligned} &\Rightarrow \int_{-1}^0 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) dx \\ &+ \int_0^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) dx \end{aligned}$$

科目: 微分积分

章节: § 3.3

做题人: 姜又健

(3)

$$\iint_D |xy| dx dy, D = \{(x, y) | x^2 + y^2 \leq R^2\}$$

$$= 4 \int_0^R dx \int_0^{\sqrt{R^2 - x^2}} xy dy$$

$$= 4 \int_0^R \frac{(R^2 - x^2)x}{2} dx$$

$$= \frac{R^4}{2}$$

(4)

$$\iint_D x \cos(xy) dx dy, D = \{(x, y) | x^2 + y^2 \leq R^2\}$$

$\because x \cos(xy)$ 关于 x 为奇函数

D 关于 y 轴对称

$$\therefore \iint_D x \cos(xy) dx dy = 0$$

$$(5) \iint_D (x^2 + y^2) dx dy, D = (y=x) \cup (y=x+1) \cup (y=1) \cup (y=0)$$

$$= \int_0^1 dy \int_{y-1}^y (x^2 + y^2) dx$$

$$= \frac{71}{2}$$

(6)

$$\iint_D e^{x+y} dx dy, D = \{(x, y) | |x| + |y| \leq 1\}$$

$$= \int_{-1}^0 dx \int_{-x}^{x+1} e^{x+y} dy + \int_0^1 dx \int_{x-1}^x e^{x+y} dy$$

$$= e - e^{-1}$$

(7)

$$\iint_D \cos(x+y) dx dy, D = \{(x, y) | 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \frac{\pi}{2}\}$$

$$= \int_0^{\frac{\pi}{2}} dx \int_0^{\frac{\pi}{2}} \cos(x+y) dy$$

$$= -x$$

(8)

$$\iint_D |\cos(x+y)| dx dy, D = [0, 1]^2$$

$$= \int_0^{\frac{\pi}{2}-1} dx \int_0^1 \cos(x+y) dy + \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} dx \int_0^{\frac{\pi}{2}-x} \cos(x+y) dy$$

$$- \int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} dx \int_{\frac{\pi}{2}-x}^1 \cos(x+y) dy$$

$$= 3 - \pi + 2\cos 1 + \cos 2$$

(9)

$$\iint_D y^2 dx dy, D \text{ 由 } \begin{cases} x = a(1 - \sin t) \\ y = a(1 - \cos t) \end{cases} t \in [0, 2\pi] \text{ 及 } x \text{ 轴围成}$$

由曲线的参数方程求其弧长

记 $y = f(x)$

$$\therefore I_2(\lambda) = \int_0^{2\pi a} dx \int_0^{f(x)} y^2 dy$$

$$dx = a(1 - \cos t) dt$$

$$\therefore I_2(\lambda) = \int_0^{2\pi} y \cdot a dt \cdot \int_0^y y^2 dy$$

$$= \frac{a^4}{3} \int_0^{2\pi} (1 - \cos t)^3 dt$$

$$= \frac{25}{12} \pi a^4$$

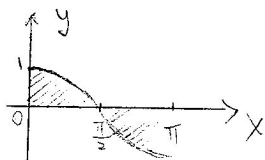
科目: 微积分

章节: § 3.3

做题人: 李双

15)

$$\int_0^{\pi} dx \int_0^{\cos x} f(x,y) dy$$



$$\Rightarrow \int_0^1 dy \int_{\pi}^{\arccos y} f(x,y) dx + \int_0^1 dy \int_0^{\arccos y} f(x,y) dx$$

6.

$$1) \iint_D x y^2 dx dy, D = \{(x,y) | 4x \geq y^2, x \leq 1\}$$

$$\therefore \int_0^1 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} f(x,y) dy$$

$$= \int_0^1 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} x y^2 dy$$

$$= \frac{32}{21}$$

$$2) \iint_D \frac{1}{\sqrt{2a-x}} dx dy, D = \{(x,y) | (x-a)^2 + (y-a)^2 \leq 1, 0 \leq x, y \leq a\}$$

$$0 \text{ 若 } |a| \leq \frac{\sqrt{2}}{2}$$

$$\iint_D \frac{1}{\sqrt{2a-x}} dx dy = 2a^{\frac{3}{2}}(\sqrt{2}-1)$$

$$0 \text{ 若 } \frac{\sqrt{2}}{2} \leq a \leq 1$$

$$D = D_1 \cup D_2$$

$$\text{其中 } D_1 = \{(x,y) | 0 \leq y \leq a, (x-a)^2 + (y-a)^2 \leq 1, 0 \leq x \leq a - \sqrt{1-a^2}\}$$

$$D_2 = \{(x,y) | 0 \leq y \leq a, a - \sqrt{1-a^2} \leq x \leq a\}$$

$$\therefore \iint_D f(x,y) d\sigma$$

$$= \int_0^{a-\sqrt{1-a^2}} \frac{\sqrt{1-(x-a)^2}}{\sqrt{2a-x}} dx + 2a \left((a + \sqrt{1-a^2})^{\frac{1}{2}} - \sqrt{a} \right)$$

$$③ \text{ 若 } a \geq 1$$

$$\iint_{D_1} f(x,y) d\sigma = \int_{a-1}^a \frac{\sqrt{1-(x-a)^2}}{\sqrt{2a-x}} dx$$

注: 由于水平所限, ② ③ 中的定积分暂无法求解, 求积出解的③ 同学予以补上.

科目: 数学分析

章节: § 3.3

做题人: 李亚健

(10)

$$\iint_D [x+y] dx dy \quad D = [0,1]^2$$

$$= \int_0^1 dx \int_0^1 [x+y] dy$$

$$= \int_0^1 dx \left(\int_0^{1-x} [x+y] dy + \int_{1-x}^1 [x+y] dy \right)$$

$$= \frac{1}{2}$$

7.

记区域 D 在 $y \leq 0$ 的部分为 D_2

$$\therefore D_1 \cap D_2 = \{y=0\}$$

构成 0 面积集

$$\therefore \iint_D f(x,y) d\sigma = \iint_{D_1} f(x,y) d\sigma + \iint_{D_2} f(x,y) d\sigma$$

~~$$f(x,-y) = f(x,y)$$~~

$$\text{在 } \iint_{D_2} f(x,y) d\sigma \text{ 中, 记 } y = -y \text{ 而}$$

$$\iint_{D_2} f(x,y) d\sigma = \iint_{D_1} f(x,-y) d\sigma$$

$$\text{由 } f(x,-y) = f(x,y)$$

$$\text{有 } \iint_{D_2} f(x,y) d\sigma = \iint_{D_1} f(x,y) d\sigma$$

$$\text{由 } f(x,-y) = -f(x,y)$$

$$\text{有 } \iint_D f(x,y) d\sigma = 0$$

当平面对称于轴对称时

$$\iint_D f(x,y) dx dy = \begin{cases} 2 \iint_{D_1} f(x,y) dx dy & f(-x,y) = f(x,y) \\ 0 & f(-x,y) = -f(x,y) \end{cases}$$

其中 D_1 为 D 在 $x \geq 0$ 的部分

$$8. D = \{(x,y) \mid x^2 + y^2 \leq R^2\}$$

 $\therefore x^2 y^3$ 为关于 y 的奇函数 D 关于 x 轴对称

$$\therefore \iint_D x^2 y^3 dx dy = 0$$

例 2

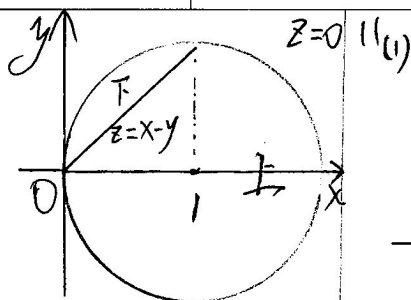
$$\iint_D \sqrt{R^2 - x^2} \sin y dx dy = 0$$

科目: 微积分

章节: 3.3

做题人: 杨志灿

9.



$$\vec{V} = \iint_{D_{xy}} z \, dx \, dy = \iint_{D_{xy}} (1 + r \cos \theta - r \sin \theta) r \, dr \, d\theta$$

$$\begin{cases} x = 1 + r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$\vec{V} = \int_0^1 dr \int_0^{2\pi} r [1 + r \cos \theta - r \sin \theta] d\theta$$

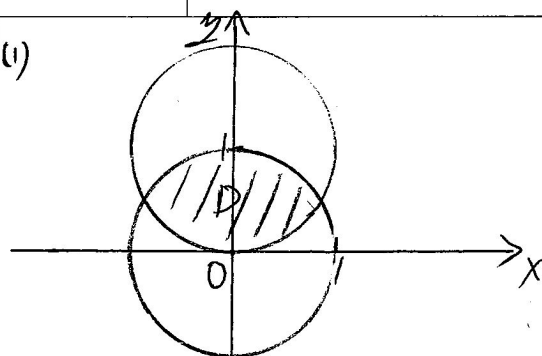
$$= \pi$$

$$\vec{V}_F = \iint_{D_{xy}} z \, dx \, dy = \int_0^1 dx \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy (x - y)$$

$$= \int_0^1 (x\sqrt{1-x^2} - \frac{1}{2}) dx = -\frac{1}{6}$$

$$\vec{V}_\perp = \vec{V} - \vec{V}_F = \pi + \frac{1}{6}$$

$$V_\perp = |\vec{V}_\perp| = \pi + \frac{1}{6} \quad V_F = |\vec{V}_F| = \frac{1}{6}$$

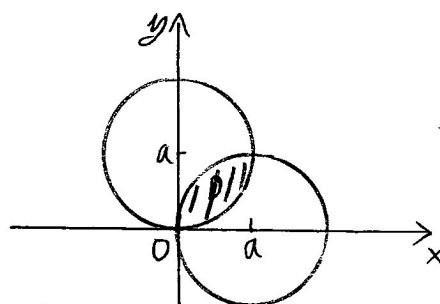


$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{cases} r^2 \leq 1, r^2 - 2r \sin \theta \leq 0 \\ \downarrow \\ 0 \leq r \leq 1, \sin \theta \geq \frac{r}{2} \end{cases}$$

$$\iint_D f(x,y) \, dx \, dy = \iint_{D_{r\theta}} f(x,y) r \, dr \, d\theta =$$

$$\int_0^1 dr \int_{\arcsin(\frac{r}{2})}^{\pi - \arcsin(\frac{r}{2})} f(x,y) r \, d\theta$$

(2)



$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \begin{cases} r^2 - 2arsin\theta \leq 0, r^2 - 2ar\cos\theta \leq 0 \\ r \leq 2a\sin\theta, r \leq 2a\cos\theta \end{cases}$$

$$\iint_D f(x,y) \, dx \, dy = \iint_{D_{r\theta}} f(x,y) r \, dr \, d\theta =$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\min(2a\sin\theta, 2a\cos\theta)} f(x,y) r \, dr$$

$$= \int_0^{\frac{\pi}{4}} d\theta \int_0^{2a\sin\theta} f(x,y) r \, dr +$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} f(x,y) r \, dr$$

10.

$$V = \iint_{D_{xy}} z \, dx \, dy = \int_{-1}^1 dx \int_{x^2}^1 (x^2 + y^2) dy$$

$$= \int_{-1}^1 (x^2 + \frac{1}{3} - x^4 - \frac{1}{3}x^6) dx = \frac{88}{105}$$

科目: 微积分

章节: 3.3

做题人: 杨志灿

12(1) 变量代换:

$$\begin{cases} x = r \cos \theta & 2 \cos \theta \leq r \leq 4 \cos \theta \\ y = r \sin \theta \end{cases}$$

$$\iint_D (x^2 + y^2) dx dy = \iint_{D_r \theta} r^2 \cdot r dr d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_{2 \cos \theta}^{4 \cos \theta} r^3 dr$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cdot 60 \cos^4 \theta$$

$$= \frac{45}{2} \pi$$

(2)

$$\begin{cases} x = r \cos \theta & r^2 \leq 1 \text{ 且 } r^2 \leq 2r \cos \theta \\ y = r \sin \theta & \text{即 } 0 \leq r \leq 1 \text{ 且 } r \leq 2 \cos \theta \end{cases}$$

$$\iint_D \sqrt{x^2 + y^2}^3 dx dy = \iint_{D_r \theta} r^3 \cdot r dr d\theta$$

$$= 2 \int_0^{\frac{\pi}{3}} d\theta \int_0^1 r^4 dr + 2 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\theta \int_0^{2 \cos \theta} r^4 dr$$

$$= \frac{2}{15} \pi + \frac{64}{5} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^5 \theta d\theta$$

$$= \frac{2}{15} \pi + \frac{512}{75} - \frac{98\sqrt{3}}{25}$$

(3) $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$

$$x^2 + y^2 = x + y$$

$$\text{即 } r = \cos \theta + \sin \theta$$

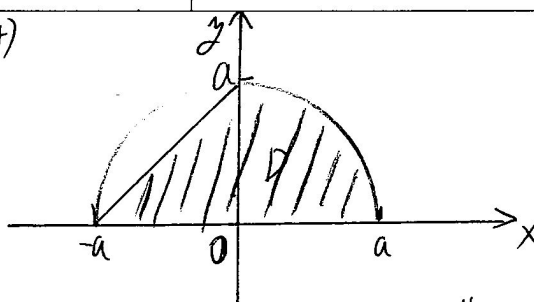
$$\iint_D (x+y) dx dy = \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} d\theta \int_0^{\sin \theta + \cos \theta} r^2 (\sin \theta + \cos \theta) dr$$

$$= \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} d\theta \cdot \frac{1}{3} (\sin \theta + \cos \theta)^4$$

$$= \int_{-\frac{\pi}{4}}^{\frac{3}{4}\pi} \frac{1}{3} (1 + 2 \sin 2\theta + \frac{1 - \cos 4\theta}{2}) d\theta$$

$$= \frac{\pi}{2}$$

(4)



$$\iint_D (y-x)^2 dx dy = \int_{-a}^0 dx \int_0^{x+a} dy \cdot (y-x)^2 = \frac{a^4}{4}$$

 D_R

$$= \int_{-a}^0 dx \cdot \frac{1}{3} x^3 = -\frac{a^4}{12}$$

$$\iint_D (y-x)^2 dx dy = \int_0^a dr \int_0^{\frac{\pi}{2}} d\theta \cdot (r \cos \theta - r \sin \theta)^2 \cdot r$$

 D_R

$$= \int_0^a dr \cdot \int_0^{\frac{\pi}{2}} r^3 (1 - \sin 2\theta) d\theta$$

$$= \int_0^a r^3 (\frac{\pi}{2} - 1) dr$$

$$= \frac{1}{8} (\pi - 2) a^4$$

$$\iint_D (y-x)^2 dx dy = \frac{1}{8} \pi a^4$$

 D

$$(5) \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$\iint_D \arctan \frac{y}{x} dx dy$$

 D

$$= \int_0^1 dr \int_{\frac{\pi}{6}}^{\frac{3}{2}\pi} r \cdot \arctan(\tan \theta) d\theta$$

$$= \int_0^1 dr \int_{\frac{\pi}{6}}^{\frac{3}{2}\pi} r \cdot (\theta - \pi) d\theta$$

$$= \int_0^1 \frac{1}{8} \pi^2 r dr$$

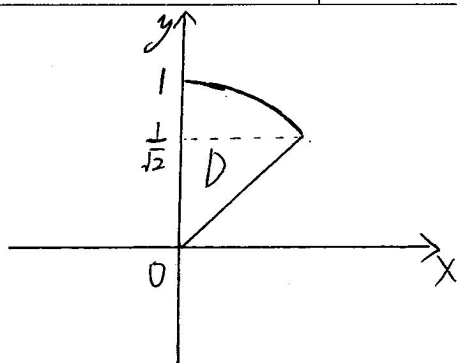
$$= \frac{\pi^2}{16}$$

科目: 微积分

章节: 3.3

做题人: 杨志灿

12(6)



$$\text{原式} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^1 dr \cdot e^{-r^2} \cdot r$$

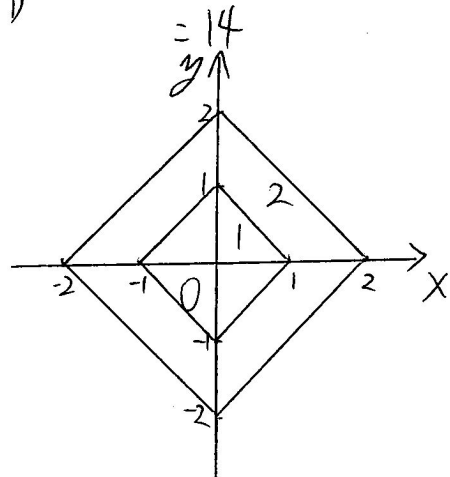
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(e^{-1} - 1) d\theta$$

$$= -\frac{\pi}{8} \left(\frac{1}{e} - 1 \right)$$

$$= \frac{\pi}{8} \left(1 - \frac{1}{e} \right)$$

(7)

$$\iint_D f(x,y) dx dy = 1 \times (\sqrt{2})^2 + 2 \times \left[(\sqrt{8})^2 - (\sqrt{2})^2 \right]$$



13. (1) 解: $D_{\rho\theta} = \{(\rho, \theta) \mid a \leq \rho \leq a\sqrt{2\cos 2\theta}, \theta \in [-\frac{\pi}{6}, \frac{\pi}{6}] \cup [\frac{5\pi}{6}, \frac{7\pi}{6}]\}$

$$\begin{aligned} \therefore \text{面积} \iint_D d\sigma &= \iint_{D_{\rho\theta}} \rho d\theta d\rho = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\theta \int_a^{a\sqrt{2\cos 2\theta}} \rho d\rho + \int_{\frac{5\pi}{6}}^{\frac{7\pi}{6}} d\theta \int_a^{a\sqrt{2\cos 2\theta}} \rho d\rho \\ &= (\sqrt{3} - \frac{\pi}{3}) a^2. \end{aligned}$$

(2) 解: $D_{\rho\theta} = \{(\rho, \theta) \mid \rho \leq \sqrt{3}a \sin \theta, \rho \leq a(1 + \cos \theta), 0 \leq \theta \leq \pi\}$
 $= \{(\rho, \theta) \mid \rho \leq \sqrt{3}a \sin \theta, \theta \in [0, \frac{\pi}{3}] \cup [\pi - \frac{\pi}{6}, \pi]\} \cup$
 $\{(\rho, \theta) \mid \rho \leq a(1 + \cos \theta), \theta \in [\frac{\pi}{3}, \pi]\}.$

$$\begin{aligned} \therefore \text{面积} \iint_D d\sigma &= \int_0^{\frac{\pi}{3}} d\theta \int_0^{\sqrt{3}a \sin \theta} \rho d\rho + \int_{\frac{\pi}{3}}^{\pi} d\theta \int_0^{a(1 + \cos \theta)} \rho d\rho \\ &= \frac{3}{2} a^2 \int_0^{\frac{\pi}{3}} \sin^2 \theta d\theta + \frac{1}{2} a^2 \int_{\frac{\pi}{3}}^{\pi} (1 + \cos \theta)^2 d\theta \\ &= \frac{3}{2} a^2 \int_0^{\frac{\pi}{3}} \frac{1 - \cos 2\theta}{2} d\theta + \frac{1}{2} a^2 \int_{\frac{\pi}{3}}^{\pi} (\frac{3}{2} + 2\cos \theta + \frac{1}{2} \cos 2\theta) d\theta \\ &= \frac{3\pi - 3\sqrt{3}}{4} a^2. \end{aligned}$$

14. (1) 解: $D = \{(x, y) \mid 2 \leq xy \leq 4, x \leq y \leq 3x, x \geq 0, y \geq 0\}.$

作代换 $\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases}$ 则 $D^* = \{(u, v) \mid 2 \leq u \leq 4, 1 \leq v \leq 3\}.$

$$\frac{D(u, v)}{D(x, y)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \frac{2y}{x} = 2v \neq 0, \therefore \frac{D(x, y)}{D(u, v)} = \frac{1}{2v}.$$

$$\therefore \iint_D x^2 y^2 dx dy = \int_2^4 du \int_1^3 \frac{u^2}{2v} dv = \frac{\ln 3}{2} \int_2^4 u^2 du = \frac{28}{3} \ln 3.$$

(2) 解: $D = \{(x, y) \mid 1 \leq xy \leq 2, 1 \leq x^2 - y^2 \leq 2, x, y \geq 0\}.$

作代换 $\begin{cases} u = xy \\ v = x^2 - y^2 \end{cases}$ $\frac{D(u, v)}{D(x, y)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = -2(x^2 + y^2) \neq 0, \frac{D(x, y)}{D(u, v)} = \frac{-1}{2(x^2 + y^2)}$

$$\therefore \iint_D (x^2 + y^2) dx dy = \int_1^2 du \int_1^2 (x^2 + y^2) \frac{-1}{2(x^2 + y^2)} dv = \frac{1}{2}.$$

$$(3) \iint_D (x^2+y^2) dx dy, D = \{(x,y) \mid |x|+|y| \leq 1\}$$

解: $D = \{(x,y) \mid -1 \leq x+y \leq 1, -1 \leq x-y \leq 1\}$. 作 $\begin{cases} u=x+y \\ v=x-y \end{cases}$ $\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$

$$\therefore \frac{D(x,y)}{D(u,v)} = -\frac{1}{2}. \therefore \iint_D (x^2+y^2) dx dy = \int_{-1}^1 du \int_{-1}^1 \frac{u^2+v^2}{2} \cdot \frac{1}{2} dv = \frac{2}{3}.$$

$$(4) \text{解: } D = \{(x,y) \mid \frac{1}{3} \leq y \leq 2, 2-2y \leq y^2-x \leq 1+y\}. \text{ 作 } \begin{cases} u=y^2-x \\ v=y \end{cases}$$

$$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} -1 & 2y \\ 0 & 1 \end{vmatrix} = -1. \therefore \frac{D(x,y)}{D(u,v)} = -1.$$

$$\iint_D (x-y^2) dx dy = \iint_{D^*} (-u) \cdot 1 du dv = \int_{\frac{1}{3}}^2 dv \int_{2-2v}^{1+v} (-u) du = -\frac{175}{54}.$$

15. 求面积.

$$(1) \text{解: } \begin{cases} u=a_1x+b_1y+c_1 \\ v=a_2x+b_2y+c_2 \end{cases} \quad \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0.$$

$$\therefore \frac{D(x,y)}{D(u,v)} = \frac{1}{a_1b_2 - a_2b_1} \therefore \text{面积} \iint_D db = \iint_{u^2+v^2 \leq 1} \frac{1}{|a_1b_2 - a_2b_1|} du dv = \frac{\pi}{|a_1b_2 - a_2b_1|}.$$

$$(2) \text{解: } \begin{cases} u=\sqrt{x} \\ v=\sqrt{y} \end{cases} \quad \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} \frac{1}{2\sqrt{x}} & 0 \\ 0 & \frac{1}{2\sqrt{y}} \end{vmatrix} = \frac{1}{4\sqrt{xy}} = \frac{1}{4uv} \neq 0. \quad \frac{D(x,y)}{D(u,v)} = 4uv.$$

$$\therefore \iint_D db = \iint_{\substack{u+v \leq \sqrt{a} \\ u,v \geq 0}} 4uv du dv = \frac{1}{6}a^2.$$

$$16. \text{证明: (1) 作代换 } \begin{cases} u=x+y \\ v=x-y \end{cases} \text{ (注意到 } D = \{(x,y) \mid -1 \leq x+y \leq 1, -1 \leq x-y \leq 1\})$$

$$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \therefore \frac{D(x,y)}{D(u,v)} = \frac{1}{2} \therefore \iint_{|x|+|y| \leq 1} f(x+y) dx dy = \int_{-1}^1 du \int_{-1}^1 \frac{1}{2} f(u) dv$$

$$= \int_{-1}^1 f(u) du = \int_{-1}^1 f(t) dt. \text{ 得证.}$$

$$(2) D = \{(x,y) \mid 1 \leq xy \leq 2, 1 \leq \frac{y}{x} \leq 4, x>0, y>0\}. \text{ 作 } \begin{cases} u=xy \\ v=\frac{y}{x} \end{cases}$$

$$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{2y}{x} = 2v \neq 0. \therefore \frac{D(x,y)}{D(u,v)} = \frac{1}{2v}.$$

$$\therefore \iint_D f(xy) dx dy = \iint_{\substack{1 \leq u \leq 2 \\ 1 \leq v \leq 4}} f(u) \frac{1}{2v} du dv = \int_1^2 f(u) du \int_1^4 \frac{1}{2v} dv = \ln 2 \int_1^2 f(u) du$$

$$= \ln 2 \int_1^2 f(t) dt \text{ 得证.}$$

有点意思... ←

17. 解: 由对称性, $\iint_{x^2+y^2 \leq R^2} \frac{f(x)}{f(x)+f(y)} dx dy = \iint_{x^2+y^2 \leq R^2} \frac{f(y)}{f(x)+f(y)} dx dy = A.$

$$\therefore 2A = \iint_{x^2+y^2 \leq R^2} \frac{f(x)+f(y)}{f(x)+f(y)} dx dy = \iint_{x^2+y^2 \leq R^2} dx dy = \pi R^2, \quad A = \frac{1}{2} \pi R^2.$$

$$\therefore \text{原式} = aA + bA = \frac{a+b}{2} \pi R^2.$$

18. 解: $F(x) = \int_0^x dt \int_{t^2}^{x^2} f(t, s) ds.$ 记 $g(x, t) = \int_{t^2}^{x^2} f(t, s) ds.$ 则 $F(x) = \int_0^x g(x, t) dt.$

$$F(x+\Delta x) - F(x) = \int_0^{x+\Delta x} g(x+\Delta x, t) dt - \int_0^x g(x, t) dt$$

$$= \int_0^x [g(x+\Delta x, t) - g(x, t)] dt + \int_x^{x+\Delta x} g(x+\Delta x, t) dt.$$

$$\therefore F'(x) = \lim_{\Delta x \rightarrow 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left[\int_0^x \frac{g(x+\Delta x, t) - g(x, t)}{\Delta x} dt + \frac{\int_x^{x+\Delta x} g(x+\Delta x, t) dt}{\Delta x} \right]$$

$$\stackrel{\lim_{\Delta x \rightarrow 0}}{=} \int_0^x g'_x(x, t) dt + g(x+\Delta x, x) = \lim_{\Delta x \rightarrow 0} \left[\int_0^x 2xf(t, x^2) dt + g(x+\Delta x, x) \right]$$

$$= \int_0^x 2xf(t, x^2) dt.$$

科目: 微积分

章节: 习题 3.4

做题人: 黄子

1. 略

2. 略

$$3. (1) \text{ 锥式} = V_{\text{锥}} = \frac{1}{3} Sh = \frac{1}{3} \cdot \pi R^2 \cdot H = \frac{1}{3} \pi H^3$$

$$(2) \text{ 锥式} = V_{\text{锥}} = \frac{1}{3} Sh = \frac{1}{3} \times (1 \times 1 \times \frac{1}{2}) \times 1 = \frac{1}{6}$$


$$4. (1) \iiint_{\Omega} f(x,y,z) dx dy dz = \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^1 f(x,y,z) dz$$

$$(2) \iiint_{\Omega} f(x,y,z) dx dy dz = \int_0^1 dx \int_0^{-x+1} dy \int_0^{x^2+y^2} f(x,y,z) dz$$

$$5. (1) \text{ 锥式} = \int_0^1 dx \int_0^x dy \int_0^{xy} xy^2 z^3 dz = \int_0^1 dx \int_0^x \frac{1}{4} x^5 y^6 dy$$

$$= \frac{1}{28} \int_0^1 x^{12} dx = \frac{1}{364}$$

$$(2) \text{ 锥式} = \int_0^{e-1} dx \int_0^{-x+e-1} dy \int_e^{x+y+e} \frac{\ln(z-x-y)}{(x-e)(x+y-e)} dz$$

$$= \int_0^{e-1} 2 + \frac{2}{x-e} - \ln(e-x) dx = 2e - 5$$

$$(3) \text{ 锥式} = \int_0^{\frac{\pi}{2}} dy \int_0^y dx \int_0^{-y+\frac{\pi}{2}} x \cos(y+z) dz$$

$$= \int_0^{\frac{\pi}{2}} dy \int_0^y x - x \sin y dx = \int_0^{\frac{\pi}{2}} -\frac{1}{2} y \sin y dy + \int_0^{\frac{\pi}{2}} \frac{1}{2} y dy$$

$$= -\frac{1}{2} + \frac{\pi^2}{16}$$

$$(4) \text{ 锥式} = \iiint_{\Omega} x dv + \iiint_{\Omega} |y| dv + \iiint_{\Omega} |z| dv = 0 + 2 \iiint_{\Omega, y \geq 0} y dv + 2 \iiint_{\Omega, z \geq 0} z dv$$

$$2 \iiint_{\Omega, y \geq 0} y dv = 2 \int_0^1 dy \int_{x+1}^{-x+1} dx \int_0^{1-x-y} y dz = 2 \int_0^1 dy \int_{y-1}^{-y+1} y(1-x-y) dx$$

$$= 2 \int_0^1 2y(1-y)^2 dy = \frac{1}{3}$$

$$\text{由对称性, } \iiint_{\Omega, y \geq 0} y dv = \iiint_{\Omega, z \geq 0} z dv$$

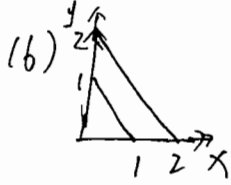
$$\therefore \text{ 锥式} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

科目: 微积分

章节: 习题 3.4

做题人: 黄子

$$(5) \text{原式} = \int_0^4 dz \iint_{D_z} \frac{\sin z}{z} d\sigma = \int_0^4 \frac{\sin z}{z} \cdot \pi z^2 dz = \pi(-4\cos 4 + \sin 4)$$



由几何意义, 该积分为 $V_{\text{大锥}} - V_{\text{小锥}}$

$$= 2 \times 2 \times \frac{1}{2} \times 2 \times \frac{1}{3} - 1 \times 1 \times \frac{1}{2} \times 1 \times \frac{1}{3}$$

$$= \frac{7}{6}$$

科目: 微积分

章节: §3.4

做题人: 马志远

6. 解: 原式 = $\iiint_{\Omega} \frac{\cos z}{1-z^2} dz$, 其中

$$\Omega = \{0 \leq z \leq y \leq x \leq 1\}$$

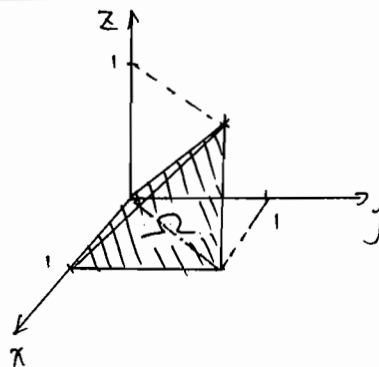
$$\therefore \text{原式} = \int_0^1 dz \int_z^1 dy \int_y^1 \frac{\cos z}{(1-z)^2} dx$$

$$= \int_0^1 dz \int_z^1 \frac{\cos z}{(1-z)^2} (1-y) dy$$

$$= \int_0^1 \frac{\cos z}{(1-z)^2} \left(y - \frac{1}{2}y^2\right) \Big|_z^1 dz$$

$$= \frac{1}{2} \int_0^1 \frac{\cos z}{(1-z)^2} (1-z)^2 dz$$

$$= \frac{\sin 1}{2}$$

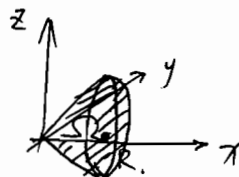


7. 解:

(1). 令 $\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \\ z = z \end{cases}$, 则 $\Omega \rightarrow \Omega^*: 0 \leq r \leq z \leq 1, \frac{D(x,y,z)}{D(r,\varphi,z)} = r$

$$\text{原式} = \iiint_{\Omega^*} r^2 dr d\varphi dz = \int_0^{2\pi} d\varphi \int_0^1 dr \int_r^1 r^2 dz = 2\pi \cdot \int_0^1 r^2 (1-r) dr = \frac{\pi}{6}$$

(2). 令 $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \sin \varphi \\ z = r \sin \theta \cos \varphi \end{cases}$ 则 $\Omega \rightarrow \Omega^*: 0 \leq r \leq R, 0 \leq \theta \leq \frac{\pi}{4}$



$$\therefore \text{原式} = \iiint_{\Omega^*} r^4 \sin \theta dr d\theta d\varphi = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{4}} \sin \theta d\theta \int_0^R r^4 dr$$

$$= 2\pi \cdot \left(1 - \frac{1}{2}\right) \cdot \frac{R^5}{5} = \frac{2\pi R^5}{5}$$

(3). 原式 = $\iint_D dx dy \int_0^{\sqrt{x^2+y^2}} \frac{z}{x^2+y^2} dz = \iint_D \frac{x^2+y^2}{2} dx dy$, 其中 $D = \{(x,y) | x,y \geq 0, x+y \leq 1\}$

由对称性, 原式 = $\iint_D x^2 dx dy = \int_0^1 dx \int_0^{1-x} x^2 dy = \int_0^1 x^2 (1-x) dx = \frac{1}{12}$

(4). 令 $\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$ 则 $\Omega \rightarrow \Omega^*: 0 \leq r \leq |a|, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq \frac{\pi}{2}$

$$\text{原式} = \iiint_{\Omega^*} r \sin \theta \cos \varphi e^{\frac{r^2}{a^2}} \cdot r^2 \sin \theta dr d\theta d\varphi = \int_0^{\frac{\pi}{2}} \cos \varphi d\varphi \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta \int_0^{|a|} r^3 e^{\frac{r^2}{a^2}} dr$$

$$= \frac{\pi}{4} \cdot \int_0^{\frac{a^2}{2}} \frac{1}{2} u e^{\frac{u}{a^2}} du = \frac{\pi a^2}{8} \int_0^1 t e^t dt = \frac{\pi a^2}{8} (te^t - e^t) \Big|_0^1 = \frac{\pi a^2}{8}$$

科目: 微积分

章节: §3.4

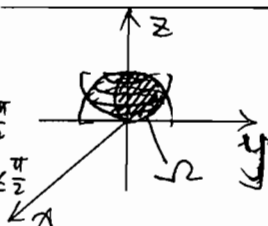
做题人: 马志远

$$(5). \text{ 令 } \begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases}$$

则 $\Omega \rightarrow \Omega_1 + \Omega_2$

$$\Omega_1: 0 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq \varphi \leq \frac{\pi}{2}$$

$$\Omega_2: 0 \leq r \leq 4 \cos \theta, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq \frac{\pi}{2}$$



$$\begin{aligned} \therefore \text{原式} &= \iiint_{\Omega_1} r^5 \sin^3 \theta \cos \theta \sin \varphi \cos \varphi d\theta d\varphi dr + \iiint_{\Omega_2} r^5 \sin^3 \theta \cos \theta \sin \varphi \cos \varphi dr d\theta d\varphi \\ &= \int_0^{\frac{\pi}{2}} \sin \varphi \cos \varphi d\varphi \int_0^{\frac{\pi}{4}} \sin^3 \theta \cos \theta d\theta \int_0^2 r^5 dr + \int_0^{\frac{\pi}{2}} \sin \varphi \cos \varphi d\varphi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3 \theta \cos \theta d\theta \int_0^{4 \cos \theta} r^5 dr \\ &= \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{2^6}{6} + \frac{1}{2} \cdot \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3 \theta \cos \theta d\theta \cdot \frac{(4 \cos \theta)^6}{6} d\theta \\ &= \frac{1}{3} + \frac{32\sqrt{2}}{15} \cdot \frac{6}{5} = \frac{29}{15} \end{aligned}$$

$$(6). \text{ 令 } \begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \quad \text{则 } \Omega \rightarrow \Omega^*: 0 \leq r \leq \cos \theta, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq 2\pi$$

$$\begin{aligned} \text{原式} &= \iiint_{\Omega^*} r^4 \sin \theta dr d\theta d\varphi = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin \theta d\theta \int_0^{\cos \theta} r^4 dr \\ &= 2\pi \cdot \int_0^{\frac{\pi}{2}} \sin \theta \cdot \frac{\cos^5 \theta}{5} d\theta = \frac{2\pi}{5} \int_0^1 t^5 dt = \frac{\pi}{15} \end{aligned}$$

8. 解:

$$(1). \text{ 令 } \begin{cases} x = a \sin \theta \cos \varphi \\ y = b \sin \theta \sin \varphi \\ z = c \cos \theta \end{cases} \quad \text{则 } \Omega \rightarrow \Omega^*: 0 \leq r \leq 1, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi$$

$$\frac{D(x, y, z)}{D(r, \theta, \varphi)} = abc r^2 \sin \theta$$

$$\begin{aligned} \therefore \text{原式} &= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin \theta d\theta \int_0^1 \sqrt{1-r^2} \cdot abc r^2 dr = 2\pi abc \cdot (-\cos \theta) \Big|_0^{\pi} \cdot \int_0^1 \sqrt{1-r^2} dr \\ &= 4\pi abc \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t dt \quad \text{令 } u = \sin t \quad 4\pi abc \int_0^1 u^2 \sqrt{1-u^2} du \\ &= 4\pi abc \int_0^{\frac{\pi}{2}} \sin^2 t (1 - \sin^2 t) dt = 4\pi abc \cdot \left(\frac{\pi}{4} - \frac{3\pi}{16} \right) = \frac{\pi^2 abc}{4} \end{aligned}$$

$$(2). \text{ 令 } \begin{cases} u = x \\ v = y^2 \\ w = z \end{cases} \quad \text{则 } \Omega \rightarrow \Omega^*: \frac{w}{2} \leq u \leq w, \frac{w}{4} \leq v \leq w, 0 \leq w \leq 3.$$

$$\frac{D(x, y, z)}{D(u, v, w)} = \frac{1}{2\sqrt{v}}$$

$$\begin{aligned} \therefore \text{原式} &= \int_0^3 dw \int_{\frac{w}{2}}^w u^2 du \int_{\frac{w}{4}}^w \frac{1}{2\sqrt{v}} dv = \int_0^3 \left(\frac{3}{4} w^2 \cdot \frac{\sqrt{w}}{2} \right) dw \\ &= \frac{3}{8} \int_0^3 w^{\frac{5}{2}} dw = \frac{3}{8} \cdot \frac{2}{7} w^{\frac{7}{2}} \Big|_0^3 = \frac{162\sqrt{3}}{56} \end{aligned}$$

科目: 微积分

章节: 3.4

做题人: 杨晓成

$$Q. (1) \begin{cases} x = \rho \cos \theta \\ y = \rho \sin \theta \end{cases} \quad \frac{D(x, y)}{D(\rho, \theta)} = \rho$$

$$\begin{cases} z = 6 - x^2 - y^2 \\ z = \sqrt{x^2 + y^2} \end{cases} \Rightarrow \begin{cases} z = 2 \\ \rho = 2 \end{cases}$$

$$V = \int_0^2 dz \int_0^z d\rho \int_0^{2\pi} \left| \frac{D(x, y)}{D(\rho, \theta)} \right| d\theta + \int_2^6 dz \int_0^{\sqrt{6-z}} d\rho \int_0^{2\pi} \left| \frac{D(x, y)}{D(\rho, \theta)} \right| d\theta$$

$$= \frac{32}{3}\pi$$

$$(2) \begin{cases} x = \rho \sin \theta \cos \varphi \\ y = \rho \sin \theta \sin \varphi \\ z = \rho \cos \theta \end{cases} \quad \frac{D(x, y, z)}{D(\rho, \theta, \varphi)} = \rho^2 \sin \theta$$

$$\begin{cases} x^2 + y^2 + z^2 \leq 2az \\ x^2 + y^2 \leq z^2 \end{cases} \Rightarrow \begin{cases} -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4} \\ \rho \leq 2 \cos \theta \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

$$V = \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_0^{2\cos\theta} \rho^2 \sin \theta d\rho$$

$$= 4\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_0^{2\cos\theta} \rho^2 \sin \theta d\rho$$

$$= 2\pi$$

$$(3) V = \int_0^a dx \int_0^{\sqrt{a^2-x^2}} dy \int_0^{\sqrt{a^2-x^2}} dz$$

$$= \int_0^a (a^2 - x^2) dx$$

$$= \frac{2}{3}a^3$$

科目:

章节:

做题人:

$$\begin{aligned}
 (4) \quad V &= 4 \int_0^a dx \int_0^{\sqrt{a^2-x^2}} dz \int_0^{a-x} dy \\
 &= 4 \int_0^a (a-x) \sqrt{a^2-x^2} dx \\
 &= 4a \int_0^a \sqrt{a^2-x^2} dx - 4 \int_0^a x \sqrt{a^2-x^2} dx \\
 &= 4a \cdot \frac{1}{4} \pi a^2 + \frac{4}{3} (a^2-x^2)^{\frac{3}{2}} \Big|_0^a \\
 &= \left(\pi - \frac{4}{3}\right) a^3
 \end{aligned}$$

$$(5) \quad \begin{cases} x = \rho \sin \theta \cos \varphi \\ y = \rho \sin \theta \sin \varphi \\ z = \rho \cos \theta \end{cases} \quad \begin{aligned} \frac{D(x, y, z)}{D(\rho, \theta, \varphi)} &= \rho^2 \sin \theta \\ \frac{\pi}{2} &\leq \theta \leq \pi \\ 0 &\leq \rho \leq a^3 \sqrt{\cos \theta} \end{aligned}$$

$$\begin{aligned}
 V &= \int_0^{2\pi} d\varphi \int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^{a^3 \sqrt{\cos \theta}} \rho^2 \sin \theta d\rho = 4\pi \int_0^{\frac{\pi}{2}} d\theta \int_0^{a^3 \sqrt{\cos \theta}} \rho^2 \sin \theta d\rho \\
 &= \frac{4\pi}{3} a^3 \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta \\
 &= \frac{4\pi}{3} a^3 \cdot \frac{2\pi}{3} a^3
 \end{aligned}$$

$$(6) \quad \begin{cases} \alpha = a_{11}x + a_{12}y + a_{13}z \\ \beta = a_{21}x + a_{22}y + a_{23}z \\ \gamma = a_{31}x + a_{32}y + a_{33}z \end{cases} \quad \frac{D(\alpha, \beta, \gamma)}{D(x, y, z)} = |A| \quad \frac{D(x, y, z)}{D(\alpha, \beta, \gamma)} = \frac{1}{|A|}$$

$$\begin{aligned}
 V &= \int_{-|h_1|}^{|h_1|} d\alpha \int_{-|h_2|}^{|h_2|} d\beta \int_{-|h_3|}^{|h_3|} d\gamma \frac{1}{|A|} \\
 &= 8 |h_1 h_2 h_3| |A| \cdot \frac{1}{|A|}
 \end{aligned}$$

$$(7) \quad \begin{cases} \alpha = a_{11}x + a_{12}y + a_{13}z \\ \beta = a_{21}x + a_{22}y + a_{23}z \\ \gamma = a_{31}x + a_{32}y + a_{33}z \end{cases} \quad \frac{D(\alpha, \beta, \gamma)}{D(x, y, z)} = |A| \quad \begin{cases} \alpha = \rho \sin \theta \cos \varphi \\ \beta = \rho \sin \theta \sin \varphi \\ \gamma = \rho \cos \theta \end{cases} \quad \begin{aligned} \frac{D(\alpha, \beta, \gamma)}{D(\rho, \theta, \varphi)} &= \rho^2 \sin \theta \end{aligned}$$

科目:

章节:

做题人:

$$V = \int_{-r}^r dr \int_{\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} d\beta \int_{\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} \frac{D(x,y,z)}{D(\rho,\theta,\varphi)} = \frac{\rho^2 \sin \theta}{|A|}$$

$$V = \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta \int_0^r \frac{\rho^2}{|A|} d\rho = \frac{4\pi r^3}{3|A|}$$

$$10. \begin{cases} x = \rho \sin \theta \cos \varphi \\ y = \rho \sin \theta \sin \varphi \\ z = \rho \cos \theta \end{cases} \quad \begin{matrix} t \geq 0 \text{ 时} \\ f(t) = 3 \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta \int_0^t \rho^2 f(\rho) d\rho + t^3 \\ = 12\pi \int_0^t \rho^2 f(\rho) d\rho + t^3 \end{matrix}$$

$$f'(t) = 12\pi t^2 f(t) + 3t^2 \Rightarrow f(t) = (e^{4\pi t^3} - \frac{1}{4\pi})$$

$$\text{另外 } f(0) = 0 \therefore C = \frac{1}{4\pi}$$

$$f(t) \text{ 在 } (-\infty, +\infty) \text{ 上连续} \Rightarrow f(t) = \frac{1}{4\pi} (e^{4\pi t^3} - 1)$$

$$11. \begin{cases} x = \rho \sin \theta \cos \varphi \\ y = \rho \sin \theta \sin \varphi \\ z = \rho \cos \theta \end{cases}$$

$$\text{原式} = \lim_{r \rightarrow 0^+} \frac{1}{r^3} \int_0^{2\pi} d\varphi \int_0^\pi \sin \theta d\theta \int_0^r \rho^2 \sin \theta f(\rho \sin \theta \cos \theta, \rho \sin \theta \sin \varphi, \rho \cos \theta) d\rho$$

$$= \frac{4\pi}{3} \lim_{r \rightarrow 0^+} \int_0^r \rho^2 f(\rho \sin \theta \cos \theta, \rho \sin \theta \sin \varphi, \rho \cos \theta) d\rho$$

$$= \frac{4\pi}{3} \lim_{r \rightarrow 0^+} \frac{r^2 f(r \sin \theta \cos \theta, r \sin \theta \sin \varphi, r \cos \theta)}{3r^2}$$

~~显然~~ 显然 $\rho^2 \sin \theta f(\rho \sin \theta \cos \theta, \rho \sin \theta \sin \varphi, \rho \cos \theta)$ 为连续函数

$$\text{原式} = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \lim_{r \rightarrow 0^+} \frac{\int_0^r \rho^2 \sin \theta f(\rho \sin \theta \cos \theta, \rho \sin \theta \sin \varphi, \rho \cos \theta) d\rho}{r^3}$$

$$= \int_0^{2\pi} d\varphi \int_0^\pi d\theta \lim_{r \rightarrow 0^+} \frac{r^2 \sin \theta f(r \sin \theta \cos \theta, r \sin \theta \sin \varphi, r \cos \theta)}{3r^2}$$

$$= \int_0^{2\pi} d\varphi \int_0^\pi d\theta \cdot \frac{\sin \theta}{3} f(0,0,0) = \frac{4\pi}{3} f(0,0,0)$$

科目: 微积分

章节: 3-5

做题人: 游宇铭

1. (1) 无准确解.

(2). 该部分关于 xy 面对称. 由 xy 面分为两部分. S^+, S^- .则 $S = 2S^+$.对 S^+ : $z = \sqrt{a^2 - x^2}$ $(x, y) \in D$, $D: x^2 + y^2 \leq a^2$.

$$\text{则 } \frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = 0.$$

$$\therefore S = 2 \iint_D \sqrt{1 + \frac{x^2}{z^2}} dx dy = 2 \iint_D \frac{a}{\sqrt{a^2 - x^2}} dx dy.$$

$$= 2 \int_{-a}^a dx \int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \frac{a}{\sqrt{a^2 - x^2}} dy = 8a^2.$$

(3) ~~圆锥面面积微元为 $\sqrt{2} d\sigma$, $d\sigma$ 为所求部分在 xy 面的投影面积~~
~~分析可得 $d\sigma$ 为 $xy = x^2$ $x \in [0, 2]$ 的面积~~
~~(见图)~~

~~$$S = \sqrt{2} \int_0^2 \frac{x^2}{2} dx = \sqrt{2} \int_0^2 2\sqrt{2x - x^2} dx =$$~~

(3) 锥面面积微元为 $\sqrt{2} d\sigma$. $d\sigma$ 为面积微元在 xy 面投影面积分析得 $d\sigma$ 为 $xy = x^2$ 的面积 ($x \in [0, 2]$).

$$\therefore S = \sqrt{2} \cdot \frac{1}{6} \cdot 2^3 = \frac{4\sqrt{2}}{3}.$$

(4). 两曲面交于曲线: $x^2 + y^2 = a^2$.

$$\therefore S = S_{\text{底}} + S_{\text{侧}}$$

$$= \pi a^2 + \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy \quad D = \{(x, y) | x^2 + y^2 \leq a^2\}.$$

$$= \pi a^2 + \iint_D \sqrt{1 + \frac{4}{a^2}(x^2 + y^2)} dx dy.$$

极坐标代换.

$$= \pi a^2 + \int_0^{2\pi} d\theta \int_0^a \sqrt{1 + \frac{4}{a^2} \rho^2} \cdot \rho d\rho.$$

$$= \pi a^2 + 2\pi \frac{a^2}{8} \cdot \frac{2}{3} (\sqrt{5} - \sqrt{3})$$

$$= \frac{\pi a^2 (\sqrt{5} + 1)}{6} = \frac{5\pi a^2 (\sqrt{5} + 1)}{6}.$$

[由于均匀, 不妨全 $\rho(x, y, z) = 1$].

2. (1).

$$\text{伸缩变换 } \begin{cases} x = au \\ y = bv \\ z = cw \end{cases} \quad \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = abc.$$

$$\text{再球变换 } \begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \quad \left| \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} \right| = abc r^2 \sin \theta.$$

科目: 微积分

章节: 3-5

做题人: 游宇铭

$$\text{可得 } M = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \sin\theta d\theta \int_0^1 r^2 dr \cdot abc = \frac{\pi}{6} abc.$$

$$\cancel{M_{yz}} = \cancel{AA}$$

$$M_{yz} = \int_0^{\frac{\pi}{2}} \cos\varphi d\varphi \int_0^{\frac{\pi}{2}} \sin^2\theta d\theta \int_0^1 r^3 dr \cdot abc = \frac{\pi-1}{4} abc.$$

$$\text{同理 } M_{xz} = \frac{\pi-1}{4} abc, \quad M_{xy} = \frac{\pi}{16} abc^2.$$

$$\therefore \text{重心为 } \left(\frac{3(\pi-2)}{4\pi} a, \frac{3(\pi-2)}{4\pi} b, \frac{3}{8} c \right).$$

(2) [计算小块的重心]

$$\text{柱坐标: } \Omega = \{(p, \theta, z) \mid \frac{\sqrt{2}z}{\sqrt{2}\cos(\frac{\pi}{4}-\theta)} \leq p \leq \frac{\sqrt{2}z}{\sqrt{2}\cos(\frac{\pi}{4}+\theta)}, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq z \leq 2\}.$$

$$M = \int_0^2 dz \int_0^{\frac{\pi}{2}} d\theta \int_{\frac{\sqrt{2}z}{\sqrt{2}\cos(\frac{\pi}{4}-\theta)}}^{\frac{\sqrt{2}z}{\sqrt{2}\cos(\frac{\pi}{4}+\theta)}} p dp = \frac{4}{3} \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2\theta} \sin\theta d\theta$$

$$= \frac{2\sqrt{2}}{3} \int_0^{\frac{\pi}{2}} \sqrt{2} dz \int_0^{\frac{\pi}{2}} \sin\theta \cdot \left[\frac{1}{2\sqrt{2}\cos^2(\frac{\pi}{4}-\theta)} \right] d\theta$$

$$= \frac{32}{15} (1+\sqrt{2}). \quad M = \int_0^2 dz \int_0^{\sqrt{2}z} dy \int_{\sqrt{2}z-y}^{\sqrt{2}z} dx.$$

$$\cancel{M_{yz}} = \cancel{M_{xz}} = \pi - 2.$$

$$M_{yz} = \int_0^2 dz \int_0^{\sqrt{2}z} dy \int_{\sqrt{2}z-y}^{\sqrt{2}z} x dy = \frac{16}{15}. \quad \text{同理 } M_{xz} = \frac{16}{15}.$$

$$M_{xy} = \frac{8}{3} \left(\frac{\pi}{2} - 1 \right). \quad \therefore \text{重心为 } \left(\frac{16}{15(\pi-2)}, \frac{16}{15(\pi-2)}, \frac{4}{3} \right).$$

$$(3). M = \int_0^6 dy \int_0^4 dz \int_0^{\sqrt{4-z}} dx = 6 \cdot \frac{16}{3} = 32. \quad \text{同理}$$

$$M_{yz} = 24 \quad M_{xz} = 96 \quad M_{xy} = \frac{416}{5} \quad \text{重心为 } \left(\frac{3}{4}, 3, \frac{13}{5} \right).$$

$$3. (1). M = \int_0^1 dx \int_0^1 dy \int_0^1 (x+y+z) dz = 3 \int_0^1 dx \int_0^1 dy \int_0^1 x dz = 3 \cdot \frac{1}{2} \cdot 1 \cdot 1 = \frac{3}{2}.$$

$$\text{由对称性 } M_{xz} = M_{yz} = M_{xy} = \frac{5}{6}.$$

$$\therefore \text{重心为 } \left(\frac{5}{9}, \frac{5}{9}, \frac{5}{9} \right).$$

$$(2) \text{ 设球坐标 } \begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$

$$\Omega^* = \{(r, \theta, \varphi) \mid 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq 2\pi, 0 \leq r \leq 2a \cos\theta\}.$$

科目: 微积分

章节: 3-5

做题人: 游宇铭

$$\begin{aligned}\therefore M &= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} \frac{1}{r} \cdot r^2 \sin\theta dr \\ &= \frac{1}{3} \cdot 2\pi \cdot 2a^2 = \frac{4\pi a^2}{3}. \quad \text{由对称性.}\end{aligned}$$

$$M_{yz} = M_{xz} = 0. \quad (\text{即 } \bar{x} = \bar{y} = 0).$$

$$M_{xy} = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} \frac{1}{r} \cdot r^2 \sin\theta \cdot r \cos\theta dr = \frac{16a^3}{15} \pi.$$

$$\therefore \text{质心为 } (0, 0, \frac{4}{15}a).$$

$$(3). \text{ 设 } \rho(x, y) = \frac{k}{\sqrt{x^2 + y^2}}. \text{ 当 } \sqrt{x^2 + y^2} = 4 \text{ 时 } \rho(x, y) = 1 \text{ 则 } k = 4.$$

$$\text{极坐标 } \begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \therefore M = \int_0^{2\pi} d\theta \int_4^8 \frac{4}{\rho} \cdot \rho^2 d\rho = 2\pi \cdot 96 = 192\pi.$$

科目: 微积分

章节: 3.5

做题人: 刘改宁

4. (1) 做变换 $\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \\ z = z \end{cases}$, $\left| \frac{\partial(x, y, z)}{\partial(u, v, z)} \right| = \frac{1}{2}$, $\Omega: -1 \leq u \leq 1, -1 \leq v \leq 1, 0 \leq z \leq \frac{1}{2}(u^2+v^2)$

$$\begin{aligned} \therefore J_2 &= \iiint_{\Omega} x^2 + y^2 \, dv = \iiint_{\Omega} \frac{1}{2}(u^2 + v^2) \cdot \frac{1}{2} \, du \, dv \, dz \\ &= \int_{-1}^1 du \int_{-1}^1 dv \int_0^{\frac{1}{2}(u^2+v^2)} \frac{1}{4}(u^2 + v^2) \, dz = \int_{-1}^1 du \int_{-1}^1 \frac{1}{8}(u^2 + v^2) \, dv \\ &= \frac{28}{45} \end{aligned}$$

(2) 做球坐标变换 $\begin{cases} x = \rho \cos \theta \cos \varphi \\ y = \rho \cos \theta \sin \varphi \\ z = \rho \sin \theta \end{cases}$, $\left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)} \right| = \rho^2 \cos \theta$

$$\Omega \rightarrow \Omega^*: 0 \leq \rho \leq \sqrt{2}, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}.$$

$$\begin{aligned} \therefore J &= \iiint_{\Omega} (x^2 + y^2) \, dv = \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_0^{\sqrt{2}} \rho^2 \cos \theta \cdot \rho^2 \cos \theta \, d\rho \\ &= 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^3 \theta \, d\theta \int_0^{\sqrt{2}} \rho^4 \, d\rho = \frac{8}{3}\pi. \end{aligned}$$

5. 不妨设求沿 z 轴的转动惯量, 做球坐标变换

$$\begin{cases} x = \rho \sin \theta \cos \varphi \\ y = \rho \sin \theta \sin \varphi \\ z = \rho \cos \theta \end{cases}, \left| \frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)} \right| = \rho^2 \sin \theta$$

$$\begin{aligned} J &= \iiint (x^2 + y^2) K \sqrt{x^2 + y^2 + z^2} \, dv = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^R \rho^2 \sin^3 \theta \cdot K \rho \cdot \rho^2 \sin \theta \, d\rho \\ &= 2\pi K \int_0^{\pi} \sin^4 \theta \, d\theta \cdot \int_0^R \rho^5 \, d\rho \\ &= 2K\pi \cdot 2 \cdot \frac{2}{3} \cdot \frac{1}{6} R^6 \\ &= \frac{4}{9} K\pi R^6. \end{aligned}$$

科目: 微积分

章节: 3.5

做题人: 刘政宇

6. 做极坐标变换: $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}, \quad \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = r.$

$$\therefore M = \iint_{\Omega} \rho dv = \int_0^{\frac{\pi}{2}} d\theta \int_0^R r(1+r^2) dr = \frac{\pi}{8}(2R^2 + R^4)$$

$$M_x = M_y = \iint_{\Omega} x \rho dv = \int_0^{\frac{\pi}{2}} d\theta \int_0^R r^2 \cos \theta (1+r^2) dr = \frac{1}{3} R^5 + \frac{1}{3} R^3$$

$$\therefore \bar{x} = \bar{y} = \frac{M_y}{M} = \frac{8(3R^3 + 5R)}{15\pi(2 + R^2)}$$

$$\therefore \text{重心} \left(\frac{8(3R^3 + 5R)}{15\pi(2 + R^2)}, \frac{8(3R^3 + 5R)}{15\pi(2 + R^2)} \right)$$

$$(2) J = \iint_{\Omega} \rho y^2 dv = \int_0^{2\pi} d\theta \int_0^R r(1+r^2) r^2 \sin^2 \theta dr = \left(\frac{1}{3} R^6 + \frac{1}{5} R^8 \right) \pi.$$

7. 做柱坐标变换: $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = h \end{cases}, \quad \left| \frac{\partial(x, y, h)}{\partial(r, \theta, h)} \right| = r.$

$$\therefore F = \iint_{\Omega} \rho g h ds = \int_0^h dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho g z d\theta = \frac{1}{2} \pi \rho g h^2$$

$$\Omega = -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq z \leq h$$

$$8. F = \iint_{\Omega} \rho g h \cos \theta ds = \int_0^h dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \rho g z \cos \theta d\theta = \frac{1}{2} \pi \rho g h^2$$

8. 以球心为原点建立球坐标系

$$F_{\perp} = \iint_{\Omega} \rho g (h - \rho \cos \theta) \cos \theta \cdot \rho d\theta \cdot \rho \sin \theta d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \rho g \rho^2 (h - \rho \cos \theta) \cos \theta \sin \theta d\theta$$

$$= 2\pi \rho g \rho^2 \left(h \cdot \frac{\pi}{2} - \rho \cdot \frac{2}{3} \right) = \pi \rho g a^2 \left(\pi h - \frac{4}{3} a \right).$$

$$F_{\perp} = \iint_{\Omega} \rho g (h + \rho \cos \theta) \cos \theta \cdot \rho d\theta \cdot \rho \sin \theta d\phi = \pi \rho g a^2 \left(\pi h + \frac{4}{3} a \right).$$

科目: 微积分

章节: 3.5

做题人: 刘政宇

9. (1) 做柱坐标变换: $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}, \Omega = \{(z, r, \theta) \mid 0 \leq z \leq R, 0 \leq r \leq R, 0 \leq \theta \leq 2\pi\}$

由对称性知, F 沿 z 轴方向, 令 $\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$

$$\therefore F = \iiint \frac{Gm\rho(z-a)r}{(r^2+(z-a)^2)^{\frac{3}{2}}} dv = \int_0^{2\pi} d\theta \int_R^R dz \int_{-\frac{R-z}{2}}^{\frac{R-z}{2}} \frac{Gm\rho(z-a)r}{(r^2+(z-a)^2)^{\frac{3}{2}}} dr$$

$$= \cancel{4\pi} \int_{-R}^R \frac{Gm\rho}{\sqrt{R^2+(z-a)^2}} dz$$

$$= \frac{4\pi}{2} Gm\rho \int_{-R}^R \left(\frac{a-z}{\sqrt{R^2+(z-a)^2}} - 1 \right) dz$$

$$= \frac{4\pi}{2} Gm\rho \int_{-R}^R \left(\frac{a}{\sqrt{R^2+(z-a)^2}} - \frac{z}{\sqrt{R^2+(z-a)^2}} - 1 \right) dz$$

$$= \frac{4\pi}{2} Gm\rho \left[\frac{-a\sqrt{R^2+(z-a)^2}}{R} \Big|_{-R}^R + \frac{1}{3R^2} (R^2+(z-a)^2)^{\frac{3}{2}} \Big|_{-R}^R - 2R \right]$$

$$= \frac{4\pi}{2} Gm\rho \left[-2R + \frac{2R}{3R^2} - 2R \right]$$

$$= \frac{GmM}{a^2}$$

1. 证明: 先证明 $\iint_D f(x,y) dx dy \geq 0$.

$\because f \in R(D)$ 故 Riemann 和 $\sum_j \sum_i f(\xi_j, \eta_i) \Delta \sigma_{ij}$ 与分割 T 和取点无关

又 $f > 0$, $f(\xi_j, \eta_i) > 0$, 上述 Riemann 和式 > 0 .

$\therefore |T| \rightarrow 0$, 由极限的保号性, $\iint_D f(x,y) dx dy \geq 0$.

又 f 在 I 上的不连续点集为零面积集, 故一定 $\exists D \subset I$, 使得 $f \in C(D)$.
且 $\sigma(D) > 0$, $\iint_D f(x,y) dx dy = \iint_D f(x,y) dx dy + \iint_{I \setminus D} f(x,y) dx dy$.

考察 $\iint_D f(x,y) dx dy$ (可将 D 取为圆形闭域, 故 D 是有界闭集)

又 $f \in C(D)$, 故 f 在 D 上有最小值 $m > 0$.

$$\iint_D f(x,y) dx dy \geq \iint_D m dx dy = m \sigma(D) > 0.$$

$$\text{又 } \iint_{I \setminus D} f(x,y) dx dy \geq 0 \text{ (前证)}.$$

$$\Rightarrow \iint_I f(x,y) dx dy > 0. \text{ 证毕!}$$

2. 证明: 先证充分性: 只证有唯一间断点的情况. (其余可类似).

设 (ξ, η) 为 f 在 I 上的间断点, 则 f 在 $I \setminus (\xi, \eta)$ 上连续.

又 $I \setminus (\xi, \eta)$ 为有限闭集, 故 f 在 $I \setminus (\xi, \eta)$ 上一致连续.

$\forall \varepsilon > 0, \exists \delta > 0$, 使得 $\forall x', x'' \in I \setminus (\xi, \eta)$, 只要 $\|x' - x''\| < \delta$, 就有 $|f(x') - f(x'')| < \frac{\varepsilon}{2(b-a)(d-c)}$.

故 $\exists$ 分割 T , 使得 $U(f, T) - L(f, T) < \varepsilon$.

记 ω 为 f 在区域 D 上的振幅. ($D \subset I$). 即 $\omega_0 = M_0 - m_0$, (M_0, m_0 分别为 f 在 D 上的最大值和最小值).

再记 $\omega(f, T) = \sum_j \sum_i \omega_{ij} \Delta \sigma_{ij}$. 显然有 $\omega(f, T) = U(f, T) - L(f, T)$ (由上式结果).

而双重 n 号各, 把 I 分成 n^2 个小矩形, 每个区域直径为 $\sqrt{\left(\frac{b-a}{n}\right)^2 + \left(\frac{d-c}{n}\right)^2} = \frac{\sqrt{(b-a)^2 + (d-c)^2}}{n}$.

对 $[a, b] \times [c, d]$ 及 $n > \frac{\sqrt{(b-a)^2 + (d-c)^2}}{\delta}$, 则小区域直径 $< \delta$. 由一致连续性.

在 I_{ij} 上, $\omega_{ij} < \frac{\varepsilon}{2(b-a)(d-c)}$ (假设 i, j 不含 (ξ, η)).

而若含 (ξ, η) , 它至多落在一个小矩形里, 为 f 在此小矩形里的上界为 M .

再将 n 放大 (若有必要的话), 使得 $(M - m) \frac{(b-a)(d-c)}{n^2} < \frac{\varepsilon}{2(b-a)(d-c)}$.

故 $\omega(f, T) = \sum_j \sum_i \omega_{ij} \Delta \sigma_{ij} < \sum_j \sum_i \frac{\varepsilon}{2(b-a)(d-c)} \Delta \sigma_{ij} < \frac{\varepsilon}{2} < \varepsilon$ 故 $f \in R(I)$.

再记必要性质: 记 $\omega_{x_0}(\delta) = \sup_{x, x' \in B(x_0, \delta)} |f(x) - f(x')|$, $\omega(x_0) = \lim_{\delta \rightarrow 0^+} \omega_{x_0}(\delta)$.

则显然 f 在 x_0 处连续 $\Leftrightarrow \omega(x_0) = 0$. 因此 f 的间断点集合 $D = \{(x, y) | \omega(x, y) > 0\}$

记 $D_n = \{(x, y) | \omega(x, y) > \frac{1}{n}\}$. 则 $D = \bigcup_{n=1}^{\infty} D_n$, 而 $D_1 \subset D_2 \subset \dots \subset D_n \subset \dots$

D 为零测度集 $\Leftrightarrow$ 每个 D_n 为零测度集.

已知 $f \in R(D)$, f 有界. 任给 $\omega_0 > 0$, 任给 $\varepsilon > 0$, 存在 I 的分割 T , 使 $U(f, T) - L(f, T) < \omega_0 \varepsilon$

$$\begin{aligned} \text{而 } U(f, T) - L(f, T) &= \sum_j \sum_i f(\xi_{ij}, \eta_{ij}) \Delta \sigma_{ij} = \sum_j \sum_i f(\xi_{ij}, \eta_{ij}) \Delta \sigma_{ij} + \sum_j \sum_i f(\xi_{ij}, \eta_{ij}) \Delta \sigma_{ij} \\ &\quad \left(I_{ij} \cap D(\omega_0) = \emptyset \right) \quad \left(I_{ij} \cap D(\omega_0) \neq \emptyset \right) \\ &\geq \sum_j \sum_i f(\xi_{ij}, \eta_{ij}) \Delta \sigma_{ij} \geq \sum_j \sum_i \omega_0 \Delta \sigma_{ij} = \omega_0 \sum_j \sum_i \Delta \sigma_{ij} \end{aligned}$$

从而 $\sum_j \sum_i \Delta \sigma_{ij} < \varepsilon$. 即 $D(\omega_0)$ 可被有限个边长小于 ε 的正方形覆盖 $\Rightarrow D(\omega_0)$ 为零测度集.
综上, $D = \bigcup_{n=1}^{\infty} D_n$ 为零测度集.

3. 证明: 对 $[a, b], [c, d]$ 均作 n 等分. 则 I 被分成 n^2 个矩形区域

若每块矩形区域 I_{ij} 上振幅 $\omega_{ij} = f(\xi_{ij}, \eta_{ij}) - f(\xi_{i-1}, \eta_{j-1})$

$$\omega(f, T) = \sum_j \sum_i \omega_{ij} \Delta \sigma_{ij} \quad \Delta \sigma_{ij} = \frac{\sigma(I)}{n^2}$$

$$\begin{aligned} &= \frac{\sigma(I)}{n^2} \sum_j \sum_i \omega_{ij} = \frac{\sigma(I)}{n^2} \left[\sum_{i=1}^n f(\xi_i, c) + \sum_{j=1}^n f(b, \eta_j) - \sum_{i=1}^n f(a, \eta_j) - \sum_{j=1}^n f(\xi_i, d) \right] \\ &\leq \frac{\sigma(I)}{n^2} \left[n f(b, c) + (n-1) f(b, c) \right] \\ &\leq \frac{2n-1}{n^2} f(b, c) \sigma(I) \leq \frac{2}{n} \sigma(I) f(b, c) \end{aligned}$$

故 $n > \frac{2 \sigma(I) f(b, c)}{\varepsilon}$. 则 $\omega(f, T) < \varepsilon$. 故 $f \in R(I)$. 证毕!

4. 证明:

1) 取 $\varepsilon_0 = \frac{1}{10}$. $\forall$ 分割 T : 将 T 分解成 $T_x \times T_y$

$T_x: 0 = x_0 < x_1 < \dots < x_n = 1$. $T_y: 0 = y_0 < y_1 < \dots < y_n = 1$.

而依法①: 若每个子矩形区域都无有理点 (ξ, η) . 即 $f(\xi, \eta) = 1$. 构成 Riemann 和 R_1 .

而依法②: 若 $y_i < \frac{2}{3}$ 时. 无有理点. 若 $y_i \geq \frac{2}{3}$ 时. 有有理点 (ξ, η) .

此时 $f(\xi, \eta) = 3y^2 \geq \frac{4}{9}$. 构成 Riemann 和 R_2 .

$$R_2 - R_1 = \sum_{\substack{j \\ (y_j < \frac{2}{3})}} (1-1) \sigma_{ij} + \sum_{\substack{j \\ (y_j \geq \frac{2}{3})}} (3y_j^2 - 1) \sigma_{ij} \geq \frac{1}{3} \sum_{\substack{j \\ (y_j \geq \frac{2}{3})}} \sigma_{ij} = \frac{1}{9} > \varepsilon_0$$

故 $\iint_D f(x, y) dx dy$ 不存在.

$$12) \text{ 记 } F(x) = \int_0^1 f(x, y) dy$$

$$\text{当 } x \in \mathbb{R} \text{ 时, } f(x, y) = 3y^2, \quad F(x) = \int_0^1 3y^2 dy = 1$$

$$\text{当 } x \notin \mathbb{R} \text{ 时, } f(x, y) = 1, \quad F(x) = \int_0^1 1 dy = 1$$

故 $F(x) \equiv 1$, 在 $x \in [0, 1]$ 上连续, 故可积

$$\int_0^1 \left(\int_0^1 f(x, y) dy \right) dx = \int_0^1 1 dx = 1$$

$$13) \text{ 记 } F(y) = \int_0^1 f(x, y) dx$$

固定 $y \in [0, 1]$, $f(x, y)$ 关于 x 处处不连续, 间断点不是零测集.

故 $f(x, y)$ 关于 x 不可积, $F(y)$ 不存在

故 $\int_0^1 \left(\int_0^1 f(x, y) dx \right) dy$ 不存在

$$\begin{aligned} 5. 1) \iint_D |y-x^2| dx dy &= \int_{-1}^1 dx \int_0^2 (y-x^2) dy + \int_{-1}^0 dx \int_0^x (x^2-y) dy + \int_0^1 dx \int_0^x (x^2-y) dy \\ &= \int_{-1}^1 \left(\frac{1}{2}y^2 - xy \right) \Big|_0^2 dx + \int_{-1}^0 \left(xy - \frac{1}{2}y^2 \right) \Big|_0^x dx \\ &= \int_{-1}^1 \left(\frac{1}{2}x^2 - 2x^2 + 2 \right) dx + \int_{-1}^0 \frac{1}{2}x^2 dx \\ &= \int_{-1}^1 (x^2 - 2x^2 + 2) dx = \frac{46}{15} \end{aligned}$$

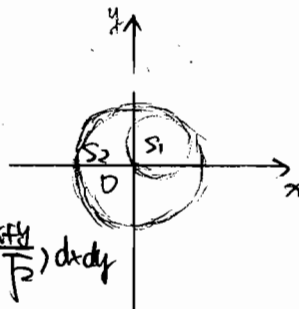
$$12) \frac{x+y}{\sqrt{2}} - x^2 - y^2 \geq 0$$

$$(x - \frac{\sqrt{2}}{2})^2 + (y - \frac{\sqrt{2}}{2})^2 \leq \frac{1}{4}, \text{ 即圆 } S_1 \text{ 内部.}$$

$$\iint_D \left| \frac{x+y}{\sqrt{2}} - x^2 - y^2 \right| dx dy = \iint_{S_1} \left(\frac{x+y}{\sqrt{2}} - x^2 - y^2 \right) dx dy + \iint_{S_2} \left(x^2 + y^2 - \frac{x+y}{\sqrt{2}} \right) dx dy$$

作极坐标变换 $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$

$$\begin{aligned} \text{原式} &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_0^{\frac{1}{\sqrt{2}(\sin \theta + \cos \theta)}} \left(\frac{r}{\sqrt{2}} (\sin \theta + \cos \theta) - r^2 \right) dr + \int_0^{2\pi} d\theta \int_{\frac{1}{\sqrt{2}(\sin \theta + \cos \theta)}}^1 \left(r^2 - \frac{r}{\sqrt{2}} (\sin \theta + \cos \theta) \right) dr \\ &= \frac{1}{\sqrt{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sin \theta + \cos \theta)^3 d\theta + \int_0^{2\pi} \left[\frac{1}{3} (\sin \theta + \cos \theta) - \frac{1}{3} - \frac{1}{\sqrt{2}} (\sin \theta + \cos \theta) \right] d\theta \\ &= \frac{9\pi}{16} \end{aligned}$$



科目: 微积分

章节: 第三章 总复习

做题人: 阮哲

6. 解: 作极坐标变换 $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$

$$D \rightarrow D^*: \{(r, \theta) \mid 0 \leq \theta \leq 2\pi, \frac{1}{2} \leq r \leq 1\}$$

$$\begin{aligned} \lim_{t \rightarrow 0} \iint_D \ln(x^2 + y^2) dx dy &= \lim_{t \rightarrow 0} \int_0^{2\pi} d\theta \int_{\frac{1}{2}}^1 r \ln r^2 dr = 4\pi \lim_{t \rightarrow 0} \int_{\frac{1}{2}}^1 r \ln r dr \\ &= 4\pi \lim_{t \rightarrow 0} \left[\frac{1}{2} r^2 \ln r \Big|_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 \frac{1}{2} r^2 \cdot \frac{1}{r} dr \right] \\ &= 4\pi \lim_{t \rightarrow 0} \left[\frac{1}{2} r^2 \ln r \Big|_{\frac{1}{2}}^1 - \frac{1}{4} r^2 \Big|_{\frac{1}{2}}^1 \right] \\ &= 4\pi \lim_{t \rightarrow 0} \left[\frac{1}{2} r^2 \ln r \Big|_{\frac{1}{2}}^1 - \frac{1}{4} + \frac{1}{4} r^2 \right] \\ &= -\pi \end{aligned}$$

7. 解:

1) 极坐标变换.

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} d\theta \int_0^{+\infty} r e^{-r^2} dr \\ &= \frac{\pi}{2} \cdot \frac{1}{2} \int_0^{+\infty} e^{-x} dx = \frac{\pi}{4} e^{-x} \Big|_{+\infty}^0 = \frac{\pi}{4} \end{aligned}$$

2) 极坐标变换

$$I = \int_0^{2\pi} d\theta \int_0^{+\infty} r e^{-r^2} \sin^2 \theta dr = \pi \int_0^{+\infty} e^{-u} \sin^2 u du$$

$$\text{设 } I_1 = \int_0^{+\infty} e^{-u} \sin u du, \quad I_2 = \int_0^{+\infty} e^{-u} \cos u du$$

$$I_1 = -e^{-u} \sin u \Big|_0^{+\infty} + \int_0^{+\infty} e^{-u} \cos u du = I_2$$

$$I_2 = -e^{-u} \cos u \Big|_0^{+\infty} - \int_0^{+\infty} e^{-u} \sin u du = -1 - I_1$$

$$\text{故 } I_1 = I_2 = -\frac{1}{2}$$

$$\text{即 } I = \pi I_1 = -\frac{\pi}{2}$$

科目: 微积分

章节: 第三章 重积分

做题人: 叶哲宇

13) 考虑线性变换

$$\begin{cases} x = au + bv \\ y = cu + dv \end{cases} \quad \text{使得 } 2xy - x^2 - y^2 \Rightarrow au^2 + bu^2$$

得到一维变换: $\begin{cases} x = u + v \\ y = 3u + \frac{1}{3}v \end{cases} \quad \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} 1 & 1 \\ 3 & \frac{1}{3} \end{vmatrix} = -\frac{5}{3}$

$$2xy - x^2 - y^2 = -5u^2 - \frac{1}{3}v^2$$

$$I = \iint_{\mathbb{R}^2} e^{-5u^2 - \frac{1}{3}v^2} \cdot \frac{5}{3} du dv$$

作极坐标变换

$$\begin{cases} u = \frac{\sqrt{3}}{5} r \cos \theta \\ v = \frac{\sqrt{3}}{5} r \sin \theta \end{cases} \quad \frac{D(u,v)}{D(r,\theta)} = \frac{3}{5} r$$

$$I = \int_0^{2\pi} d\theta \int_0^{+\infty} e^{-r^2} \cdot \frac{3}{5} \cdot \frac{5}{3} r dr$$

$$= 2\pi \int_0^{+\infty} r e^{-r^2} dr$$

$$= \pi \int_0^{+\infty} e^{-u} du = \pi$$

8. 证明: 若 $X = (x, y)^T$, $U = (t, s)^T$ 作正交变换 $X = QU$, 其中 $Q^T = I$, 且 $U = Q^T X$, 中

$$\text{则 } X^T X \leq 1 \Rightarrow U^T Q^T Q U \leq 1 \Rightarrow U^T U \leq 1, \quad t = \frac{1}{\sqrt{a^2+b^2}}(ax+by)$$

$$\text{即 } s^2 + t^2 \leq 1$$

$$I = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} ds \int_{-\sqrt{a^2+b^2-t^2}}^{\sqrt{a^2+b^2-t^2}} f(t\sqrt{a^2+b^2-t^2}+c) dt$$

$$= \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} f(t\sqrt{a^2+b^2-t^2}+c) dt$$

科目: 微积分

章节: 第三章 定积分

做题人: 阮哲言

9. 证明: 作变换 $\begin{cases} t = xy \\ u = y \end{cases}$ $\frac{D(t, u)}{D(x, y)} = \begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix} = y$
 $\Rightarrow \frac{D(x, y)}{D(t, u)} = \frac{1}{y}$

$\therefore 0 \leq x \leq 1, 0 \leq y \leq 1.$

$\Rightarrow 0 \leq t \leq 1.$

又 $u = y = \frac{t}{x} \geq t.$

故 $t \leq u \leq 1$

$$I = \int_0^1 dx \int_t^1 t^x \frac{1}{u} du$$

$$= - \int_0^1 t^x \ln t dx$$

考虑 $I' = \int_0^1 t^x dx - I = \int_0^1 t^x (\ln t + 1) dx$

则 $(t^x)' = t^x (\ln t + 1)$

故 $I' = \int_0^1 d(t^x) = t^x \Big|_0^1 = 0$

即 $I = \int_0^1 t^x dx$. 得证

11. 证: $\iint_D f(x,y) \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx dy$

$$= \iint_D f(x,y) \frac{\partial^2 f}{\partial x^2} dx dy + \iint_D f(x,y) \frac{\partial^2 f}{\partial y^2} dx dy$$

$$\iint_D f(x,y) \frac{\partial^2 f}{\partial x^2} dx dy = \int_{-1}^1 dy \cdot \int_{f(y)}^{f(y)} f(x,y) \frac{\partial^2 f}{\partial x^2} dx$$

$$= \int_{-1}^1 dy \cdot \int_{f(y)}^{f(y)} f(x,y) d\left(\frac{\partial f}{\partial x}\right)$$

$$= \int_{-1}^1 dy \cdot \left(f(x,y) \cdot \frac{\partial f}{\partial x} \Big|_{f(y)}^{f(y)} - \int_{f(y)}^{f(y)} \frac{\partial f}{\partial x} dx \right)$$

由于 $f(x,y)$ 在 ∂D 上为 0

$$\text{故 } \iint_D f(x,y) \frac{\partial^2 f}{\partial x^2} dx dy = - \int_{-1}^1 dy \cdot \int_{f(y)}^{f(y)} \frac{\partial^2 f}{\partial x^2} dx$$

$$\text{由于 } \left| \frac{\partial f}{\partial x} \right|^2 \geq 0 \text{ 故 } - \int_{-1}^1 dy \cdot \int_{f(y)}^{f(y)} \frac{\partial^2 f}{\partial x^2} dx \leq 0$$

$$\text{同理 } \iint_D f(x,y) \frac{\partial^2 f}{\partial y^2} dx dy \leq 0$$

$$\text{故, 综上, } \iint_D f(x,y) \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx dy \leq 0$$

12. 证: 若 $\exists$ 点 $A(x_0, y_0, z_0)$ 使得 $f(A) \neq 0$.

不妨设 $f(A) > 0$. 则由 $f(x,y,z)$ 在 Ω 内连续

故, 取 $f(A) = 2\varepsilon$. 则, 对于 $\varepsilon = \frac{f(A)}{2}$, $\exists r$, 使得 $S(A, r)$ 中点

$B(x,y,z)$ 有 $f(B) > \varepsilon$

故, 取 $\Omega_1 = \Omega \cap S(A, r)$ $\iiint_{\Omega_1} f(x,y,z) dx dy dz > 0$. 矛盾.

故 $f(A) \leq 0$ 同理 $f(A) \geq 0$ 故 $f(A) = 0$.

由 A 的任意性, 得 $f(x,y,z) = 0$, $(x,y,z) \in \Omega$.

科目: 微积分

章节: 第三章总复习题

做题人: 杨本欣

$$\begin{aligned}
 13. \text{解: } I &= \int_0^1 dx \cdot \int_0^{1-x} dz \cdot \int_0^{1-z-x} (1-y) \cdot e^{-(1-y-z)^2} \cdot dy \\
 &= \int_0^1 dy \int_0^{1-y} dz \int_0^{1-y-z} (1-y) \cdot e^{-(1-y-z)^2} \cdot dx \\
 &= \int_0^1 dy \int_0^{1-y} (1-y) \cdot (1-y-z) \cdot e^{-(1-y-z)^2} \cdot dz \\
 &= \frac{1}{2} \int_0^1 (1-y) \cdot (1 - e^{-(1-y)^2}) \cdot dy \\
 &= \frac{1}{4} \int_0^1 (1 - e^{-t}) dt \quad (t = (1-y)^2) \\
 &= \frac{1}{4} + \frac{1}{4e} - 1 = \frac{1}{e} - \frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 14. \text{证: } & \int_0^1 dx \cdot \int_x^1 f(x) \cdot f(y) dy = \int_0^1 dx \cdot \int_0^x f(x) \cdot f(y) dy \\
 \text{故} & \int_0^1 dx \cdot \int_x^1 f(x) \cdot f(y) dy = \frac{1}{2} \int_0^1 dx \cdot \int_0^1 f(x) \cdot f(y) dy \\
 &= \frac{1}{2} \int_0^1 f(x) \cdot dx \cdot \int_0^1 f(y) dy \\
 &= \frac{1}{2} \cdot \left(\int_0^1 f(x) \cdot dx \right)^2
 \end{aligned}$$

对于 $A(x, y, z)$ 满足 $0 \leq z \leq y \leq x \leq a$.

考虑 $A_1(x, z, y), A_2(y, x, z), A_3(y, z, x),$
 $A_4(z, x, y), A_5(z, y, x).$

有 $f(A_i) = C \quad (i=1, 2, \dots, 5)$

$$\begin{aligned}
 \text{故} & \int_0^a dx \cdot \int_0^x dy \cdot \int_0^y f(x) \cdot f(y) \cdot f(z) dz \\
 &= \frac{1}{6} \int_0^a dx \cdot \int_0^a dy \cdot \int_0^a f(x) \cdot f(y) \cdot f(z) \cdot dx \cdot dy \cdot dz \\
 &= \frac{1}{6} \cdot \left(\int_0^a f(x) \cdot dx \right)^3.
 \end{aligned}$$

科目: 微积分

章节: 第三章 总复习题

做题人: 齐本优

15. 解: 设 $Q = Q^T$

$$H(x) = x Q A_1 Q^T x^T$$

$$A_1 = \begin{bmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2^2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^2 \end{bmatrix} \quad |A_1| = |A|.$$

$$\text{设 } x \cdot Q = (y_1, y_2, y_3).$$

$$I = \iiint_{H(x) \leq 1} e^{\sqrt{H(x)}} \cdot dx_1 \cdot dy_1 \cdot dz_1$$

$$= \iiint e^{\sqrt{a_1^2 y_1^2 + a_2^2 y_2^2 + a_3^2 y_3^2}} \cdot |Q| dy_1 dy_2 dy_3.$$

$$= \frac{1}{a_1 a_2 a_3} \iiint e^{\rho} \cdot \rho^2 \sin \theta \cdot d\rho \cdot d\theta \cdot d\varphi.$$

$$= \frac{1}{a_1 a_2 a_3} \cdot \iiint e^{\rho} \cdot \rho^2 \sin \theta \cdot d\rho \cdot d\theta \cdot d\varphi.$$

$$= \frac{1}{a_1 a_2 a_3} \cdot 2\pi \cdot \int_0^{\pi} e^{\rho} \cdot \rho^2 \sin \theta \cdot d\rho \cdot d\theta$$

$$= \frac{2\pi \sqrt{|A|}}{a_1 a_2 a_3} (e - 2 \int_0^1 \rho e^{\rho} d\rho)$$

$$= 2\pi \sqrt{|A|} \cdot (e - 2e + 2e - 2)$$

$$= 2\pi (e - 2) \sqrt{|A|}$$

16. 解: ① $V = \iiint dx dy dz$

$$= \iiint \rho^2 \sin \theta \cdot d\rho \cdot d\theta \cdot d\varphi$$

$$= \int_1^2 \rho^2 \cdot d\rho \cdot \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin \theta \cdot d\theta \cdot \int_0^{2\pi} d\varphi$$

$$= \frac{7}{3} \cdot \frac{\sqrt{2}}{2} \cdot 2\pi = \frac{7\sqrt{2}}{3} \pi$$

② 由对称性知 $\bar{x}$ 在 x 轴上. $y = a \rho \cos \theta, z = b \rho \sin \theta.$

$$\bar{x} = \frac{\iiint x \cdot dV}{\iiint dV} = \frac{ab \int_0^1 \rho d\rho \int_0^{2\pi} d\varphi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \rho^2 \sin \theta \cdot d\theta}{ab \int_0^1 \rho d\rho \int_0^{2\pi} d\varphi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \rho^2 \sin \theta \cdot d\theta} = \frac{5}{16}$$

科目: 微积分

章节: 第三章总复习题

做题人: 杨本优

17.

$$17. \text{解} \quad I_n = \int_{[0,1]^n} (x_1^2 + x_2^2 + \dots + x_n^2) dx_1 \dots dx_n.$$

$$= \int_{[0,1]^n} (x_1^2 + x_2^2 + \dots + x_{n-1}^2 + \frac{1}{3}) dx_1 \dots dx_{n-1}$$

$$= \int_{[0,1]^{n-2}} \int_{[0,1]^{n-1}} (x_1^2 + x_2^2 + \dots + x_{n-2}^2 + \frac{2}{3}) dx_1 \dots dx_{n-1}$$

$$= \int_{[0,1]} (x_1^2 + \frac{n-1}{3}) dx_1$$

$$= \frac{n}{3}$$

$$a_n = \int_{[0,1]^n} (x_1 + x_2 + \dots + x_n)^2 dx_1 \dots dx_n$$

$$= \int_{[0,1]^{n-1}} \int_{[0,1]} ((x_1 + x_2 + \dots + x_{n-1}) + \frac{1}{3} + (x_1 + x_2 + \dots + x_{n-1})^2) dx_1 \dots dx_{n-1}$$

$$= a_{n-1} + \frac{n-1}{2} + \frac{1}{3} = a_{n-1} + \frac{n}{2} - \frac{1}{6}$$

$$\text{又 } a_1 = \int_0^1 x_1^2 dx_1 = \frac{1}{3}$$

$$\text{故 } a_n = \frac{1}{3} + \frac{n(n+1)-1}{2} - \frac{n}{6} + \frac{1}{6}$$

$$= \frac{n^2}{2} - \frac{n}{3}$$

科目: 微积分

章节: 第三章总复习题

做题人: 桥本优

$$\begin{aligned}
 18. \text{解: 由 } V &= \int_0^{a_1} \int_0^{a_2 - \frac{a_1}{a_2}x} \cdots \int_0^{a_n - a_{n-1}(\frac{x_1}{a_1} + \frac{x_2}{a_2} + \cdots + \frac{x_{n-1}}{a_{n-1}})} dx_n \cdots \\
 &= a_n \int_0^{a_1} dx_1 \cdots \int_0^{a_{n-1} - a_{n-1}(\frac{x_1}{a_1} + \cdots + \frac{x_{n-2}}{a_{n-2}})} dx_{n-1} \\
 &= \frac{a_n \cdot a_{n-1}}{2} \int_0^{a_1} dx_1 \cdots \int_0^{a_{n-2} - a_{n-2}(\frac{x_1}{a_1} + \cdots + \frac{x_{n-3}}{a_{n-3}})} dx_{n-2} \\
 &\quad \cdots \\
 &= \frac{a_n a_{n-1} \cdots a_2}{2 \cdot 3 \cdots (n-1)} \int_0^{a_1} (1 - \frac{x_1}{a_1})^{n-1} dx_1 \\
 &= \frac{a_1 a_2 \cdots a_n}{n!}
 \end{aligned}$$

$$\Rightarrow V = \frac{1}{n!} \quad (\text{由上小题结论可得}).$$