

数学作业纸

(科目: 微积分)

班级: 计 5

姓名: 王 琦

编号: 2013011398

第 1 页

习题 1.2.

1. (1). 设圆锥底半径为 r , $r = \sqrt{l^2 - h^2}$. $V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi h (l^2 - h^2)$.

(2). 由题 $x^2 + y^2 + z^2 = 4$. $S = 2xy + 2(x+y)\sqrt{4-x^2-y^2}$.

(3). 由题, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. $V = 8xy\sqrt{c^2 - \frac{c^2x^2}{a^2} - \frac{c^2y^2}{b^2}} = 8cxy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$.

(4). $b = \sqrt{(x-1)^2 + y^2 + (z+1)^2} - \sqrt{2}$

1. (1). 定义域: $D = \{(x, y) | 4x^2 + y^2 \geq 1\}$.

(2). 定义域: $D = \{(x, y) | xy > 0\}$.

(3). 定义域: $D = \{(x, y, z) | y^2 \geq 1 \text{ 且 } x^2 + y^2 + z^2 \leq 4\}$.

(4). 定义域: $D = \{(x, y) | x^2 + y^2 \neq 0\}$.

2. 令 $u = x+y$, $v = \frac{y}{x}$. 则 $x = \frac{u}{v+1}$, $y = \frac{uv}{v+1}$. 有 $f(u, v) = \frac{u^2}{(v+1)^2} - \frac{u^2 v^2}{(v+1)^2} = \frac{u^2(1-v)}{1+v}$.

于 $f(x, y) = \frac{x^2(1-y)}{1+y}$.

4. (1). $f(tx, ty, tz) = f(x, y, z)$. 齐次式, $k=0$.

(2). $f(tx, ty, tz) = t^{\frac{3}{2}}(\sqrt{x^2+y^2+z^2} + t^{\frac{3}{2}}xyz)$. 不是齐次式.

(3). $f(tx_1, tx_2, \dots, tx_n) = t^2 f(x_1, x_2, \dots, x_n)$. 齐次式, $k=2$.

5. 略.

6. 由题知 F 的值域 $F(D) = \{(u, v) | 1 \leq u \leq 4, 1 \leq v \leq 2\}$. 将 $x = at^2$, 有 $u = -\frac{v^2}{a^2} + a^2$. 映射为抛物线.

7. (1). 由题 $\begin{cases} u = \frac{x}{R_1^2} \\ v = \frac{y}{R_2^2} \end{cases}$ 且 $x^2 + y^2 = R^2$. 则 $u^2 + v^2 = \frac{1}{R^2}$. 映射为圆.

(2). 设 $y = x = t$. 则 $\begin{cases} u = \frac{1}{2t} \\ v = \frac{1}{2t} \end{cases}$. 由于 $u=v$. 映射为直线.

数学作业纸

(科目: 微积分)

班级: 计25

姓名: 王奇

编号: 2013011318

第 2 页

习题 1.3

(10). 设 $\rho = x^2 + y^2$. 则 $(x, y) \rightarrow (0, 0) \Leftrightarrow \rho \rightarrow 0$. 注意到 $\lim_{\rho \rightarrow 0} \frac{\cos \rho}{\rho} = 1$. 所以所求极限存在, 且为 1.

(11). 注意到 $\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| = \left| \frac{x}{\sqrt{x^2+y^2}} \right| \cdot |y| \leq |y|$. 故所求极限存在, 且为 0.

(12). $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2}{e^{x+y}} = \frac{\lim_{(x,y) \rightarrow (0,0)} (x^2+y^2)}{\lim_{(x,y) \rightarrow (0,0)} e^{x+y}} = \frac{0}{1} = 0$. 所求极限存在, 且为 0.

(13). 若 (x, y) 沿直线 $y = kx$ ($k > 0$) 趋于 $(0, 0)$ 时.

$$\textcircled{1} x > 0. \quad \frac{x+y}{|x|+|y|} = \frac{x+kx}{x+kx} = 1. \quad \textcircled{2} x < 0. \quad \frac{x+y}{|x|+|y|} = \frac{x+kx}{-x-kx} = -1.$$

当 x 从不同方向趋于 0 时导致不同的极限值, 故极限不存在.

(14). 若 (x, y) 沿直线 $y = kx$ 趋于 $(0, 0)$ 时, $f(x, y) = \frac{x^2 - k^2 x^2}{x^2 + k^2 x^2} = \frac{1 - k^2}{1 + k^2}$. 不同的 k 值导致不同的极限.

故极限不存在.

(15). 设 $t = x^2$. 则原式可化为 $\frac{ty}{t^2+y^2}$. 由例题知极限不存在.

(16). 注意到 $\left| \frac{x^2 - y^2}{x^2 + y^2} - 0 \right| = |x - y| \cdot \left| \frac{x + y}{x^2 + y^2} \right| < 2|x - y|$. 而当 $(x, y) \rightarrow (0, 0)$ 时, $2|x - y| \rightarrow 0$.

故极限存在, 且为 0.

(17). $\lim_{(x,y) \rightarrow (0,0)} \frac{1 - \cos(xy)}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{\frac{1}{2}(xy)^2}{x^2 + y^2} = \frac{1}{2} \left(\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} \right)^2 = 0$. 极限存在, 且为 0.

(18). 若 (x, y) 沿直线 $y = kx$ 趋于 $(0, 0)$ 时, $\frac{x^2}{x^2 + y^2} = \frac{1}{1 + k^2}$. 不同的 k 值导致不同的极限. 故极限不存在.

(19). 设 $t = xy$. 则 $(x, y) \rightarrow (0, 0)$ 时, $t \rightarrow 0$.

$$\lim_{t \rightarrow 0} \frac{t - \sin t}{t(1 - \cos t)} = \lim_{t \rightarrow 0} \frac{t - (t - \frac{1}{6}t^3)}{t \cdot (\frac{1}{2}t^2)} = \frac{1}{3}. \text{ 极限存在, 且为 } \frac{1}{3}.$$

(20). 若 (x, y) 沿曲线 $x = ky^2$ 趋于 $(0, 0)$ 时, $\frac{x^2 + y^4}{(x^2 + y^4)^3} = \frac{k^2 y^4}{(k^2 + 1)^2 y^{12}} = \frac{k^2}{(k^2 + 1)^2}$.

不同的 k 值导致不同的极限. 故极限不存在.

(21). 设 $t = x^2 y$. 则当 $(x, y) \rightarrow (0, 0)$ 时, $t \rightarrow 0$.

$$\lim_{t \rightarrow 0} \frac{\sin t - \arcsin t}{t^3} = \lim_{t \rightarrow 0} \frac{(t - \frac{1}{6}t^3) - (t + \frac{1}{6}t^3)}{t^3} = -\frac{1}{3}. \text{ 极限存在, 且为 } -\frac{1}{3}.$$

(22). $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{\ln(x + \sin y)}{\sqrt{x^2 + y^2}} = \frac{\ln(3 + \sin 0)}{\sqrt{3^2 + 0^2}} = \frac{\ln 3}{3}$

(23). 注意到 $\left| \frac{x+y}{x^2+y^2+y^2} - 0 \right| \leq \frac{1}{2} \left| \frac{1}{x+y} \right|$. 而当 $x \rightarrow \infty, y \rightarrow \infty$ 时, $\frac{1}{x+y} \rightarrow 0$. 所以所求极限为 0.

(24). $\lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} (x^2 + y^2) e^{y-x} = \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \frac{x^2 + y^2}{e^{x-y}} \quad \alpha \quad \lim_{\substack{x \rightarrow +\infty \\ y \rightarrow +\infty}} \frac{(x+y)^2}{e^{x+y}} = \lim_{t \rightarrow +\infty} \frac{t^2}{e^t} = 0.$

而 $0 < \frac{x^2 + y^2}{e^{x+y}} < \frac{(x+y)^2}{e^{x+y}}$. 由夹逼定理知其极限为 0.

数学作业纸

(科目: 微积分)

班级: 计25

姓名: 马奇

编号: 2013011398

第 2 页

(4). 设 $\rho = \sqrt{x^2+y^2}$. 则 $\lim_{x \rightarrow 0, y \rightarrow 0} \frac{\ln(x^2+y^2)}{\sqrt{x^2+y^2}} = \lim_{\rho \rightarrow 0} \frac{\ln \rho^2}{\rho} = 0$.

(5). 注意到 $\frac{xy}{x^2+y^2} \leq \frac{1}{2}$. 而 $\lim_{x \rightarrow 0} (\frac{1}{2})^{x^2} = 0$. ~~$\frac{xy}{x^2+y^2}$~~

又 $-\frac{1}{2} \leq \frac{xy}{x^2+y^2}$. 而 $\lim_{x \rightarrow 0} (-\frac{1}{2})^{x^2} = 0$. 由夹逼定理 $\lim_{x \rightarrow 0} (\frac{xy}{x^2+y^2})^{x^2} = 0$

(6). $\lim_{x \rightarrow 0, y \rightarrow 0} \frac{x+y}{xy} = \lim_{x \rightarrow 0} \frac{1}{x} + \lim_{y \rightarrow 0} \frac{1}{y} = 0$.

3. (1). $\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} \sin \frac{\pi x}{2x+y} = \lim_{x \rightarrow 0} (\lim_{y \rightarrow 0} \sin \frac{\pi x}{2x+y}) = \lim_{x \rightarrow 0} 0 = 0$.

$\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} \sin \frac{\pi x}{2x+y} = \lim_{y \rightarrow 0} (\lim_{x \rightarrow 0} \sin \frac{\pi x}{2x+y}) = \lim_{y \rightarrow 0} \frac{\pi}{2} = \frac{\pi}{2}$.

由于上述两极限不等. 故 $\lim_{x \rightarrow 0, y \rightarrow 0} \sin \frac{\pi x}{2x+y}$ 不存在.

(2). $\lim_{x \rightarrow 0, y \rightarrow 0} \frac{xy}{1+xy} = \lim_{x \rightarrow 0} (\lim_{y \rightarrow 0} \frac{xy}{1+xy}) = \lim_{x \rightarrow 0} \frac{1}{1+1} = \frac{1}{2}$.

$\lim_{y \rightarrow 0} (\lim_{x \rightarrow 0} \frac{xy}{1+xy}) = \lim_{y \rightarrow 0} 1 = 1$.

由于上述两极限不等. 故 $\lim_{x \rightarrow 0, y \rightarrow 0} \frac{xy}{1+xy}$ 不存在.

(3). $\lim_{x \rightarrow 0} \lim_{y \rightarrow 0} (x+y) \sin \frac{1}{x} \sin \frac{1}{y} = \lim_{x \rightarrow 0} (\lim_{y \rightarrow 0} (x+y) \sin \frac{1}{x} \sin \frac{1}{y})$ 不存在.

同理. $\lim_{y \rightarrow 0} \lim_{x \rightarrow 0} (x+y) \sin \frac{1}{x} \sin \frac{1}{y}$ 不存在.

注意到 $|(x+y) \sin \frac{1}{x} \sin \frac{1}{y}| \leq |x+y|$. 而当 $(x, y) \rightarrow (0, 0)$ 时, $|x+y| \rightarrow 0$.

从而 $\lim_{x \rightarrow 0, y \rightarrow 0} (x+y) \sin \frac{1}{x} \sin \frac{1}{y} = 0$.

4. 略.

5. 略.

6. (1). $\lim_{x \rightarrow 0, y \rightarrow 0} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{x^2+y^2 \rightarrow 0} \frac{\sin(x^2+y^2)}{x^2+y^2} \lim_{x \rightarrow 0, y \rightarrow 0} \frac{x^2+y^2}{x^2+y^2} = \lim_{x^2+y^2 \rightarrow 0} \frac{x^2+y^2}{x^2+y^2} = \frac{\lim_{x \rightarrow 0, y \rightarrow 0} (x^2+y^2) \lim_{x \rightarrow 0, y \rightarrow 0} \frac{x^2+y^2}{x^2+y^2}}{\lim_{x \rightarrow 0, y \rightarrow 0} (x^2+y^2)}$

注意到 $|\frac{x^2+y^2}{x^2+y^2} - 0| = |x+y| \cdot |\frac{x^2+y^2-xy}{x^2+y^2}| < 2|x+y|$. 而当 $x^2+y^2 \rightarrow 0$ 时, $|x+y| \rightarrow 0$. 故

$\lim_{x \rightarrow 0, y \rightarrow 0} f(x, y) = f(0, 0)$ 连续.

(2). 显然不连续.

(3). $(x, y) \rightarrow (0, 0) \Leftrightarrow \sqrt{x^2+y^2} \rightarrow 0$. 设 $\rho = \sqrt{x^2+y^2}$. 注意到 $|\frac{x^2+y^2}{(\sqrt{x^2+y^2})^3}| \leq \frac{\rho^2}{\rho^3} = \frac{1}{\rho} \rightarrow 0$. 故

$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0)$. 连续.

(4). $t = y^2$. 显然不连续.

数学作业纸

(科目: 微积分)

班级: 计 25

姓名: 王 衡

编号: 2013011398

第 4 页

7. (1) 当 $x+y \neq 0$ 时, $f(x, y)$ 是初等连续函数. 若存在 t , 使 $f(x, y)$ 在点 $(t, -t)$ 处连续.

①. $t \neq 0, 1$. $\lim_{\substack{x \rightarrow t \\ y \rightarrow -t}} \frac{x-y^2}{x^2+y^2} = \frac{t-t^2}{t^2+t^2+t^2} = \frac{t-t^2}{t^2+t^2+t^2}$. $\lim_{\substack{x \rightarrow t \\ y \rightarrow -t}} \frac{1}{x+y} = \frac{1-t}{t}$. $\lim_{\substack{x \rightarrow t \\ y \rightarrow -t}} \frac{1}{x+y}$. 极限不存在, 不连续.

②. $t=0$. 当 (x, y) 沿直线 $x=ky^2$ 趋于 $(0, 0)$ 时 $(k \neq 1)$.

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x-y^2}{x^2+y^2} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{k-1}{k^2y^2+y} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{k-1}{k^2y^2+y}$$

极限不存在, 不连续.

③. $t=1$. 设 $x=u+1, y=v-1$. $\lim_{\substack{x \rightarrow 1 \\ y \rightarrow -1}} \frac{x-y^2}{x^2+y^2} = \lim_{\substack{u \rightarrow 0 \\ v \rightarrow 0}} \frac{u+1-v^2}{u^2+u+v^2} = \lim_{\substack{u \rightarrow 0 \\ v \rightarrow 0}} \frac{u+1-v^2}{u+v}$. 当 (u, v) 沿直线 $u=kv$ 趋于 $(0, 0)$ 时, 易验证其极限不存在.

综上, f 在 $D = \{(x, y) | x+y \neq 0\}$ 上连续.

(2) 显然当 $y \neq 0$ 时, $f(x, y)$ 是连续的. 当 $y=0, x \neq 0$ 时, 考虑 $f(x, y)$ 在点 $(x_0, 0)$ 处的连续性. $(x_0 \neq 0)$

$$\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow 0}} \frac{x}{y} e^{-\frac{x^2}{y^2}} = \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow 0}} \frac{t^2 e^{-t^2}}{t} = \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow 0}} \frac{t^2}{x e^{t^2}} = \frac{1}{x_0} \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow 0}} \frac{t^2}{e^{t^2}} = 0 = f(x_0, 0)$$

故连续.

再考虑 $f(x, y)$ 在 $(0, 0)$ 处的连续性. 当 (x, y) 沿直线 $x=ky^2$ 趋于 $(0, 0)$ 时.

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x}{y} e^{-\frac{x^2}{y^2}} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} k e^{-k^2 y^2} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{k}{e^{k^2 y^2}} = k$$

其极限根据 k 的值而变化. 极限不存在, 不连续.

综上, f 在 $D = \{(x, y) | x^2+y^2 \neq 0\}$ 上连续.

(3) 直线族 $\{x+y=k, k \in \mathbb{Z}\}$ 将 $\mathbb{R}^2$ 分为若干区域. 在每个区域中的点均连续. 于是 f 在 $D = \{(x, y) | x+y \neq 0\}$ 上连续.

(4) 若 $f(x, y)$ 在 (x_0, y_0) 点连续且 $x_0+y_0 \neq 0$.

①. $x_0, y_0 \in \mathbb{Q}$. 则 $f(x_0, y_0) = x_0+y_0 \neq 0$. 而 $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y)$ 不存在. 所以不连续.

②. $x_0 \notin \mathbb{Q}$ 或 $y_0 \notin \mathbb{Q}$. 则 $f(x_0, y_0) = 0$. 与上同理知不连续.

综上, f 在 $D = \{(x, y) | x^2+y^2 \neq 0\}$ 上连续.

8. 由题知 $\exists M, N, A$ 当 $|x| > M$ 且 $|y| > N$ 时, $f(x, y) > A - \max\{f(-M, -N), f(-M, N), f(M, -N), f(M, N)\}$.

8. 由题知 $\exists M > 0, N > 0, A > 0$. 当 $|x| > M$ 且 $|y| > N$ 时, $f(x, y) > \max\{f(M, N), f(-M, -N), f(-M, N), f(M, -N)\}$. 再利用最值定理即可.

9. 略.

10. 略.

11. (1) 2 阶. (2) 1 阶. (3) 无. (4) 无. (5) 无.

清华大学数学作业纸

(科目)

班级 计25

姓名 吴之琛

编号

第 1 页

习题 1.4

$$1. (1) \frac{\partial z}{\partial x} = 2axy + by^2, \quad \frac{\partial z}{\partial y} = ax^2 + 2bxy.$$

$$(2) \frac{\partial z}{\partial x} = 2x \cdot 2 \tan(x^2+y^2) \cdot \frac{1}{\cos^2(x^2+y^2)} = \frac{4x \tan(x^2+y^2)}{\cos^2(x^2+y^2)}$$

$$\frac{\partial z}{\partial y} = \frac{4y \tan(x^2+y^2)}{\cos^2(x^2+y^2)}$$

$$(3) \frac{\partial z}{\partial x} = \frac{1}{y} - \frac{y}{x^2}, \quad \frac{\partial z}{\partial y} = -\frac{x}{y^2} + \frac{1}{x}$$

$$(4) \frac{\partial z}{\partial x} = -\frac{2y}{x^3} \cdot \frac{x^4}{x^4+x^2y^2} = -\frac{2xy}{x^2+x^2y^2}, \quad \frac{\partial z}{\partial y} = \frac{1}{x^2} \cdot \frac{4x^4}{x^4+y^2} = \frac{x^2}{x^4+y^2}$$

$$(5) \frac{\partial z}{\partial x} = \left(\frac{2x-1}{\sqrt{x^2+y^2}} + 1 \right) \cdot \frac{1}{x+\sqrt{x^2+y^2}} = \frac{x+\sqrt{x^2+y^2}}{x\sqrt{x^2+y^2}+x^2+y^2}$$

$$\frac{\partial z}{\partial y} = -\frac{y}{x\sqrt{x^2+y^2}+x^2+y^2}$$

$$(6) \frac{\partial z}{\partial x} = e^{-y} - ye^{-x}, \quad \frac{\partial z}{\partial y} = -xe^{-y} + e^{-x}$$

$$(7) \frac{\partial z}{\partial x} = -2^{xy} \cdot y \cdot \ln 2 \cdot \sin(1+2^{xy}), \quad \frac{\partial z}{\partial y} = -2^{xy} \cdot x \cdot \ln 2 \cdot \sin(1+2^{xy})$$

$$(8) \frac{\partial u}{\partial x} = -\frac{y}{x^2} \operatorname{ch} \frac{y}{x}, \quad \frac{\partial u}{\partial y} = \frac{1}{x} \operatorname{ch} \frac{y}{x} + z \operatorname{sh} \frac{y}{x}, \quad \frac{\partial u}{\partial z} = y \operatorname{sh} \frac{y}{x} \quad (\text{题中应为 } u = \operatorname{sh} \frac{y}{x} + \operatorname{ch} yz)$$

$$(9) xy > 0 \text{ 时 } \frac{\partial z}{\partial x} = \frac{y}{2\sqrt{xy}}, \quad \frac{\partial z}{\partial y} = \frac{x}{2\sqrt{xy}}$$

$$xy < 0 \text{ 时, } \frac{\partial z}{\partial x} = -\frac{y}{2\sqrt{-xy}}, \quad \frac{\partial z}{\partial y} = -\frac{x}{2\sqrt{-xy}}$$

$$2. (1) \text{ 在原点处 } \lim_{\Delta x \rightarrow 0} \frac{f(x,0) - f(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{\Delta x} \cos y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\cos y}{\sqrt{\Delta x}} \text{ 此极限不存在, 则不可微.}$$

$$(2) \frac{\partial f}{\partial x} = \frac{2y}{\sqrt{x^2+y^2}} + 2xy \cdot 2x \cdot \left(-\frac{1}{2}\right) \cdot \frac{1}{(x^2+y^2)^{3/2}} = \frac{2y^3}{(x^2+y^2)^{3/2}}$$

$$\lim_{x \rightarrow 0} \frac{2y^3}{(x^2+y^2)^{3/2}} = \frac{2y^3}{\pm y^3} = \pm 2, \text{ 极限不存在则不可微.}$$

$$(3) \frac{\partial f}{\partial x} = \frac{2xy^2 - x^2y^2}{(x^2+y^2)^{5/2}}, \quad \lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial x} \text{ 不存在, 则不可微.}$$

$$(4) \frac{\partial f}{\partial x} = \frac{x}{|x|} \varphi(x,y) + |x-y| \frac{\partial \varphi}{\partial x}(x,y), \quad \lim_{(x,y) \rightarrow (0,0)} \frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial x}(0,0), \text{ 所以 } \frac{\partial f}{\partial x} \text{ 在原点处连续.}$$

清华大学数学作业纸

(科目)

班级计35

姓名 吴之豫

编号

第 2 页

同理可得 $\frac{\partial f}{\partial y}$ 也在原点处连续, 所以 f 在坐标原点可微.

3. 偏导存在但不可微

$$f(x, y) = \begin{cases} x^2 + \frac{xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

可微但偏导不连续

$$f(x, y) = \begin{cases} x^2 \sin \frac{1}{x} + y^2 \sin \frac{1}{y} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

$$4. (1) \frac{\partial u}{\partial x} \left(\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2} \right) = -\frac{1}{2} \cos 1 \quad \frac{\partial u}{\partial x} \left(\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2} \right) = -\frac{\sqrt{2}}{2} \cos 1 \quad \frac{\partial u}{\partial y} \left(\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2} \right) = \frac{1}{2} \cos 1$$

$$du \left(\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2} \right) = -\frac{\sqrt{2}}{2} \cos 1 dx - \frac{1}{2} \cos 1 dy + \frac{1}{2} \cos 1 dz.$$

$$(6) dz = -\frac{1}{\sqrt{1-e^{2xy}}} y e^{xy} dx - \frac{1}{\sqrt{1-e^{2xy}}} x e^{xy} dy$$

$$(2) du = e^{xy+z} (dx + dy + dz)$$

$$(3) dz = (2x + 2y) (dx + dy)$$

$$(4) du = \frac{x dx + y dy + z dz}{\sqrt{1+x^2+y^2+z^2}}$$

$$(5) dz = \frac{2y}{(x+y)^2} dx - \frac{2x}{(x+y)^2} dy$$

$$(7) du = \frac{2(x dx + y dy + z dz)}{1+x^2+y^2+z^2}$$

$$(8) \frac{\partial z}{\partial x_i} = 2 \sum_{j=1}^n x_j, \quad dz = 2 \sum_{j=1}^n \left(\sum_{i=1}^n x_j dx_i \right)$$

$$5. V = \frac{1}{3} (\pi R^2 \frac{hR}{R-r} - \pi r^2 \frac{hr}{R-r}) = f(R, r, h)$$

$$dV = \frac{\partial V}{\partial r} \Delta r + \frac{\partial V}{\partial R} \Delta R + \frac{\partial V}{\partial h} \Delta h$$

$$= \frac{\partial V}{\partial r} (30, 20, 40) \times 0.4 + \frac{\partial V}{\partial R} (30, 20, 40) \times 0.3 + \frac{\partial V}{\partial h} (30, 20, 40) \times 0.2 \approx 2576 \text{ cm}^3.$$

清华大学数学作业纸

(科目)

班级 1125

姓名 任张永

编号

第 页

$$6. S = \frac{2\pi R^2}{360} \quad \frac{\partial S}{\partial R} = \frac{\pi}{180} R \quad \frac{\partial S}{\partial \alpha} = \frac{\pi}{360} R^2$$

$$\Delta S = \frac{\pi}{180} R \Delta R + \frac{\pi}{360} R^2 \Delta \alpha$$

$$\therefore 0 = \frac{\pi}{180} \times 60 \times 20 \Delta R + \frac{\pi}{360} \times 20^2 \Delta \alpha \quad \Delta \alpha = 1$$

$$\therefore \Delta R = -\frac{1}{6} \text{ cm}$$

$$7. \Delta u = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

$$= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) + f(x_0 + \Delta x, y_0) - f(x_0, y_0)$$

由一元函数微分中值定理可知:

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) = \frac{\partial f}{\partial y}(x_0 + \Delta x, y_0 + \theta \Delta y) \Delta y$$

$$\because \frac{\partial f}{\partial y} \text{ 连续} \therefore \Delta x \rightarrow 0, \Delta y \rightarrow 0 \text{ 时, 上式} = \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + o(\Delta y)$$

$$\because \frac{\partial f}{\partial x}(x_0, y_0) \text{ 存在 由导数定义: } f(x_0 + \Delta x, y_0) - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + o(\Delta x)$$

$$\therefore \Delta u = \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + o(\Delta x) + o(\Delta y) = \frac{\partial f}{\partial x}(x_0, y_0) \Delta x + \frac{\partial f}{\partial y}(x_0, y_0) \Delta y + o(\sqrt{\Delta x^2 + \Delta y^2})$$

$\therefore f(x, y)$ 在 (x_0, y_0) 处可微.

清华大学数学作业纸

(科目)

班级计35 姓名任森. 编号

第 页

8. ①. $\lim_{|(x,y)| \rightarrow (0,0)} \sqrt[3]{xy} = 0 = f(0,0) \therefore f$ 在原点连续

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(0,0) - f(0,0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0.$$

同理: $\frac{\partial f}{\partial y} = 0 \therefore$ 偏导数存在.

而沿方向 $l = (a,b) (ab \neq 0)$ 时

$$\frac{\partial f}{\partial x} = \lim_{t \rightarrow 0} \frac{f(\frac{ta}{t}, \frac{tb}{t}) - f(0,0)}{t} = \lim_{t \rightarrow 0} \frac{\sqrt[3]{t^2 ab} - 0}{t} = \lim_{t \rightarrow 0} \frac{\sqrt[3]{t^2 ab}}{t} = \infty \text{ 不存在.}$$

9. 令 $f = kx^3$

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^3}{y} = \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^3}{kx^3} = \frac{1}{k} \therefore \text{在原点处不连续.}$$

($u^2 + v^2 = 1$)

方向导数: $\lim_{t \rightarrow 0} \frac{f(tu, tv) - f(0,0)}{t} = \frac{(tu)^3}{tu \cdot t} = t \frac{u}{v}$

$v \neq 0$ 时: $\lim_{t \rightarrow 0} t \frac{u}{v} = 0.$

$v = 0$ 时: $\therefore f(x,y) = 0 (y=0)$ $v=0$ 即是在 y 方向的^{方向}导数 $\therefore$ 为 0.

$\therefore$ 沿任何方向导数均存在.

10. 令单位向量 $\lambda = (u,v).$

则: $\lim_{t \rightarrow 0} \frac{f(0+tu, 0+tv) - f(0,0)}{t} = \frac{\sqrt{1+t^2 u^2 - t^2 v^2}}{t} = \sqrt{u^2 - v^2}$

$\therefore$ 沿所有方向方向导数都存在.

微积分 1.4

多元函数的全 清华大学数学作业纸

微分及偏导数. 陈雨兰.

~~王森~~ ~~吴之霖~~ ~~陈雨兰~~.

(科目)

班级

姓名

编号

第

页

11. (1). $z = \cos(x+y)$. $P_0 = (0, \frac{\pi}{5})$, $l = (3, -4)$.

$$\begin{aligned}\frac{\partial z}{\partial l}(P_0) &= \frac{\partial z}{\partial x}(P_0) \cdot \frac{3}{5} + \frac{\partial z}{\partial y}(P_0) \cdot (-\frac{4}{5}) \\ &= -\sin(0+\frac{\pi}{5}) \cdot \frac{3}{5} + (-\sin(0+\frac{\pi}{5})) \cdot (-\frac{4}{5}) \\ &= \frac{1}{5}\end{aligned}$$

(2) $z = \frac{1}{\sqrt{x^2+y^2+z^2}}$. $P_0 = (1, 2, -2)$, $l = (-2, 1, 2)$

$$\begin{aligned}\frac{\partial z}{\partial l}(P_0) &= \frac{\partial z}{\partial x}(P_0) \cdot (-\frac{2}{3}) + \frac{\partial z}{\partial y}(P_0) \cdot \frac{1}{3} + \frac{\partial z}{\partial z}(P_0) \cdot \frac{2}{3} \\ &= \frac{4}{81}\end{aligned}$$

(3). $z = \sum_{i=1}^n \sum_{j=1}^n x_i x_j$. $P_0 = (1, 1, \dots, 1)$. $l = (-1, -1, \dots, -1)$

$$\frac{\partial z}{\partial x_i}(P_0) = 2 \sum_{j=1}^n x_j = 2n. \quad \cos \alpha_i = -\frac{1}{\sqrt{n}}$$

$$\begin{aligned}\frac{\partial z}{\partial l}(P_0) &= \sum_{i=1}^n \frac{\partial z}{\partial x_i}(P_0) \cdot \cos \alpha_i \\ &= \sum_{i=1}^n (-2\sqrt{n})\end{aligned}$$

$$= -2n\sqrt{n}$$

(4). $z = \ln(x_1 + x_2 + \dots + x_n)$. $P_0 = (1, 0, \dots, 0)$. $l = (1, 1, \dots, 1)$

$$\frac{\partial z}{\partial x_i}(P_0) = \frac{1}{x_1 + x_2 + \dots + x_n} \Big|_{(1, 0, \dots, 0)} = 1. \quad \cos \alpha_i = \frac{1}{\sqrt{n}}$$

$$\begin{aligned}\frac{\partial z}{\partial l}(P_0) &= \sum_{i=1}^n \frac{\partial z}{\partial x_i}(P_0) \cos \alpha_i \\ &= \sqrt{n}\end{aligned}$$

清华大学数学作业纸

(科目)

班级

姓名

编号

第

页

$$12(1) u(x, y) = \sqrt{x^2 + y^2}$$

$$\begin{aligned} \text{grad} u(x, y) &= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \\ &= \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) \end{aligned}$$

$$(2) u(x, y, z) = \frac{xyz}{x+y+z}$$

$$\begin{aligned} \text{grad} u(x, y, z) &= \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right) \\ &= \left(\frac{y^2 z + y z^2}{(x+y+z)^2}, \frac{x z^2 + x^2 z}{(x+y+z)^2}, \frac{xy^2 + x^2 y}{(x+y+z)^2} \right) \end{aligned}$$

$$(3) u(x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i$$

$$\begin{aligned} \text{grad} u(x_1, x_2, \dots, x_n) &= \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right) \\ &= (1, 1, \dots, 1) \end{aligned}$$

$$(4) u(x_1, \dots, x_n) = \sum_{i=1}^n \sum_{j=1}^n x_i x_j$$

$$\frac{\partial u}{\partial x_i} = 2 \sum_{j=1}^n x_j$$

$$\text{grad} u(x_1, \dots, x_n) = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right)$$

$$= 2 \sum_{j=1}^n x_j (1, 1, \dots, 1)$$

$$13. \text{ 设 } \lambda: (\cos \alpha_1, \cos \alpha_2, \cos \alpha_3)$$

$$\begin{aligned} \frac{\partial u}{\partial \lambda}(p) &= \frac{\partial u}{\partial x}(p) \cos \alpha_1 + \frac{\partial u}{\partial y}(p) \cos \alpha_2 + \frac{\partial u}{\partial z}(p) \cos \alpha_3 \\ &= 2 \cos \alpha_2 + 2 \cos \alpha_3 \end{aligned}$$

清华大学数学作业纸

(科目)

班级

姓名

编号

第

页

$$|\cos \alpha_2 + \cos \alpha_3| \leq 2\sqrt{\frac{1}{2}} = \sqrt{2}.$$

$$\therefore \frac{\partial u}{\partial \lambda}(P)_{\max} = \sqrt{2} \quad \lambda(0, \frac{\pi}{4}, \frac{\pi}{4})$$

$$\frac{\partial u}{\partial \lambda}(P)_{\min} = -\sqrt{2} \quad \lambda(0, -\frac{\pi}{4}, -\frac{\pi}{4})$$

$$\frac{\partial u}{\partial \lambda}(P) = 0 \Rightarrow \cos \alpha_2 + \cos \alpha_3 = 0.$$

$$\lambda: \lambda \perp (0, 1, 1)$$

$$14. (1) \quad u = \cos^2(ax - by)$$

$$\frac{\partial u}{\partial x} = -2a \sin(2ax - 2by)$$

$$\frac{\partial u}{\partial y} = 2b \sin(2ax - 2by)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d}{dx}(-2a \sin(2ax - 2by)) = -2a^2 \cos(2ax - 2by)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{d}{dy}(2b \sin(2ax - 2by)) = -2b^2 \cos(2ax - 2by)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(2b \sin(2ax - 2by)) = 2ab \cos(2ax - 2by)$$

$$(2) \quad u = e^{-\alpha x} \sin \beta y$$

$$\frac{\partial u}{\partial x} = -\alpha e^{-\alpha x} \sin \beta y, \quad \frac{\partial u}{\partial y} = \beta e^{-\alpha x} \cos \beta y$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(-\alpha e^{-\alpha x} \sin \beta y) = \alpha^2 e^{-\alpha x} \sin \beta y$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(\beta e^{-\alpha x} \cos \beta y) = -\beta^2 e^{-\alpha x} \sin \beta y$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(\beta e^{-\alpha x} \cos \beta y) = -\alpha \beta e^{-\alpha x} \cos \beta y$$

清华大学数学作业纸

(科目)

班级

姓名

编号

第

页

(3). $u = xe^{-xy}$

$$\frac{\partial u}{\partial x} = (1-xy)e^{-xy} \quad \frac{\partial u}{\partial y} = -x^2e^{-xy}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} ((1-xy)e^{-xy}) = (-2y+xy^2)e^{-xy}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} (-x^2e^{-xy}) = x^3e^{-xy}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} (-x^2e^{-xy}) = (-2x+x^2y)e^{-xy}$$

(4). $u = \ln(x + \sqrt{1-y^2})$

$$\frac{\partial u}{\partial x} = \frac{1}{x + \sqrt{1-y^2}} \quad \frac{\partial u}{\partial y} = \frac{\frac{-y}{\sqrt{1-y^2}}}{x + \sqrt{1-y^2}} = \frac{-y}{x\sqrt{1-y^2} + 1-y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{1}{x + \sqrt{1-y^2}} \right) = -\frac{1}{(x + \sqrt{1-y^2})^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{-y}{x\sqrt{1-y^2} + 1-y^2} \right) = \frac{-(x\sqrt{1-y^2} + 1-y^2) + y(\frac{-xy}{\sqrt{1-y^2}} - 2y)}{(x\sqrt{1-y^2} + 1-y^2)^2} = -\frac{x + (1+y^2)\sqrt{1-y^2}}{\sqrt{1-y^2}(x\sqrt{1-y^2} + 1-y^2)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{-y}{x\sqrt{1-y^2} + 1-y^2} \right) = \frac{y}{\sqrt{1-y^2}(x + \sqrt{1-y^2})^2}$$

15. (1). $u = 2\cos^2(x - \frac{y}{2})$ 满足 $2\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = 0$

$$\frac{\partial u}{\partial y} = \sin(2x-y).$$

$$2\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = -2\cos(2x-y) + 2\cos(2x-y) = 0$$

得证.

清华大学数学作业纸

(科目)

班级

姓名

编号

第

页

(2) $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ 满足 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(-x(x^2+y^2+z^2)^{-\frac{3}{2}} \right) = -(x^2+y^2+z^2)^{-\frac{3}{2}} + 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= -3(x^2+y^2+z^2)^{-\frac{3}{2}} + 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-\frac{5}{2}} \\ &= -3(x^2+y^2+z^2)^{-\frac{3}{2}} + 3(x^2+y^2+z^2)^{-\frac{3}{2}} \\ &= 0. \end{aligned}$$

得证.

(3) $\begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$ 满足 $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y, \quad \frac{\partial v}{\partial x} = e^x \sin y, \quad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y, \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\frac{\partial^2 v}{\partial x^2} = e^x \sin y, \quad \frac{\partial^2 v}{\partial y^2} = -e^x \sin y$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

清华大学数学作业纸

(科目)

班级

姓名

编号

第

页

(4). $n > 2$. $u = (\sqrt{x_1^2 + x_2^2 + \dots + x_n^2})^{2-n}$. 满足 $\frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} = 0$.

$$\frac{\partial u}{\partial x_i} = \frac{2-n}{2} (x_1^2 + \dots + x_n^2)^{-\frac{n}{2}} \cdot 2x_i = (2-n)x_i (x_1^2 + \dots + x_n^2)^{-\frac{n}{2}}$$

$$\frac{\partial^2 u}{\partial x_i^2} = (2-n)(x_1^2 + \dots + x_n^2)^{-\frac{n}{2}} + (2-n)x_i \left(-\frac{n}{2}\right) (x_1^2 + \dots + x_n^2)^{-\frac{n+2}{2}} \cdot 2x_i$$

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = n(2-n)(x_1^2 + \dots + x_n^2)^{-\frac{n}{2}} - n(2-n)(x_1^2 + \dots + x_n^2)(x_1^2 + \dots + x_n^2)^{-\frac{n}{2}-1}$$
$$= 0.$$

得证.

清华大学数学作业纸

(科目 微积分)

班级 计35 姓名 蒋文明

编号 2013011388

第 页

习题 1.5

$$1. \quad \langle 1 \rangle \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

$$\text{由 } \frac{\partial(u, v)}{\partial(x, y)} \neq 0 \quad \text{得} \quad \begin{cases} x \neq 0 \\ y \neq 0 \end{cases}$$

$$\langle 2 \rangle \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

$$\text{由 } \frac{\partial(u, v)}{\partial(x, y)} \neq 0 \quad \text{得}$$

$$\langle 3 \rangle \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{-2xy}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{pmatrix}$$

$$\text{由 } \frac{\partial(u, v)}{\partial(x, y)} \neq 0 \quad \text{得} \quad \begin{cases} x \neq 0 \\ y \neq 0 \end{cases}$$

$$2. \quad \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} = \begin{pmatrix} \sin \theta \cos \varphi & r \cos \theta \cos \varphi & -r \sin \theta \sin \varphi \\ \cos \theta \cos \varphi & -r \sin \theta \cos \varphi & -r \cos \theta \sin \varphi \\ \sin \varphi & 0 & r \cos \varphi \end{pmatrix}$$

$$3. \quad \langle 1 \rangle. \quad \frac{\partial z}{\partial x} = \frac{2xv - uy}{v^2 + u^2} \quad \frac{\partial z}{\partial y} = \frac{2yv - ux}{v^2 + u^2}$$

= 阶偏导略.

清华大学数学作业纸

(科目)

班级 计25

姓名 蒋文明

编号

第

页

$$(2) \quad \frac{\partial z}{\partial x} = 3f_u + 4f_v \quad \frac{\partial z}{\partial y} = 2f_u - 2f_v$$

$$\frac{\partial^2 z}{\partial x^2} = 9f_{uu} + 24f_{uv} + 16f_{vv} \quad \frac{\partial^2 z}{\partial y^2} = 4f_{uu} - 8f_{uv} + 4f_{vv}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6f_{uu} + 2f_{uv} - 8f_{vv}$$

$$(3) \quad \frac{\partial z}{\partial x} = 2x f_u + y e^{xy} f_v \quad \frac{\partial z}{\partial y} = x e^{xy} f_v - 2y f_u$$

= 阶偏导略

$$(4) \quad \frac{\partial z}{\partial x} = f_u + f_v + f_w \quad \frac{\partial z}{\partial y} = f_v - f_w$$

$$\frac{\partial^2 z}{\partial x^2} = f_{uu} + 2f_{uv} + 2f_{uw} + 2f_{vw} + f_{vv} + f_{ww}$$

$$\frac{\partial^2 z}{\partial y^2} = f_{vv} - 2f_{vw} + f_{ww}$$

$$\frac{\partial^2 z}{\partial x \partial y} = f_{uv} + f_{vw} - f_{uw} - f_{vw}$$

$$(5) \quad \frac{\partial z}{\partial x} = y + \frac{y^2}{x} f_u - \frac{y}{x^2} f(u) \quad \frac{\partial z}{\partial y} = x + \frac{1}{x} f(u) + y f_u$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2}{x} f_{uu} + \frac{2y}{x^2} f(u) - \frac{2y}{x^2} f_u$$

$$\frac{\partial^2 z}{\partial y^2} = 2f_u + xy f_{uu}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 1 + \frac{y}{x} f_u - \frac{1}{x^2} f(u) + y^2 f_{uu}$$

$$(6) \quad \frac{\partial z}{\partial x} = f_u + 2f_v \quad \frac{\partial z}{\partial y} = -f_v$$

$$\frac{\partial^2 z}{\partial x^2} = (\ln x + 1)^2 f_{uu} + 4(\ln x + 1) f_{uv} + 4f_{vv} + \frac{1}{x} f_u$$

$$\frac{\partial^2 z}{\partial y^2} = f_{vv} \quad \frac{\partial^2 z}{\partial x \partial y} = -(\ln x + 1) f_{uv} - 2f_{vv}$$

数学作业纸

(科目: 微积分)

班级: 计 25

姓名: 王博

编号: 2013011398

第

页

$$1. \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} = \cos \theta \cdot \frac{\partial u}{\partial x} + \sin \theta \cdot \frac{\partial u}{\partial y}.$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} = -r \sin \theta \frac{\partial u}{\partial x} + r \cos \theta \frac{\partial u}{\partial y}. \quad \text{代入即可.}$$

$$6. \frac{\partial u}{\partial x} = y + f\left(\frac{y}{x}\right) + x \frac{\partial}{\partial x} \left(f\left(\frac{y}{x}\right)\right), \quad x \frac{\partial u}{\partial x} = u + x^2 \frac{\partial}{\partial x} \left(f\left(\frac{y}{x}\right)\right) = u + x^2 \frac{\partial f}{\partial t} \cdot \frac{\partial t}{\partial x} = u - y \frac{\partial f}{\partial t} \quad (t = \frac{y}{x})$$

$$\frac{\partial u}{\partial y} = x + x \frac{\partial}{\partial y} \left(f\left(\frac{y}{x}\right)\right) = x + \frac{\partial f}{\partial t}, \quad y \frac{\partial u}{\partial y} = xy + y \frac{\partial f}{\partial t}. \quad \text{代入即可.}$$

$$7. \text{设 } v = \frac{x}{x^2+y^2}, \quad w = \frac{y}{x^2+y^2}. \quad \text{则 } u(x, y) = f(v, w).$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} \frac{\partial f}{\partial v} - \frac{xy}{(x^2+y^2)^2} \frac{\partial f}{\partial w}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{y^2-x^2}{(x^2+y^2)^2} \frac{\partial f}{\partial v} - \frac{xy}{(x^2+y^2)^2} \frac{\partial f}{\partial w} \right] = \frac{2x^2-4xy^2}{(x^2+y^2)^3} \frac{\partial f}{\partial v} + \frac{y^2-x^2}{(x^2+y^2)^3} \frac{\partial^2 f}{\partial v^2} \cdot \frac{\partial v}{\partial x} - \frac{xy^3-6xy^2}{(x^2+y^2)^3} \frac{\partial f}{\partial w} - \frac{xy}{(x^2+y^2)^3} \frac{\partial^2 f}{\partial v \partial w} \cdot \frac{\partial v}{\partial x} - \frac{xy}{(x^2+y^2)^3} \frac{\partial^2 f}{\partial w^2} \cdot \frac{\partial w}{\partial x}.$$

$$\text{同理求 } \frac{\partial^2 u}{\partial y^2}. \quad \text{代入即可.}$$

$$9. (11). \quad J = \begin{bmatrix} 1 & 1 \\ u_2 & u_1 \\ u_2 & 1 \\ -\frac{1}{u_2} & \frac{1}{u_1} \end{bmatrix} \begin{bmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & -\frac{xy}{(x^2+y^2)^2} \\ \frac{-2x}{(x^2+y^2)^2} & \frac{y^2-x^2}{(x^2+y^2)^2} \end{bmatrix} = \begin{bmatrix} 1 & y^2-x^2-xy & x^2-y^2-xy \\ (x^2+y^2)^2 & u_1 y^2 - u_2 x^2 - 2u_1 x & u_1 x^2 - u_2 y^2 - 2u_1 y \\ (x^2+y^2)^2 & \frac{y^2-x^2}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y(xy+y^2)}{x^2} & \frac{x(xy+x^2)}{x^2} \end{bmatrix} \begin{bmatrix} \frac{y^2-x^2}{(x^2+y^2)^2} & -\frac{xy}{(x^2+y^2)^2} \\ \frac{-2x}{(x^2+y^2)^2} & \frac{y^2-x^2}{(x^2+y^2)^2} \end{bmatrix} \quad dY = J(dX, dY)^T.$$

$$(12). \quad J = \begin{bmatrix} u_1 & u_2 \\ u_1 & -u_2 \end{bmatrix} \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} = \frac{1}{x^2+y^2} \begin{bmatrix} 2x \ln(x^2+y^2) - 2y \arctan \frac{y}{x} & 2y \ln(x^2+y^2) + 2x \arctan \frac{y}{x} \\ 2x \ln(x^2+y^2) + 2y \arctan \frac{y}{x} & 2y \ln(x^2+y^2) - 2x \arctan \frac{y}{x} \end{bmatrix}$$

$$dY = J(dX, dY)^T$$

$$(13). \quad J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad dY = J(dX, dY)^T.$$

清华大学数学作业纸

(科目)

班级

姓名 郑光衡 编号

第 页

习题 1-6

2. (1)

解: 记 $f(x, y) = (x^2 + y^2)^2 + a(x^2 - y^2)$

$$\therefore \frac{\partial f}{\partial x} = 2x(x^2 + y^2) + 2ax$$

$$\frac{\partial f}{\partial y} = 4y(x^2 + y^2) - 2ay$$

$$\therefore \frac{dy}{dx} = - \frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}} = - \frac{2y(x^2 + y^2) - 2ay}{2x(x^2 + y^2) + 2ax}$$

2. (2)

解: 记 $f(x, y, z) = e^{-(x+y+z)} - (x+y+z)$

$$\therefore \frac{\partial f}{\partial x} = -e^{-(x+y+z)} - 1$$

$$\frac{\partial f}{\partial y} = -e^{-(x+y+z)} - 1$$

$$\frac{\partial f}{\partial z} = -e^{-(x+y+z)} - 1$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = -1$$

2. (3)

解: 记 $f(x, y, z) = \sin xy + \sin yz + \sin zx$

$$\therefore \frac{\partial f}{\partial x} = y \cos xy + z \cos xz$$

$$\frac{\partial f}{\partial y} = x \cos xy + z \cos yz$$

$$\frac{\partial f}{\partial z} = y \cos yz + x \cos zx$$

$$\therefore \frac{\partial z}{\partial x} = - \frac{y \cos xy + z \cos xz}{x \cos xy + y \cos yz}$$

$$\frac{\partial z}{\partial y} = - \frac{x \cos xy + z \cos yz}{y \cos xy + x \cos zx}$$

3. (1)

解: $\frac{\partial f}{\partial x} = af_1', \frac{\partial f}{\partial y} = af_2', \frac{\partial f}{\partial z} = -cf_1' - bf_2'$

$$\therefore \frac{\partial z}{\partial x} = \frac{cf_1' + bf_2'}{af_1' + bf_2'}, \frac{\partial z}{\partial y} = \frac{af_2'}{cf_1' + bf_2'}$$

$$\therefore \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = a$$

3. (2)

解: 记 $F(x, y, z) = y + z - x f(x^2 - z^2)$

$$\therefore \frac{\partial F}{\partial x} = -f(x^2 - z^2) - x \cdot 2x \cdot f'$$

$$\frac{\partial F}{\partial y} = 1$$

$$\frac{\partial F}{\partial z} = 1 + x \cdot 2z f'$$

$$\therefore \frac{\partial z}{\partial x} = \frac{f + 2xz f'}{1 + 2zx f'}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + 2zx f'}$$

$$\therefore x \frac{\partial z}{\partial x} + z \frac{\partial z}{\partial y} = \frac{y + 2xz f'}{1 + 2zx f'}$$

3. (3)

解: $\frac{\partial f}{\partial x} = f_1' + f_2' + f_3'$

$$\frac{\partial f}{\partial y} = f_2' + f_3'$$

$$\frac{\partial f}{\partial z} = f_3'$$

$$\therefore \frac{\partial z}{\partial x} = \frac{-f_1' - f_2' - f_3'}{f_3'}, \frac{\partial z}{\partial y} = \frac{-f_2' - f_3'}{f_3'}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = \frac{f_1'' + f_2'' + f_3''}{(f_3')^2} + \frac{f_3''}{f_3'^3}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = \frac{1}{f_3'^3} = \frac{1}{f_3'} (f_1'' + f_2'' + f_3'') \left(1 + \frac{\partial z}{\partial x} \right)$$

清华大学数学作业纸

(科目)

班级

姓名

编号

第

页

$$\therefore \frac{\partial^2 z}{\partial x^2} = \frac{\partial \left(\frac{-f'_1 + f'_2 + f'_3}{f_3} \right)}{\partial x}$$

$$= \frac{1}{(f_3)^2} \left[-f_3' \left[f_{11}'' + f_{21}'' + f_{31}'' \left(1 + \frac{\partial z}{\partial x} \right) + f_{12}'' + f_{22}'' + f_{32}'' \left(1 + \frac{\partial z}{\partial x} \right) - (f_1' + f_2') \frac{\partial z}{\partial x} (f_{31}'' + f_{32}'' + f_{33}'') \left(1 + \frac{\partial z}{\partial x} \right) \right] \right]$$

$$= \frac{1}{f_3^2} \left[f_{11}'' + f_{21}'' + f_{31}'' \left(1 + \frac{\partial z}{\partial x} \right) + f_{12}'' + f_{22}'' + f_{32}'' \left(1 + \frac{\partial z}{\partial x} \right) + \left(1 + \frac{\partial z}{\partial x} \right) \left[f_{31}'' + f_{32}'' + f_{33}'' \left(1 + \frac{\partial z}{\partial x} \right) \right] \right]$$

3. (4)

解: 取局部 $\begin{cases} x = u \cos v \\ y = u \sin v \end{cases}$, 记 $f(x, y, u, v) = \begin{cases} w_1 = x - u \cos v \\ w_2 = y - u \sin v \end{cases}$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial (x, y)} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \frac{\partial f}{\partial (u, v)} = \begin{bmatrix} -\cos v & +u \sin v \\ -\sin v & -u \cos v \end{bmatrix}$$

$$\therefore \left(\frac{\partial f}{\partial (u, v)} \right)^T = \frac{1}{u} \begin{bmatrix} u \cos v & -u \sin v \\ \sin v & -\cos v \end{bmatrix} \therefore \frac{\partial (u, v)}{\partial (x, y)} = \frac{1}{u} \begin{bmatrix} u \cos v & u \sin v \\ \sin v & -\cos v \end{bmatrix}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = -\frac{\sin v}{u}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\cos v}{u}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = \frac{-\sin v}{u^2} = \frac{-\cos v - \sin v - \sin v - \cos v}{u^2} = \frac{\sin 2v}{u^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin^2 v - \cos^2 v}{u^2} = -\frac{\cos 2v}{u^2}$$

4. 证明: $\frac{\partial f}{\partial x} = f'_1 \cdot \frac{\partial x}{\partial x}, \frac{\partial f}{\partial y} = f'_2 \cdot \frac{\partial y}{\partial y}, \frac{\partial f}{\partial z} = f'_3 \cdot \frac{\partial z}{\partial z}, \frac{\partial f}{\partial u} = u(f'_1 + f'_2 + f'_3)$

$$\therefore \frac{\partial u}{\partial x} = \frac{x f'_1}{u(f'_1 + f'_2 + f'_3)}, \quad \frac{\partial u}{\partial y} = \frac{y f'_2}{u(f'_1 + f'_2 + f'_3)}, \quad \frac{\partial u}{\partial z} = \frac{z f'_3}{u(f'_1 + f'_2 + f'_3)}$$

$$\therefore \frac{1}{x} \frac{\partial u}{\partial x} + \frac{1}{y} \frac{\partial u}{\partial y} + \frac{1}{z} \frac{\partial u}{\partial z} = \frac{1}{u}$$

证毕

数学作业纸

(科目:)

班级:

姓名: 马志明

编号:

第

页

解: 习题 1.6
5. 由 $\begin{cases} x = u+v \\ y = u-v \end{cases} \Rightarrow \begin{cases} u = \frac{x+y}{2} \\ v = \frac{x-y}{2} \end{cases}$

故 $z = uv = \left(\frac{x+y}{2}\right) \cdot \left(\frac{x-y}{2}\right) = \frac{(x^2 - y^2)}{4}$

$\therefore \frac{\partial z}{\partial x} = \frac{(x^2 - y^2)}{4} \cdot 2x = \frac{x(x^2 - y^2)}{2}$

$\frac{\partial z}{\partial y} = \frac{x^2 - y^2}{4} \cdot (-2y) = -\frac{y(x^2 - y^2)}{2}$

6. 解: 对方程组直接微分得:

$\begin{cases} dx + dy + dz + 2zdz = 0 \\ dx + 2ydy + dz + 3z^2dz = 0 \end{cases}$

$\therefore \begin{cases} 1 + y' + (1 + 2z)z' = 0 \\ 1 + 2y \cdot y' + (1 + 3z^2)z' = 0 \end{cases}$

解得: $z' = \frac{2y-1}{1+3z^2-2y-4yz}$

$y' = -1 - \frac{(2y-1)(1+2z)}{1+3z^2-2y-4yz}$

$\therefore z'(-1) = -1 \quad y'(-1) = 0$

7. 解: 对方程组直接微分得:

$\begin{cases} 2x dx + 2y dy = z dz \\ dx + dy + dz = 0 \end{cases} \therefore \begin{cases} 2x \cdot \frac{dx}{dz} + 2y \cdot \frac{dy}{dz} = z \\ \frac{dx}{dz} + \frac{dy}{dz} + 1 = 0 \end{cases}$

解得: $\frac{dy}{dz} = \frac{z+2x}{2y-2x} \quad \frac{dx}{dz} = \frac{z+2y}{2x-2y}$

$\frac{d^2y}{dz^2} = \frac{1}{2y-2x} \quad \frac{d^2x}{dz^2} = \frac{1}{2x-2y}$

故在 $(1, -1, 2)$ 处有 $\frac{dy}{dz} = -1 \quad \frac{dx}{dz} = 0 \quad \frac{d^2y}{dz^2} = -\frac{1}{4} \quad \frac{d^2x}{dz^2} = \frac{1}{4}$

8. 定理 1.6.4

叶子明

证明: 若在 B 邻域内可逆 (可逆的证明在后面补证)则有 $y_1 = f_1(x_1, \dots, x_n)$ $\vdots$ $y_n = f_n(x_1, \dots, x_n)$ 每个式子对每个 y_i 求偏导, 有

$$I = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \frac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)}$$

$$\text{即 } Jg(Y) = Jf(X)^{-1}$$

$$(1) \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial f}{\partial(x,y)} \right)^{-1} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}$$

$$\text{行列式} = \frac{1}{4x^2+4y^2}$$

$$(2) \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial f}{\partial(x,y)} \right)^{-1} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}^{-1} = \frac{1}{e^x} \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix}$$

$$\text{行列式} = \frac{1}{e^{2x}}$$

$$(3) \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial f}{\partial(x,y)} \right)^{-1} = \begin{bmatrix} 3x^2 & -3y^2 \\ y^2 & 2xy \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3x^2}{6x^3y+3y^4} & \frac{3y^2}{6x^3y+3y^4} \\ \frac{-y^2}{6x^3y+3y^4} & \frac{2xy}{6x^3y+3y^4} \end{bmatrix}$$

$$\text{行列式} = \frac{1}{6x^3y+3y^4}$$

$$(4) \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial f}{\partial(x,y)} \right)^{-1} = \begin{bmatrix} \cosh x & \sinh y \\ -\sinh x & \cosh y \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\cosh x}{\cosh(x+y)} & \frac{-\sinh y}{\cosh(x+y)} \\ \frac{\sinh x}{\cosh(x+y)} & \frac{\cosh y}{\cosh(x+y)} \end{bmatrix}$$

$$\text{行列式} = \frac{1}{\cosh(x+y)}$$

$$(5) \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial f}{\partial(x,y)} \right)^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{a}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{d}{ad-bc} \end{bmatrix}$$

$$\text{行列式} = \frac{1}{ad-bc}$$

$$(6) \frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial f}{\partial(x,y)} \right)^{-1} = \begin{bmatrix} 3x^2 & -1 \\ 1 & 3y^2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3x^2}{9x^2y^2+1} & \frac{1}{9x^2y^2+1} \\ \frac{-1}{9x^2y^2+1} & \frac{3y^2}{9x^2y^2+1} \end{bmatrix}$$

$$\text{行列式} = \frac{1}{9x^2y^2+1}$$

$$0. (1) J_1 = \frac{\partial(u, v)}{\partial(\xi, \eta)} = \begin{bmatrix} 2\xi & -2\eta \\ -2\eta & 2\xi \end{bmatrix}, \text{行列式在 } (\xi_0, \eta_0) \text{ 处值为 } 4e^2 \text{ 不为 } 0$$

$$J_2 = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{bmatrix} e^{x \cos y} & -e^{x \sin y} \\ e^{x \sin y} & e^{x \cos y} \end{bmatrix} \text{行列式在 } (x_0, y_0) \text{ 处值为 } e^2 \text{ 不为 } 0$$

则可以确定逆向量函数

$$(2) J_1 = \frac{\partial(u, v)}{\partial(\xi, \eta)} = \begin{bmatrix} e^{\xi+\eta} & e^{\eta+\xi} \\ e^{\xi-\eta} & e^{\xi-\eta} \end{bmatrix} \text{在 } (\xi_0, \eta_0) \text{ 处值为 } -2e^{2\xi} \text{ 不为 } 0$$

$$J_2 = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{bmatrix} \operatorname{sh} x & \operatorname{ch} y \\ \operatorname{ch} x & \operatorname{sh} y \end{bmatrix} \text{在 } (x_0, y_0) \text{ 处行列式的值为 } -1 \text{ 不为 } 0$$

则可以确定逆向量函数

$$(3) J_1 = \frac{\partial(u, v)}{\partial(\xi, \eta)} = \begin{bmatrix} 5\xi^4 & 1 \\ -1 & 5\eta^4 \end{bmatrix} \text{在 } (\xi_0, \eta_0) = (1, 1) \text{ 处值为 } 26 \text{ 不为 } 0$$

$$J_2 = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{bmatrix} 3x^2 & -3y^2 \\ 2x & 4y \end{bmatrix} \text{在 } (1, 0) \text{ 处值为 } 0$$

于是不能确定是否能确定可微逆函数

$$\begin{cases} u = (x^3 - y^3)^5 + x^2 + 2y^2 \\ v = (x + 2y^2)^5 - x^3 + y^3 \end{cases} \quad \text{则 } J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{bmatrix} 5(x^3 - y^3)^4 \cdot 3x^2 + 2x & -15(x^3 - y^3)^4 y^2 + 4y \\ 5(x + 2y^2)^4 \cdot 2x - 3x^2 & 5(x + 2y^2)^4 \cdot 4y + 3y^2 \end{bmatrix}$$

行列式为 0

依旧不能确定

7. 1.7

$$1. \quad u, \quad \frac{\partial z}{\partial x}(1,2) = 2 \quad \frac{\partial z}{\partial y}(1,2) = 4$$

$$z = 2(x-2) + 4(y-1)$$

$$(2) \quad \frac{\partial z}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{y}{x^2} = \frac{-y}{x^2 + y^2} = -\frac{1}{2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2} = \frac{1}{2}$$

$$\therefore z = -\frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

$$(3) \quad \text{if } F(x, y, z) = (2a^2 - z^2)x^2 - a^2y^2$$

$$\frac{\partial F}{\partial x} = (2a^2 - z^2) \cdot 2x = 2a^3$$

$$\frac{\partial F}{\partial y} = -2a^2y = -2a^3$$

$$\frac{\partial F}{\partial z} = -2x^2z = -2a^3$$

$$\therefore 2a^3(x-a) - 2a^3(y-a) - 2a^3(z-a)$$

$$(4) \quad F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2} \quad \frac{\partial F}{\partial y} = \frac{2y}{b^2} \quad \frac{\partial F}{\partial z} = \frac{2z}{c^2}$$

$$= \frac{2}{\sqrt{3}} \frac{1}{a} \quad = \frac{2}{\sqrt{3}} \frac{1}{b} \quad \frac{2}{\sqrt{3}} \frac{1}{c}$$

$$\frac{2}{\sqrt{3}a} \left(x - \frac{a}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}b} \left(y - \frac{b}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}c} \left(z - \frac{c}{\sqrt{3}}\right) = 1$$

$$5) \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{bmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} \cos v & \sin v \\ -\frac{\sin v}{u} & \frac{\cos v}{u} \end{bmatrix}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = a \cdot \sin v$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{a \cos v}{y}$$

$$6) \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} 1 & 1 \\ 2u & 2v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} -2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= 3u^2 \cdot 2 + 3v^2(-1)$$

$$= 6 - 12 = -6$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial v} \cdot 3 \cdot \left(-\frac{1}{2}\right) + 12 \times \frac{1}{2} = \frac{9}{2}$$

$$2. \quad \vec{v} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$$

$$= \left(\frac{2x^2}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right)$$

$$\therefore \begin{cases} x=y=z \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \end{cases}$$

~~$$x=y=z=2=\sqrt{3}$$~~

$$3. \quad F = \dots$$

$$\frac{\partial F}{\partial x} = 2x \quad \frac{\partial F}{\partial y} = 4y \quad \frac{\partial F}{\partial z} = 6z$$

~~$$x=y=z$$~~

~~$$x=\frac{1}{2}y=\frac{1}{2}z$$~~

$$x=\frac{1}{2}y=\frac{1}{2}z$$

~~$$4x^2 + 2x^2 + 2x^2 = 21$$~~

~~$$a^2 + a^2 + 12a^2$$~~ ~~$x=1$~~ $y=2$ $z=2$

$$4. \quad 1) \quad \frac{\partial F}{\partial x} = yz \quad \frac{\partial F}{\partial y} = xz \quad \frac{\partial F}{\partial z} = xy$$

$$\text{at } P(x_0, y_0, z_0)$$

$$y_0 z_0 (x - x_0) + x_0 z_0 (y - y_0) + x_0 y_0 (z - z_0) = 0$$

~~$x=3x_0$~~ $(3x_0, 0, 0)$ $(0, 3y_0, 0)$

~~$(0, 0, 3z_0)$~~ $(0, 0, 3z_0)$

$$\frac{27}{2} x_0 y_0 z_0$$

$$u) \quad \frac{\partial F}{\partial x} = \frac{1}{2\sqrt{x}} \quad \frac{\partial F}{\partial y} = \frac{1}{2\sqrt{y}} \quad \frac{\partial F}{\partial z} = \frac{1}{2\sqrt{z}}$$

$$At \quad P = (x_0, y_0, z_0)$$

$$\frac{1}{2\sqrt{x_0}}(x-x_0) + \frac{1}{2\sqrt{y_0}}(y-y_0) + \frac{1}{2\sqrt{z_0}}(z-z_0) = 0$$

$$\frac{1}{2\sqrt{x_0}}(x-x_0) + \frac{1}{2\sqrt{y_0}}(y-y_0) + \frac{1}{2\sqrt{z_0}}(z-z_0) = 0$$

$$\frac{1}{2\sqrt{x_0}}(x-x_0) - \frac{\sqrt{y_0}}{2} - \frac{\sqrt{z_0}}{2} = 0$$

$$x = \sqrt{x_0 y_0} + \sqrt{x_0 z_0} + x_0 = \sqrt{x_0} \cdot \sqrt{y_0 + z_0 + x_0}$$

⋮

$$3) \quad \left(\begin{array}{c} 0, \\ 1, \\ \frac{1}{2} \end{array} \right)$$

$$(1, 2, 0)$$

$$(2, 2, -1)$$

$$\frac{\partial z}{\partial x} = 2x \quad \frac{\partial z}{\partial y} = 2y$$

$$x=y=1 \quad z=2$$

3. 设 $F = x^2 + 2y^2 + 3z^2 - 21 = 0$

$$\frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = 4y, \quad \frac{\partial F}{\partial z} = 6z$$

由平面与 $x+4y+6z=0$ 平行

故 $\frac{2x}{1} = \frac{4y}{4} = \frac{6z}{6} = k$

代入 $x^2 + 2y^2 + 3z^2 = 21$

$\therefore k = \pm 2$

$\therefore x = \pm 1, y = \pm 1, z = \pm 1$

故切平面为

$$2(x-1) + 8(y-1) + 12(z-1) = 0 \quad \text{或} \quad -2(x+1) - 8(y+1) - 12(z+1) = 0$$

4. (1) 令 $F = xyz - a^3 = 0$

$$\frac{\partial F}{\partial x} = yz, \quad \frac{\partial F}{\partial y} = xz, \quad \frac{\partial F}{\partial z} = xy$$

取点为 (x_0, y_0, z_0)

则方程为 $yz_0(x-x_0) + x_0z_0(y-y_0) + x_0y_0(z-z_0) = 0$

三个交点为 $(3x_0, 0, 0), (0, 3y_0, 0), (0, 0, 3z_0)$

$\therefore$ 四面体体积为 $\frac{1}{3} \cdot 3x_0 \cdot 3y_0 \cdot 3z_0 = 9a^3$

(2) 令 $F = \sqrt{x} + \sqrt{y} + \sqrt{z} - \sqrt{a} = 0$

$$\frac{\partial F}{\partial x} = \frac{1}{2\sqrt{x}}, \quad \frac{\partial F}{\partial y} = \frac{1}{2\sqrt{y}}, \quad \frac{\partial F}{\partial z} = \frac{1}{2\sqrt{z}}$$

取点为 $(x_0, y_0, z_0), (R)$

则为 $\frac{1}{2\sqrt{x_0}}(x-x_0) + \frac{1}{2\sqrt{y_0}}(y-y_0) + \frac{1}{2\sqrt{z_0}}(z-z_0) = 0$

求得截距为 $\sqrt{x_0}(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}), \sqrt{y_0}(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}), \sqrt{z_0}(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0})$

$\therefore$ 截距之和为 $(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0})^2 = a$

清华大学

(3) 设 $F = x^2 + y^2 - z = 0$

$$\frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = 2y, \quad \frac{\partial F}{\partial z} = -1$$

∴ 法线为 $(2x, 2y, -1)$

$$\text{又: } \begin{cases} x+2z=1 \\ y+2z=2 \end{cases} \text{ 垂直}$$

$$\therefore \begin{cases} \frac{2x}{1} = -\frac{1}{2} \\ \frac{2y}{1} = -\frac{1}{2} \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{4} \\ y = -\frac{1}{4} \\ z = \frac{1}{8} \end{cases}$$

∴ 方程为 $-\frac{1}{2}(x+\frac{1}{4}) - \frac{1}{2}(y+\frac{1}{4}) - (z-\frac{1}{8}) = 0$

(4) $\frac{\partial f}{\partial x} = f_2, \quad \frac{\partial f}{\partial y} = f_1, \quad \frac{\partial f}{\partial z} = -a \cdot f_1 - b \cdot f_2$

∴ 法向量为 $(f_2, f_1, -a f_1 - b f_2)$

可以设直线的方向量为 $(b, a, 1)$, 故 $f(y-az, x-bz) = 0$ 与方向向量为 $(b, a, 1)$ 的直线平行

(5) 设 $F = y f(\frac{x}{y}) - z = 0$

$$\frac{\partial F}{\partial x} = f'(\frac{x}{y}) \quad \frac{\partial F}{\partial y} = f(\frac{x}{y}) - \frac{x}{y} f'(\frac{x}{y}) \quad \frac{\partial F}{\partial z} = -1$$

设点为 $(x_0, y_0, z_0), (x, y, z)$

方程为 $f'(\frac{x_0}{y_0})(x-x_0) + (f(\frac{x_0}{y_0}) - \frac{x_0}{y_0} f'(\frac{x_0}{y_0}))(y-y_0) - (z-z_0) = 0$

整理可得 $f' \cdot x + f \cdot y - \frac{x_0}{y_0} f' \cdot y - z = 0$

∴ 过定点 $(0, 0, 0)$

清华大学数学作业纸

(科目)

班级计35

姓名谢琛睿

编号2013011406

第 页

5. 记 $f(x, y, z) = x^2 + y^2 + z^2 - 6$, $g(x, y, z) = x + y + z$. 求

$$L: \begin{cases} f(x, y, z) = 0 \\ g(x, y, z) = 0 \end{cases}$$

平面 $f(x, y, z)$ 的法向量为 $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = (2x, 2y, 2z)$.

平面 $g(x, y, z)$ 的法向量为 $(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}) = (1, 1, 1)$.

$\therefore L$ 在 P 的法切向量为

$$(2x, 2y, 2z) \times (1, 1, 1) \Big|_{(1, -2, 1)} = (-6, 0, 6).$$

$$\therefore \text{切线方程} \begin{cases} x = 1-t \\ y = -2 \\ z = 1+t \end{cases} \quad \begin{array}{l} \text{法平面方程} \\ z - x = 0 \end{array} \quad \begin{array}{l} -(x-1) + (z-1) = 0, \text{ 即} \\ z - x = 0 \end{array}$$

6. 证明: 切向量 $(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}) = (-a \sin t, a \cos t, b)$.

~~z 分量为常数, 故切向量与 z 轴所成夹角~~

z 上的方向余弦为 $\frac{b}{\sqrt{(a \sin t)^2 + (a \cos t)^2 + b^2}} = \frac{b}{\sqrt{a^2 + b^2}}$ 为常数.

故切线与 z 轴成定角.

(科目: 微积分A) 数 学 作 业 纸

编号: 2013011400

班级: 计35

姓名: 高胜康

第 1. 页

 习题 1.8 / (1) $z(0,0)=1$

$$J_z(x,y) = (-2x \sin(x^2+y^2), -2y \sin(x^2+y^2)) \quad J_z(0,0) = (0,0)$$

$$H(x,y) = \begin{bmatrix} -2 \sin(x^2+y^2) - 4x^2 \cos(x^2+y^2) & -4xy \cos(x^2+y^2) \\ -4xy \cos(x^2+y^2) & -2 \sin(x^2+y^2) - 4y^2 \cos(x^2+y^2) \end{bmatrix} \quad H(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \text{Peano 余项} \quad z \stackrel{(x,y)}{=} 1 + o(x^2+y^2)$$

$$\text{Lagrange 余项} \quad z \stackrel{(x,y)}{=} 1 + \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix} H(\theta x, \theta y) \begin{pmatrix} x \\ y \end{pmatrix} + \alpha(x,y) \text{ 如上}$$

 (2) $z(0,0)=1$

$$J_z(x,y) = (2x \cdot e^{x^2-y^2}, -2y e^{x^2-y^2}) \quad J_z(0,0) = (0,0)$$

$$H(x,y) = \begin{bmatrix} (4x^2+2)e^{x^2-y^2} & -4xy e^{x^2-y^2} \\ -4xy e^{x^2-y^2} & (4y^2-2)e^{x^2-y^2} \end{bmatrix} \quad H(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\therefore \text{Peano 余项} \quad z(x,y) = 1 + x^2 - y^2 + o(x^2+y^2)$$

$$z(x,y) = 1 + \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix} H(0,0) \begin{pmatrix} x \\ y \end{pmatrix} + \alpha(x,y) \quad , \quad \alpha(x,y) = o(x^2+y^2)$$

 (3) $u(0,0,0)=0$

$$J_u(x,y,z) = \left(\frac{1}{1+x+y+z}, \frac{1}{1+x+y+z}, \frac{1}{1+x+y+z} \right) \quad J_u(0,0,0) = (1,1,1)$$

$$H(x,y,z) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \frac{-1}{(1+x+y+z)^2} \quad H(0,0,0) = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

$$\therefore u(x,y,z) = x+y+z - \frac{1}{2}(x^2+y^2+z^2) - 2xy - 2xz - 2yz + o(x^2+y^2)$$

$$u(x,y,z) = x+y+z + \frac{1}{2} \begin{pmatrix} x \\ y \\ z \end{pmatrix} H(0,0,0) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + o(x^2+y^2)$$

 2.4) $z(1,1)=1$

$$J_z(x,y) = (y x^{y-1}, x^y \ln x) \quad J_z(1,1) = (1,0)$$

$$H(x,y) = \begin{bmatrix} y(y-1)x^{y-2} & y x^{y-1} \ln x \\ y x^{y-1} \ln x & x^y \ln^2 x \end{bmatrix} \quad H(1,1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



(科目:) 数 学 作 业 纸

编号:

班级:

姓名:

第 2 页

▽

(1) $z(1,1)=1$

$$Jz(x,y) = (y x^{y-1}, x^y \ln x), \quad Jz(1,1) = (1,0)$$

$$H(x,y) = \begin{bmatrix} y(y-1)x^{y-2} & x^{y-1} + y x^{y-1} \ln x \\ x^{y-1} + y x^{y-1} \ln x & x^y \ln^2 x \end{bmatrix} \quad H(1,1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\therefore z$ 展开为 $1+x+xy$, 计算略

(2) $z(0,0)=1$

$$Jz(x,y) = \left(-\frac{\sin x}{\cos y}, \frac{-\sinh \cosh x}{\cosh^2 y} \right), \quad Jz(0,0) = (0,0)$$

$$H(x,y) = \begin{bmatrix} -\frac{\cos x}{\cos y} & \frac{\sinh \cosh y}{\cosh^2 y} \\ \frac{\sin x \sinh y}{\cosh^2 y} & \frac{(\cosh y - 2 \sinh^2 y)}{\cosh^3 y} \cosh x \end{bmatrix}, \quad H(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore z$ 展开为 $1 + \frac{1}{2}x^2 + \frac{1}{2}y^2$

(3) $z(0,0)=0$

$$Jz(x,y) = \left(-e^{-x} \ln(1+y), \frac{e^{-x}}{1+y} \right), \quad Jz(0,0) = (0,1)$$

$$H(x,y) = \begin{bmatrix} e^{-x} \ln(1+y) & -\frac{e^{-x}}{1+y} \\ -\frac{e^{-x}}{1+y} & \frac{-e^{-x}}{(1+y)^2} \end{bmatrix}, \quad H(0,0) = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$$

$\therefore z$ 展开为 $y - xy - \frac{1}{2}y^2$

清华大学

1.9

1. 研究下列函数的极值.

(1) $z = x^3 + y^3 - 3x^2 - 3y^2$

解: $\frac{\partial z}{\partial x} = 3x^2 - 6x = 0$ $\frac{\partial z}{\partial y} = 3y^2 - 6y = 0$

$$\Rightarrow \begin{cases} x=0 \text{ 或 } 2 \\ y=0 \text{ 或 } 2 \end{cases}$$

$$\frac{\partial^2 z}{\partial x^2} = 6x - 6 \quad \frac{\partial^2 z}{\partial x \partial y} = 0$$

$$\frac{\partial^2 z}{\partial y^2} = 6y - 6$$

(x, y)	(0, 0)	(0, 2)	(2, 0)	(2, 2)
Hessen	$\begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$	$\begin{bmatrix} -6 & 0 \\ 0 & 6 \end{bmatrix}$	$\begin{bmatrix} 6 & 0 \\ 0 & -6 \end{bmatrix}$	$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$
	负定	不定	不定	正定

$\therefore$ 极大值点 (0, 0) 为 0
极小值点 (2, 2) 为 -4

$u = \sin x + \sin y + \sin z - \sin(x+y+z)$

解: $\frac{\partial u}{\partial x} = \cos x - \cos(x+y+z)$ $\frac{\partial u}{\partial y} = \cos y - \cos(x+y+z)$ $\frac{\partial u}{\partial z} = \cos z - \cos(x+y+z)$

当 $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$ 时

$$\begin{cases} x=0 \text{ 或 } \pi \text{ 或 } \frac{\pi}{2} \\ y=0 \text{ 或 } \pi \text{ 或 } \frac{\pi}{2} \\ z=0 \text{ 或 } \pi \text{ 或 } \frac{\pi}{2} \end{cases}$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x + \sin(x+y+z)$$

$$\frac{\partial^2 u}{\partial y^2} = -\sin y + \sin(x+y+z)$$

$\frac{\partial^2 u}{\partial x \partial y} = \sin(x+y+z)$ $\frac{\partial^2 u}{\partial x \partial z} = \sin(x+y+z)$ $\frac{\partial^2 u}{\partial y \partial z} = \sin(x+y+z)$

(x, y, z)	(0, 0, 0)	(π, π, π)	($\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$)
Hessen	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$
			负定

$\therefore$ 极大值点 ($\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$)
值为 4

(2) $z = e^{2x}(x+y^2+2y)$

解: $\frac{\partial z}{\partial x} = 2e^{2x}(x+y^2+2y) + e^{2x} = 0$ $\frac{\partial z}{\partial y} = e^{2x}(2y+2) = 0$

$$\Rightarrow \begin{cases} x = -\frac{1}{2} \\ y = -1 \end{cases}$$

$$\frac{\partial^2 z}{\partial x^2} = 4e^{2x} + 4e^{2x}(x+y^2+2y)$$

$$\frac{\partial^2 z}{\partial y^2} = 2e^{2x}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 4e^{2x}(2y+2)$$

Hessen: $\begin{bmatrix} 2e & 4e \\ 4e & 2e \end{bmatrix}$ 不定 $\therefore$ 极小值点 $(-\frac{1}{2}, -1)$ 值为 $-\frac{1}{2}e$

(4) $z = x_1 + \frac{x_2}{x_1} + \dots + \frac{x_n}{x_{n-1}} + \frac{2}{x_n}$

解: $\frac{\partial z}{\partial x_1} = 1 - \frac{x_2}{x_1^2}$ $\frac{\partial z}{\partial x_2} = \frac{1}{x_1} - \frac{x_3}{x_2^2}$ $\dots$ $\frac{\partial z}{\partial x_{n-1}} = \frac{1}{x_{n-2}} - \frac{x_n}{x_{n-1}^2}$ $\frac{\partial z}{\partial x_n} = \frac{1}{x_{n-1}} - \frac{2}{x_n^2}$

当上式全为 0 时, $x_m = x_{m+1}$ $\Rightarrow x_1 = 2^{\frac{1}{n+1}}$

$\therefore x_1 + \frac{x_2}{x_1} + \dots + \frac{x_n}{x_{n-1}} + \frac{2}{x_n} \geq \sqrt[n+1]{2}$

$\therefore$ 最小值点为极小值点
 $\therefore$ 极小值点 $(2^{\frac{1}{n+1}}, 2^{\frac{1}{n+1}}, \dots, 2^{\frac{1}{n+1}})$
值为 $(n+1)^{\frac{1}{n+1}} \sqrt[n+1]{2}$

清华大学

(5) $u = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$ (z 应为 1 (左侧)) ($x > 0, y > 0, z > 0$)

解: $\frac{\partial u}{\partial x} = 1 - \frac{y^2}{4x^2}$ $\frac{\partial u}{\partial y} = \frac{y}{2x} - \frac{z^2}{y^2}$ $\frac{\partial u}{\partial z} = \frac{z}{y} - \frac{2}{z^2}$

上式全为 0, 得 $\begin{cases} x = \frac{1}{2} \\ y = 1 \\ z = 1 \end{cases}$

$\frac{\partial^2 u}{\partial x^2} = \frac{y^2}{2x^3}$
 $\frac{\partial^2 u}{\partial x \partial z} = 0$

$\frac{\partial^2 u}{\partial y^2} = \frac{1}{2x} + \frac{2z^2}{y^3}$ $\frac{\partial^2 u}{\partial x^2} = \frac{2}{y} + \frac{4}{z^3}$ $\frac{\partial^2 u}{\partial y \partial z} = -\frac{2z}{y^2}$

Hessian:

$(x, y, z) = (\frac{1}{2}, 1, 1)$
 $\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & 2 & -6 \end{bmatrix}$
 正定

极大值点 $(\frac{1}{2}, 1, 1)$ 值为 4

极大值点 $(\frac{1}{2}, 1, 1)$ 为 4

2. (1) $2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0$

解: 设 $F(x, y, z) = 2x^2 + 2y^2 + z^2 + 8xz - z + 8$

$\frac{\partial F}{\partial x} = 4x + 8z$ $\frac{\partial F}{\partial y} = 4y$ $\frac{\partial F}{\partial z} = 2z + 8x - 1$ $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{4x + 8z}{2z + 8x - 1} = 0$ $\frac{\partial z}{\partial y} = \frac{-4y}{2z + 8x - 1} = 0$

得出: $\begin{cases} x = -2z \\ y = 0 \end{cases}$ 代入原式: $8z^2 + z^2 - 16z^2 - z + 8 = 0 \Rightarrow z = (7z + 8)(z - 1) = 0$

$\frac{\partial^2 z}{\partial x^2} = -\frac{\frac{\partial(4x+8z)}{\partial x}(2z+8x-1) - \frac{\partial(2z+8x-1)}{\partial x}(4x+8z)}{(2z+8x-1)^2} = -\frac{4(2z+8x-1)}{(2z+8x-1)^2} = -\frac{4}{2z+8x-1}$

$\frac{\partial^2 z}{\partial y^2} = -\frac{4}{2z+8x-1}$

$\frac{\partial^2 z}{\partial x \partial y} = 0$

Hessian 矩阵
行列式为

$\begin{bmatrix} \frac{4}{15} & 0 \\ 0 & \frac{4}{15} \\ 4 & 0 \\ 0 & -\frac{4}{15} \end{bmatrix}$

极大值点 $(-2, 0, 1)$
极大值点 $(\frac{4}{7}, 0, -\frac{8}{7})$
极大值为 $-\frac{8}{7}$, 极小值为 1

(2) $x^2 + \frac{y^2}{4} + \frac{z^2}{9} - \frac{2}{3}z = 0$

解: $F(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9} - \frac{2}{3}z$

$\frac{\partial F}{\partial x} = 2x$ $\frac{\partial F}{\partial y} = \frac{y}{2}$ $\frac{\partial F}{\partial z} = \frac{2}{9}z - \frac{2}{3}$ $\frac{\partial z}{\partial x} = -\frac{2x}{\frac{2}{9}z - \frac{2}{3}} = 0$ $\frac{\partial z}{\partial y} = -\frac{\frac{y}{2}}{\frac{2}{9}z - \frac{2}{3}} = 0$

当 $x = y = 0$ 时 $z = 0$ 或 6

$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{\frac{2}{9}z - \frac{2}{3}}$

$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{\frac{2}{9}z - \frac{2}{3}}$

$\frac{\partial^2 z}{\partial x \partial y} = 0$

Hessian 矩阵 $\begin{bmatrix} 3 & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$
行列式为 $\begin{bmatrix} 3 & 0 \\ 0 & -\frac{3}{4} \end{bmatrix}$
正定 负定

极大值点 $(0, 0, 6)$ 为 6
极小值点 $(0, 0, 0)$ 为 0

清华大学

(5) $u = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$ (z 应为 1 (左侧)) ($x > 0, y > 0, z > 0$)

解: $\frac{\partial u}{\partial x} = 1 - \frac{y^2}{4x^2}$ $\frac{\partial u}{\partial y} = \frac{y}{2x} - \frac{z^2}{y^2}$ $\frac{\partial u}{\partial z} = \frac{z}{y} - \frac{2}{z^2}$

上式全为 0, 得 $\begin{cases} x = \frac{1}{2} \\ y = 1 \\ z = 1 \end{cases}$

$\frac{\partial^2 u}{\partial x^2} = \frac{y^2}{2x^3}$
 $\frac{\partial^2 u}{\partial x \partial z} = 0$

$\frac{\partial^2 u}{\partial y^2} = \frac{1}{2x} + \frac{2z^2}{y^3}$ $\frac{\partial^2 u}{\partial x^2} = \frac{2}{y} + \frac{4}{z^3}$ $\frac{\partial^2 u}{\partial y \partial z} = -\frac{2z}{y^2}$

(x, y, z)

Hessen: $\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 6 \end{bmatrix}$ $\begin{bmatrix} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -6 \end{bmatrix}$
正定 负定

~~极大值点 $(\frac{1}{2}, 1, 1)$ 值为 4~~
~~极小值点 $(\frac{1}{2}, 1, 1)$ 值为 4~~

极大值点 $(\frac{1}{2}, 1, 1)$ 为 4

2. (1) $2x^2 + 2y^2 + z^2 + 8xz - z + 8 = 0$

解: 设 $F(x, y, z) = 2x^2 + 2y^2 + z^2 + 8xz - z + 8$

$\therefore \frac{\partial F}{\partial x} = 4x + 8z$ $\frac{\partial F}{\partial y} = 4y$ $\frac{\partial F}{\partial z} = 2z + 8x - 1$ $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{4x + 8z}{2z + 8x - 1} = 0$ $\frac{\partial z}{\partial y} = \frac{-4y}{2z + 8x - 1} = 0$

得出: $\begin{cases} x = -2z \\ y = 0 \end{cases}$ 代入原式: $8z^2 + z^2 - 16z^2 - z + 8 = 0 \Rightarrow z = (7z + 8)(z - 1) = 0$

$\frac{\partial^2 z}{\partial x^2} = -\frac{\frac{\partial(4x+8z)}{\partial x}(2z+8x-1) - \frac{\partial(2z+8x-1)}{\partial x}(4x+8z)}{(2z+8x-1)^2} = -\frac{4(2z+8x-1)}{2z+8x-1} = -4$

$\frac{\partial^2 z}{\partial y^2} = -\frac{4(2z+8x-1) - \frac{\partial(2z+8x-1)}{\partial y} \cdot 4y}{(2z+8x-1)^2} = -\frac{4}{2z+8x-1}$

$\frac{\partial^2 z}{\partial x \partial y} = 0$

Hessen 矩阵
分别为

$\begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$ $\begin{bmatrix} \frac{4}{15} & 0 \\ 0 & \frac{4}{15} \end{bmatrix}$ $\begin{bmatrix} 4 & 0 \\ 15 & 0 \\ 0 & -\frac{4}{15} \end{bmatrix}$

极小: 极大值点 $(-2, 0, 1)$
极大值点 $(\frac{4}{7}, 0, -\frac{8}{7})$
极大值为 $-\frac{8}{7}$, 极小值为 1

(2) $x^2 + \frac{y^2}{4} + \frac{z^2}{9} - \frac{2}{3}z = 0$

解: $F(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9} - \frac{2}{3}z$

$\therefore \frac{\partial F}{\partial x} = 2x$ $\frac{\partial F}{\partial y} = \frac{y}{2}$ $\frac{\partial F}{\partial z} = \frac{2}{9}z - \frac{2}{3}$ $\frac{\partial z}{\partial x} = -\frac{2x}{\frac{2}{9}z - \frac{2}{3}} = 0$ $\frac{\partial z}{\partial y} = -\frac{\frac{y}{2}}{\frac{2}{9}z - \frac{2}{3}} = 0$

当 $x = y = 0$ 时 $z = 0$ 或 $z = 6$

$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{\frac{2}{9}z - \frac{2}{3}}$

$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{\frac{2}{9}z - \frac{2}{3}}$

$\frac{\partial^2 z}{\partial x \partial y} = 0$

Hessen 矩阵 $\begin{bmatrix} 2 & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$ $\begin{bmatrix} 3 & 0 \\ 0 & -\frac{3}{4} \end{bmatrix}$
分别为 正定 负定

极大值点 $(0, 0, 6)$ 为 6
极小值点 $(0, 0, 0)$ 为 0

清华大学

$$4. (1) \frac{\partial z}{\partial x} = 2xe^{-x^2-y^2} - 2x(x^2+y^2)e^{-x^2-y^2} = 0 \quad \frac{\partial z}{\partial y} = 4ye^{-x^2-y^2} - 2y(x^2+y^2)e^{-x^2-y^2} = 0$$

当上二式为0, $\begin{cases} x=0 \\ y=0 \end{cases}$ 或 $\begin{cases} x=\pm 1 \\ y=0 \end{cases}$ 或 $\begin{cases} x=0 \\ y=\pm 1 \end{cases}$

$$\frac{\partial^2 z}{\partial x \partial y} = -4xye^{-x^2-y^2} - 8xye^{-x^2-y^2} + 4xy(x^2+y^2)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = 2e^{-x^2-y^2} - 4x^2e^{-x^2-y^2} - 8xye^{-x^2-y^2} + 4xy(x^2+y^2)e^{-x^2-y^2} - (6x^3+4y^4)e^{-x^2-y^2} + 4x^2(x^2+y^2)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = 4e^{-x^2-y^2} - 8y^2e^{-x^2-y^2} - (2x^2+12y^2)e^{-x^2-y^2} + 4y^2(x^2+y^2)e^{-x^2-y^2}$$

$\therefore$ 当 x, y 为上面5点时, 经求Hessen矩阵, 得:

极大值点 $(0, \pm 1)$ 极大值 $2e^{-1}$

极小值点 $(0, 0)$ 极小值 0

$$(2) \frac{\partial z}{\partial x} = 4y - 2xy - y^2 = 0 \quad \frac{\partial z}{\partial y} = 4x - x^2 - 2xy = 0 \quad \text{此时} \begin{cases} x=4 \\ y=0 \end{cases} \text{或} \begin{cases} x=2 \\ y=2 \end{cases}$$

$$\text{当 } x+y=6 \text{ 时 设 } F = 4xy - x^2y - y^2 - \lambda(x+y-6) \quad \frac{\partial F}{\partial x} = 0 \quad \frac{\partial F}{\partial y} = 0 \quad \frac{\partial F}{\partial \lambda} = 0 \quad \text{此时} \begin{cases} x=3 \\ y=3 \end{cases}$$

$$\text{当 } x=1 \text{ 时, } z_{\max} = \frac{9}{4} \quad z_{\min} = -10$$

$$\therefore \text{综上, 求出 } z_{\min} = -18 \quad z_{\max} = \frac{9}{4}$$

5. (1) 证明:

若 $f(x, y)$ 不恒为0 则由于连续, f 在 D 上有极值. (因为边界都为0)

$\therefore$ 设有极大值 则 $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \quad \therefore kf(x, y) = 0 \quad \because k > 0 \quad \therefore f(x, y) = 0$ 矛盾

极小值类似

$\therefore$ 得证.

(2) 证明: 若 $u(x, y) \leq 0$

则 $\exists (x, y), x^2+y^2 < 1, (x, y)$ 为极小值点

$$\therefore \frac{\partial^2 u}{\partial x^2} > 0 \quad \frac{\partial^2 u}{\partial y^2} > 0 \Rightarrow u > 0 \quad \text{矛盾!}$$

$\therefore$ 得证.

清华大学数学作业纸

(科目微积分)

班级 计35

姓名 蒋文明

编号

第 页

1.9.

$$6. \because \begin{cases} \frac{\partial z}{\partial x} = -\sin x (1+e^y) = 0 \\ \frac{\partial z}{\partial y} = e^y (\cos x - 1 - y) = 0 \end{cases} \quad \text{解得} \begin{cases} x = 2k\pi + \pi \\ y = -2 \end{cases} \quad \text{或} \begin{cases} x = 2k\pi \\ y = 0 \end{cases}$$

$$\therefore H_z(x, y) = \begin{pmatrix} -\cos x (1+e^y) & -\sin x \cdot e^y \\ -\sin x \cdot e^y & e^y (\cos x - 2 - y) \end{pmatrix}$$

$$\textcircled{1} \text{ 当 } \begin{cases} x = 2k\pi + \pi \\ y = -2 \end{cases} \text{ 时 } H_z(x, y) = \begin{pmatrix} 1+e^{-2} & 0 \\ 0 & -e^{-2} \end{pmatrix} \text{ 负定. } \therefore \text{为极大值点.}$$

$$\textcircled{2} \text{ 当 } \begin{cases} x = 2k\pi \\ y = 0 \end{cases} \text{ 时 } H_z(x, y) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \text{ 负定 } \therefore \text{为极大值点.}$$

$\therefore$ 有无穷多个极大值点, 而无极小值.

$$7. \textcircled{1} \text{ 令 } L(x, y, \lambda) = x^2 + y^2 + \lambda \left(\frac{x}{a} + \frac{y}{b} - 1 \right)$$

$$\text{则由 } \begin{cases} \frac{\partial L}{\partial x} = 2x + \frac{\lambda}{a} = 0 \\ \frac{\partial L}{\partial y} = 2y + \frac{\lambda}{b} = 0 \\ \frac{\partial L}{\partial \lambda} = \frac{x}{a} + \frac{y}{b} = 1 \end{cases} \quad \text{解得} \begin{cases} x = \frac{ab^2}{a^2+b^2} \\ y = \frac{a^2b}{a^2+b^2} \\ \lambda = \frac{-2a^2b^2}{a^2+b^2} \end{cases}$$

$$\therefore \text{极小值为 } z_{\min} = \frac{a^2b^4 + a^4b^2}{(a^2+b^2)^2} = \frac{a^2b^2}{a^2+b^2}$$

$$\textcircled{2} \text{ 令 } L(x, y, z, \lambda) = x - 2y + 2z + \lambda (x^2 + y^2 + z^2 - 1)$$

$$\text{则 } \begin{cases} \frac{\partial L}{\partial x} = 1 + 2\lambda x = 0 \\ \frac{\partial L}{\partial y} = -2 + 2\lambda y = 0 \\ \frac{\partial L}{\partial z} = 2 + 2\lambda z = 0 \\ \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 = 1 \end{cases} \quad \therefore \begin{cases} x = -\frac{3}{4} \\ y = \frac{2}{3} \\ z = -\frac{2}{3} \\ \lambda = \frac{3}{2} \end{cases} \quad \text{或} \begin{cases} x = \frac{3}{4} \\ y = -\frac{2}{3} \\ z = \frac{2}{3} \\ \lambda = -\frac{3}{2} \end{cases}$$

$$\therefore \text{极小值为 } u_{\min} = -\frac{41}{12}. \quad \text{极大值为 } \frac{41}{12}.$$

清华大学数学作业纸

(科目微积分)

班级 计35

姓名 蒋文明

编号

第 页

1.9.

$$6. \because \begin{cases} \frac{\partial z}{\partial x} = -\sin x (1+e^y) = 0 \\ \frac{\partial z}{\partial y} = e^y (\cos x - 1 - y) = 0 \end{cases} \quad \text{解得} \begin{cases} x = 2k\pi + \pi \\ y = -2 \end{cases} \quad \text{或} \begin{cases} x = 2k\pi \\ y = 0 \end{cases}$$

$$\therefore H_z(x, y) = \begin{pmatrix} -\cos x (1+e^y) & -\sin x \cdot e^y \\ -\sin x \cdot e^y & e^y (\cos x - 2 - y) \end{pmatrix}$$

$$\textcircled{1} \text{ 当 } \begin{cases} x = 2k\pi + \pi \\ y = -2 \end{cases} \text{ 时 } H_z(x, y) = \begin{pmatrix} 1+e^{-2} & 0 \\ 0 & -e^{-2} \end{pmatrix} \text{ 负定. } \therefore \text{为极大值点.}$$

$$\textcircled{2} \text{ 当 } \begin{cases} x = 2k\pi \\ y = 0 \end{cases} \text{ 时 } H_z(x, y) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \text{ 负定. } \therefore \text{为极大值点.}$$

$\therefore$ 有无穷多个极大值点, 而无极小值.

$$7. \textcircled{1} \text{ 令 } L(x, y, \lambda) = x^2 + y^2 + \lambda \left(\frac{x}{a} + \frac{y}{b} - 1 \right)$$

$$\text{则由 } \begin{cases} \frac{\partial L}{\partial x} = 2x + \frac{\lambda}{a} = 0 \\ \frac{\partial L}{\partial y} = 2y + \frac{\lambda}{b} = 0 \\ \frac{\partial L}{\partial \lambda} = \frac{x}{a} + \frac{y}{b} = 1 \end{cases} \quad \text{解得} \begin{cases} x = \frac{ab^2}{a^2+b^2} \\ y = \frac{a^2b}{a^2+b^2} \\ \lambda = \frac{-2a^2b^2}{a^2+b^2} \end{cases}$$

$$\therefore \text{极小值为 } z_{\min} = \frac{a^2b^4 + a^4b^2}{(a^2+b^2)^2} = \frac{a^2b^2}{a^2+b^2}$$

$$\textcircled{2} \text{ 令 } L(x, y, z, \lambda) = x - 2y + 2z + \lambda (x^2 + y^2 + z^2 - 1)$$

$$\text{则由 } \begin{cases} \frac{\partial L}{\partial x} = 1 + 2\lambda x = 0 \\ \frac{\partial L}{\partial y} = -2 + 2\lambda y = 0 \\ \frac{\partial L}{\partial z} = 2 + 2\lambda z = 0 \\ \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 = 1 \end{cases} \quad \therefore \begin{cases} x = -\frac{3}{4} \\ y = \frac{2}{3} \\ z = -\frac{2}{3} \\ \lambda = \frac{3}{2} \end{cases} \quad \text{或} \quad \begin{cases} x = \frac{3}{4} \\ y = -\frac{2}{3} \\ z = \frac{2}{3} \\ \lambda = -\frac{3}{2} \end{cases}$$

$$\therefore \text{极小值为 } u_{\min} = -\frac{41}{12}, \quad \text{极大值为 } \frac{41}{12}.$$

清华大学数学作业纸

(科目)

班级

姓名

编号

第

页

33) 令 $L(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda \left(\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} - 1 \right)$

$$\begin{cases} \frac{\partial L}{\partial x} = 2x + \lambda \cdot \frac{2x}{16} = 0 \\ \frac{\partial L}{\partial y} = 2y + \lambda \cdot \frac{2y}{9} = 0 \\ \frac{\partial L}{\partial z} = 2z + \lambda \cdot \frac{2z}{4} = 0 \\ \frac{\partial L}{\partial \lambda} = \frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} - 1 = 0 \end{cases} \quad \text{解得} \quad \begin{cases} x = \pm 4 \\ y = 0 \\ z = 0 \\ \lambda = -16 \end{cases} \quad \begin{cases} x = 0 \\ y = \pm 3 \\ z = 0 \\ \lambda = -9 \end{cases} \quad \begin{cases} x = 0 \\ y = 0 \\ z = \pm 2 \\ \lambda = -4 \end{cases}$$

$\therefore$ 极大值为 16, 9. 极大值为 4.

4) 令 $L(x, y, z, \lambda_1, \lambda_2) = x^2 + y^2 + z^2 + \lambda_1(x + y - z - 1) + \lambda_2(x + y + z)$

$$\text{则} \begin{cases} \frac{\partial L}{\partial x} = 2x + \lambda_1 + \lambda_2 = 0 \\ \frac{\partial L}{\partial y} = 2y + \lambda_1 + \lambda_2 = 0 \\ \frac{\partial L}{\partial z} = 2z - \lambda_1 + \lambda_2 = 0 \\ \frac{\partial L}{\partial \lambda_1} = x + y - z - 1 = 0 \\ \frac{\partial L}{\partial \lambda_2} = x + y + z = 0 \end{cases} \quad \text{解得} \quad \begin{cases} x = \frac{1}{4} \\ y = \frac{1}{4} \\ z = -\frac{1}{2} \\ \lambda_1 = -\frac{3}{4} \\ \lambda_2 = \frac{1}{4} \end{cases}$$

$\therefore$ 极小值 $u_{\min} = \frac{3}{8}$.

8. 解: ① 当 $x^2 + y^2 + z^2 < 4$ 时.

$$\text{由} \begin{cases} \frac{\partial u}{\partial x} = 2x - 2y = 0 \\ \frac{\partial u}{\partial y} = 4y - 2z - 2x = 0 \\ \frac{\partial u}{\partial z} = 2z - 2y = 0 \end{cases} \quad \text{解得} \quad x = y = z$$

$\therefore$ 最小值为 $u_{\min} = 0$.

② 当 $x^2 + y^2 + z^2 = 4$ 时.

令 $L(x, y, z, \lambda) = x^2 + y^2 + z^2 - 2xy - 2yz + \lambda(x^2 + y^2 + z^2 - 4)$

$$\text{由} \begin{cases} \frac{\partial L}{\partial x} = 2x - 2y + 2\lambda x = 0 \\ \frac{\partial L}{\partial y} = 4y - 2z - 2x + 2\lambda y = 0 \\ \frac{\partial L}{\partial z} = 2z - 2y + 2\lambda z = 0 \\ \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 4 = 0 \end{cases}$$

清华大学数学作业纸

(科目)

班级

姓名

编号

第

页

$$\text{得} \begin{cases} x = \pm \frac{2}{\sqrt{3}} \\ y = \pm \frac{2}{\sqrt{3}} \\ z = \pm \frac{2}{\sqrt{3}} \\ \lambda = 0. \end{cases} \begin{cases} x = \pm 2 \\ y = 0 \\ z = 0 \\ \lambda = -1. \end{cases} \begin{cases} x = 0 \\ y = \pm 2 \\ z = 0 \\ \lambda = -1. \end{cases} \begin{cases} x = 0 \\ y = 0 \\ z = \pm 2 \\ \lambda = -1. \end{cases}$$

$\therefore$ 最大值 $u_{\max} = 4$.

$$9. \text{ (1) 令 } L(x, y, \lambda) = \frac{4-x-y}{\sqrt{2}} + \lambda \left(x^2 + \frac{y^2}{4} - 1 \right)$$

$$\begin{cases} \frac{\partial L}{\partial x} = -\frac{1}{\sqrt{2}} + 2\lambda x = 0. \\ \frac{\partial L}{\partial y} = -\frac{1}{\sqrt{2}} + \lambda \cdot \frac{2y}{4} = 0. \\ \frac{\partial L}{\partial \lambda} = x^2 + \frac{y^2}{4} - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{2}} \\ y = \frac{4}{\sqrt{2}} \\ \lambda = \frac{\sqrt{2}}{2} \end{cases} \begin{cases} x = -\frac{1}{\sqrt{2}} \\ y = -\frac{4}{\sqrt{2}} \\ \lambda = -\frac{\sqrt{2}}{2} \end{cases}$$

$\therefore$ 最大值: $\frac{4+\sqrt{2}}{\sqrt{2}}$ 最小值: $\frac{4-\sqrt{2}}{\sqrt{2}}$

$$(2) \text{ 令 } L(x, y, z, \lambda) = \frac{x+y-z-1}{\sqrt{3}} + \lambda (x^2 + y^2 - z)$$

$$\begin{cases} \frac{\partial L}{\partial x} = \frac{1}{\sqrt{3}} + 2\lambda x = 0. \\ \frac{\partial L}{\partial y} = \frac{1}{\sqrt{3}} + 2\lambda y = 0. \\ \frac{\partial L}{\partial z} = -\frac{1}{\sqrt{3}} - \lambda = 0. \\ \frac{\partial L}{\partial \lambda} = x^2 + y^2 - z = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ y = \frac{1}{2} \\ z = \frac{1}{2} \\ \lambda = -\frac{1}{\sqrt{3}} \end{cases}$$

$\therefore$ 最小值: $\frac{1}{2\sqrt{3}}$.

$$(3) \text{ 令 } L(x, y, z, \lambda) = 8xyz + \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) \quad (x, y, z > 0)$$

$$\begin{cases} \frac{\partial L}{\partial x} = 8yz + \lambda \cdot \frac{2x}{a^2} = 0. \\ \frac{\partial L}{\partial y} = 8xz + \lambda \cdot \frac{2y}{b^2} = 0. \\ \frac{\partial L}{\partial z} = 8xy + \lambda \cdot \frac{2z}{c^2} = 0. \\ \frac{\partial L}{\partial \lambda} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0. \end{cases} \Rightarrow \begin{cases} x = \frac{a}{\sqrt{3}} \\ y = \frac{b}{\sqrt{3}} \\ z = \frac{c}{\sqrt{3}} \\ \lambda = -\frac{4}{\sqrt{3}}abc \end{cases}$$

$\therefore$ 最大值: $\frac{8abc}{3\sqrt{3}}$.

清华大学数学作业纸

(科目)

班级

姓名

编号

第 页

<4> 令 $L(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(xy + x - y - z^2 + 4)$

$$\begin{cases} \frac{\partial L}{\partial x} = 2x + \lambda(y+1) = 0 \\ \frac{\partial L}{\partial y} = 2y + \lambda(x-1) = 0 \\ \frac{\partial L}{\partial z} = 2z + \lambda(-2z) = 0 \\ \frac{\partial L}{\partial \lambda} = xy + x - y - z^2 + 4 = 0 \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = 1 \\ z = \pm 1 \\ \lambda = 1 \end{cases}$$

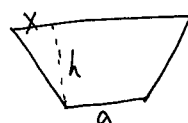
$\therefore$ 最小值为 $\sqrt{3}$.

<5> 令 $L(x, y, \lambda) = x^2 + y^2 + \lambda(x^2 - xy + y^2 - 1) = 0$

$$\begin{cases} \frac{\partial L}{\partial x} = 2x + \lambda(2x - y) = 0 \\ \frac{\partial L}{\partial y} = 2y + \lambda(-x + 2y) = 0 \\ \frac{\partial L}{\partial \lambda} = x^2 - xy + y^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \\ \lambda = -2 \end{cases}$$

$\therefore$ 最短距离为: $\sqrt{2}$.

10. <1> 令 $L(x, h, a, \lambda) = 2\sqrt{x^2 + h^2} + a + \lambda[(a+x)h - S]$



$$\begin{cases} \frac{\partial L}{\partial x} = \frac{2x}{\sqrt{x^2 + h^2}} + \lambda h = 0 \\ \frac{\partial L}{\partial h} = \frac{2h}{\sqrt{x^2 + h^2}} + \lambda \cdot x = 0 \\ \frac{\partial L}{\partial a} = 1 + \lambda h = 0 \\ \frac{\partial L}{\partial \lambda} = (a+x)h = S \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \sqrt{\frac{4S}{3\lambda}} \\ y = \frac{\sqrt{3}}{2} \sqrt{\frac{4S}{3\lambda}} \\ a = \sqrt{\frac{4S}{3\lambda}} \end{cases}$$

$\therefore$ 上、下底及腰的比例为 $2:1:1$.

<2> 令 $L(r, d, \lambda) = \frac{1}{2}\pi r^2 d + \lambda(\pi r d + 2\pi r d + \pi r^2) - \lambda S$

$$\begin{cases} \frac{\partial L}{\partial x} = \pi r d + \lambda(\pi d + 2\pi d + 2\pi r) = 0 \\ \frac{\partial L}{\partial d} = \frac{1}{2}\pi r^2 + \lambda(\pi r + 2\pi r) = 0 \\ \frac{\partial L}{\partial \lambda} = \pi r d + 2\pi r d + \pi r^2 = S \end{cases}$$

自行求解. r, d

清华大学数学作业纸

(科目)

班级

姓名

编号

第 页

<3> 令 $L(x, y, \lambda) = \pi y^2 x + \lambda(x + y - p)$

$$\begin{cases} \frac{\partial L}{\partial x} = \pi y^2 + \lambda = 0. \\ \frac{\partial L}{\partial y} = \pi x \cdot 2y + \lambda = 0. \\ \frac{\partial L}{\partial \lambda} = x + y = p \end{cases} \Rightarrow \begin{cases} x = \frac{p}{3} \\ y = \frac{2p}{3} \\ \lambda = \pi \cdot \frac{4p^2}{9} \end{cases}$$

$\therefore$ 最大值为: $\frac{4\pi p^3}{27}$

<4> 令 $L(r, x, y, \lambda) = \frac{\sqrt{3}}{4}r^2 + xy + \lambda(3r + 2x + 2y - a)$

$$\text{则} \begin{cases} \frac{\partial L}{\partial r} = \frac{\sqrt{3}}{2}r + 3\lambda = 0. \\ \frac{\partial L}{\partial x} = y + 2\lambda = 0. \\ \frac{\partial L}{\partial y} = x + 2\lambda = 0. \\ \frac{\partial L}{\partial \lambda} = 3r + 2x + 2y = a \end{cases} \Rightarrow \begin{cases} x = \frac{a}{3\sqrt{3}+4} \\ y = \frac{a}{3\sqrt{3}+4} \\ r = \frac{\sqrt{3}a}{3\sqrt{3}+4} \\ \lambda = -\frac{a}{6\sqrt{3}+8} \end{cases}$$

$\therefore$ 最大值: $\frac{a^2}{4(3\sqrt{3}+4)}$

清华大学数学作业纸

(科目)

班级

姓名 温和

编号

第 页

1.9

11. (1) 令 $L = x^2 y^2 z^2 + \lambda (x^2 + y^2 + z^2 - a^2)$

则
$$\begin{cases} \frac{\partial L}{\partial x} = 2xy^2z^2 + 2\lambda x = 0 \\ \frac{\partial L}{\partial y} = 2yx^2z^2 + 2\lambda y = 0 \\ \frac{\partial L}{\partial z} = 2zx^2y^2 + 2\lambda z = 0 \\ \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - a^2 = 0 \end{cases}$$

可解得 $x^2 = y^2 = z^2 = \frac{a^2}{3}$ 时取最大值 $\frac{a^6}{27}$
 $\therefore$ 有 $\sqrt[3]{x^2 y^2 z^2} \leq \sqrt[3]{\frac{a^6}{27}} = \frac{a^2}{3} = \frac{x^2 + y^2 + z^2}{3}$

(2) 证第一个 $\sqrt{\frac{\sum a_i^2}{n}} \geq \frac{a_1 + a_2 + \dots + a_n}{n}$: 求 $\sum a_i$ 在 $\sum a_i^2 = S^2$ 下的最大值

令 $L = \sum a_i + \lambda (\sum a_i^2 - S^2)$.

则
$$\begin{cases} 1 + 2\lambda a_i = 0, \quad i=1, 2, \dots, n \\ \sum a_i^2 - S^2 = 0 \end{cases}$$
 可解得 $a_1^2 = a_2^2 = \dots = a_n^2 = \frac{S^2}{n}$ 时取最大值 $\sqrt{\frac{S^2}{n}}$
 $\therefore$ 有 $\frac{\sum a_i}{n} \leq \sqrt{\frac{S^2}{n}} = \sqrt{\frac{\sum a_i^2}{n}}$

证第二个 $\frac{\sum a_i}{n} \geq \sqrt[n]{\prod a_i}$ 时取 $\prod a_i$ 在 $\sum a_i = S$ 时最大值. 同一可证.

证第三个 $\sqrt[n]{\prod a_i} \geq \frac{n}{\sum \frac{1}{a_i}}$ 时亦同理.

12. (1) 设 $P(x, y)$. 则要求 $\sum [(x-x_i)^2 + (y-y_i)^2]$ 最小.

令 $F = \sum [(x-x_i)^2 + (y-y_i)^2]$ 则
$$\begin{cases} \frac{\partial F}{\partial x} = \sum 2(x-x_i) = 0 \\ \frac{\partial F}{\partial y} = \sum 2(y-y_i) = 0 \end{cases}$$

$\therefore \begin{cases} x = \frac{\sum x_i}{n} \\ y = \frac{\sum y_i}{n} \end{cases}$ 即几何中心

(2) 设 $Q(x, y)$, 则要求 $\sum m_i [(x-x_i)^2 + (y-y_i)^2]$ 最小.

$$\begin{cases} \frac{\partial J}{\partial x} = \sum 2m_i (x-x_i) = 0 \\ \frac{\partial J}{\partial y} = \sum 2m_i (y-y_i) = 0 \end{cases}$$

$\therefore \begin{cases} x = \frac{\sum m_i x_i}{\sum m_i} \\ y = \frac{\sum m_i y_i}{\sum m_i} \end{cases}$ 即质心

清华大学数学作业纸

(科目)

班级

姓名

编号

第 页

$$13. \sum_{i=1}^4 (a+bx_i - y_i)^2 = \sum x_i^2 2ab + 4a^2 + b^2 \sum x_i^2 + \sum y_i^2 - 2a \sum y_i - 2b \sum x_i y_i$$

$$= 26.8ab + 4a^2 + 46.5ab^2 + 13 - 12a - 45.8b$$

$$\frac{\partial F}{\partial a} = 26.8b + 8a - 12 \quad b = 1.74$$

$$\frac{\partial F}{\partial b} = 93b + 26.8a - 45.8 \quad a = -4.329$$

$$14. \sum (ax_i^2 + bx_i + c - y_i)^2 = a^2 \sum x_i^4 + b^2 \sum x_i^2 + 5c^2 + \sum y_i^2$$

$$+ 2ab \sum x_i^3 + 2ac \sum x_i^2 - 2a \sum x_i y_i$$

$$+ 2bc \sum x_i - 2b \sum x_i y_i - 2c \sum y_i$$

$$= 979a^2 + 55b^2 + 5c^2 + 167.36$$

$$+ 450ab + 110ac - 43.8a$$

$$+ 30bc - 43.8b - 56c$$

$$\sum x_i^4 = 979$$

$$\sum x_i^3 = 225$$

$$\sum x_i^2 = 55$$

$$\sum x_i y_i = 22.9$$

$$\sum y_i^2 = 167.36$$

$$\sum x_i = 15$$

$$\sum y_i =$$

$$\frac{\partial F}{\partial a} = 1958a + 450b + 110c - 43.8 = 0$$

$$\frac{\partial F}{\partial b} = 450a + 110b + 30c - 43.8 = 0$$

$$\frac{\partial F}{\partial c} = 110a + 30b + 10c - 56 = 0$$

$$a = \frac{34600}{5600}$$

$$b = \frac{-242376}{5600}$$

$$c = \frac{-533608}{5600}$$

清华大学数学作业纸

(科目 微积分)

班级 计35 姓名 靳瑞

编号 2013011411

第 1 页

$$(1) \quad \forall X_0 \in \mathbb{R}^n, \quad \emptyset \setminus X_0 = \emptyset$$

$$\therefore B(X_0, \Delta) \cap (\emptyset \setminus X_0) = \emptyset$$

$$\therefore d(\emptyset) = \emptyset.$$

$$(2) \quad \forall X_0 \in d(A), \quad B(X_0, \Delta) \cap (A \setminus X_0) \neq \emptyset.$$

$$\because A \subseteq B \quad \therefore B(X_0, \Delta) \cap (B \setminus X_0) \neq \emptyset$$

$$\therefore X_0 \in d(B)$$

$$\therefore d(A) \subseteq d(B).$$

$$(3) \quad \forall X_0 \in d(A \cup B) \text{ 有 } \forall \Delta > 0, B(X_0, \Delta) \cap (A \cup B \setminus X_0) \neq \emptyset.$$

$$\therefore B(X_0, \Delta) \cap (A \setminus X_0) \neq \emptyset \text{ 或 } B(X_0, \Delta) \cap (B \setminus X_0) \neq \emptyset$$

$$\therefore X_0 \in d(A) \cup d(B)$$

$$\therefore d(A \cup B) = d(A) \cup d(B).$$

~~4. $\Rightarrow \Omega$ 是闭集 $\therefore \forall X_0 \in \Omega, \Delta > 0, B(X_0, \Delta) \cap (\Omega \setminus X_0) \neq \emptyset.$~~

~~$\therefore d(\Omega) \subseteq \Omega. \quad \forall X_0.$~~

清华大学数学作业纸

(科目 微积分)

班级 计35 姓名 朱新瑞 编号 201301141

第 2 页

4/ $\forall x_0 \in d(\Omega), \forall \Delta > 0, B(x_0, \Delta) \cap (\Omega \setminus x_0) \neq \emptyset$.

若 $x_0 \in \Omega$, 则 x_0 为内点或边界点.

$\therefore d(\Omega) \subseteq \Omega \Rightarrow \Omega$ 为闭集

Ω 为闭集 $\Rightarrow \forall x_0 \notin \Omega, x_0 \notin d(\Omega) \Rightarrow d(\Omega) \subseteq \Omega$.

5/ 设 $p \in d(\Omega) \Rightarrow$ 取 $\Delta_i = 2^{-i}, B(x_0, \Delta_i) \cap (\Omega \setminus x_0) = A_i$.

取 $p_i \in A_i$ (任一点), 可得 $\lim_{n \rightarrow \infty} p_n = p$.

若 $\lim_{n \rightarrow \infty} p_n = p, \forall \Delta > 0 \exists N > 0, \forall i > N, p_i \in (x_0, \Delta) \cap (\Omega \setminus x_0)$.

$\therefore p \in d(\Omega)$

6/ $\forall x_0 \in \partial\Omega$, 必有 $x_0 \in d(\Omega)$.

$\forall x_0 \in d(\Omega)$, 则 $x_0 \in \bar{\Omega}$

$\therefore \bar{\Omega} = \Omega \cup d(\Omega)$.

7/ $\therefore \bar{\Omega} = \Omega \cup d\Omega$.

$\bar{\Omega} = \bar{\Omega} \cup d\bar{\Omega}$.

$\therefore \bar{\Omega} = \Omega \cup d\Omega \cup d(\Omega) \cup d(d(\Omega)) = \bar{\Omega}$.

清华大学数学作业纸

(科目 微积分)

班级 计35

姓名 朱新瑞

编号 201301141

第3 页

5. $\because \|f(x) - f(y)\| \leq L \|x - y\| \therefore f$ 为连续函数.

取 $F(x) = f(x) - x$.

$\therefore f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ 且 $L < 1 \therefore F: \mathbb{R}^n \rightarrow \mathbb{R}^n$

$\therefore F$ 有零点 $\therefore \exists f(x^*) = x^*$.

6. $\forall \varepsilon > 0, \exists \Delta = \frac{\sqrt{\varepsilon}}{2}, \forall x \in B(x_0, \Delta), \|x - y\| \leq \|x - x_0\| + \|x_0 - y\|$
 $\leq \varepsilon + \|x_0 - y\|$

$\therefore p(x, \Omega)$ 为连续函数.

(2) $\because p(x, \Omega)$ 是闭集 由定义可得.

$\therefore \Omega$ 是闭集 $\therefore$ 若 $x \in \Omega$, 取 $x_0 = x$, 若 $x \notin \Omega$, 则 $x_0 \notin \Omega$, 且

必存在最小值.

3). 由2)可知, 若 $\Omega_1 \cap \Omega_2 \neq \emptyset$, 取公共点即可, 若 $\Omega_1 \cap \Omega_2 = \emptyset$, 边界点尝试取最小.

(科目:) 数 学 作 业 纸

编号:

班级: 210A

姓名: 杨明

第

页

7. 证明: 对 $f(x, y)$ 求偏导

$$f'_1 = g(x) \int_c^y h(s) ds, \quad f'_2 = h(y) \int_a^x g(t) dt.$$

$$\because g(x) \in C[a, b] \quad \therefore g(x, y) \in C([a, b] \times [c, d]),$$

$$\because h(x) \in C[c, d] \quad \therefore h(x, y) \in C([a, b] \times [c, d]),$$

$$\text{其中 } g(x, y) = g(x), \quad h(x, y) = h(y).$$

$$\text{又令 } G(x, y) = \int_c^y h(s) ds, \quad \text{易知 } G(x, y) \in C([a, b] \times [c, d])$$

$$H(x, y) = \int_a^x g(t) dt, \quad \text{易知 } H(x, y) \in C([a, b] \times [c, d]).$$

$$\text{故 } f'_1, f'_2 \in C([a, b] \times [c, d])$$

$\therefore$ 偏导均存在且连续, 故 $f(x, y)$ 可微.

$$\Delta f(x, y) = f'_1 \cdot \Delta x + f'_2 \cdot \Delta y + o(\sqrt{\Delta x^2 + \Delta y^2})$$

$$\text{即 } \sqrt{\Delta x^2 + \Delta y^2} \rightarrow 0 \text{ 时, } \Delta f(x, y) \rightarrow 0.$$

$\therefore f(x, y)$ 在 $[a, b] \times [c, d]$ 上连续. 证毕.

8. 解:

① 当 $x \neq 0, y \neq 0$ 时

$$\frac{\partial z}{\partial x} = \begin{cases} \frac{y}{2\sqrt{xy}}, & (x, y) \text{ 在一、三象限内} \\ \frac{-y}{2\sqrt{-xy}}, & (x, y) \text{ 在二、四象限内.} \end{cases}$$

② 当 $y = 0$ 时

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{0 - 0}{\Delta x} = 0, \quad \text{即 } \frac{\partial z}{\partial x} = 0.$$

③ 当 $x = 0$ 时 但 $y \neq 0$ 时

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{y}}{\Delta x} \text{ 不存在.}$$

综上所述, ...

9. 解: $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial^2 r}{\partial x^2} = (y^2 + z^2) / (x^2 + y^2 + z^2)^{3/2}, \quad \frac{\partial^2 r}{\partial y^2} = (x^2 + z^2) / (x^2 + y^2 + z^2)^{3/2}, \quad \frac{\partial^2 r}{\partial z^2} = (x^2 + y^2) / (x^2 + y^2 + z^2)^{3/2}$$

(科目:) 数 学 作 业 纸

编号:

班级:

姓名:

第

页

$$\therefore \Delta r = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\| \text{grad } r \| = \sqrt{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2} = 1$$

$$\frac{\partial^2(\ln r)}{\partial x^2} = \frac{r r_{xx}'' - (r_x')^2}{r^2}, \quad \frac{\partial^2(\ln r)}{\partial y^2} = \frac{r r_{yy}'' - (r_y')^2}{r^2}, \quad \frac{\partial^2(\ln r)}{\partial z^2} = \frac{r r_{zz}'' - (r_z')^2}{r^2}$$

$$\therefore \Delta(\ln r) = \frac{r \Delta r - \| \text{grad } r \|^2}{r^2} = \frac{1}{x^2+y^2+z^2}$$

$$\Delta\left(\frac{1}{r}\right) = -\frac{2r\|\text{grad } r\|^2 - r^2 \Delta r}{r^4} = 0$$

10. 设同一点在新坐标系下的坐标为 x', y', z' .

$$(x, y, z)^T = A_3 (x', y', z')^T A_2, \quad A_2 \text{ 是正交阵}$$

$$\begin{aligned} \frac{\partial f}{\partial I_1} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial I_1} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial I_1} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial I_1} \\ &= f_1' \cdot a_{11} + f_2' \cdot a_{21} + f_3' \cdot a_{31} = \nabla f \cdot \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial I_2} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial I_2} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial I_2} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial I_2} \\ &= f_1' \cdot a_{12} + f_2' \cdot a_{22} + f_3' \cdot a_{32} = \nabla f \cdot \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial I_3} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial I_3} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial I_3} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial I_3} \\ &= f_1' \cdot a_{13} + f_2' \cdot a_{23} + f_3' \cdot a_{33} = \nabla f \cdot \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \end{aligned}$$

$$\therefore \left(\frac{\partial f}{\partial I_1}\right)^2 + \left(\frac{\partial f}{\partial I_2}\right)^2 + \left(\frac{\partial f}{\partial I_3}\right)^2 = \nabla f \cdot \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}^2 + \nabla f \cdot \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}^2 + \nabla f \cdot \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}^2$$

$$= (f_1')^2 (a_{11}^2 + a_{21}^2 + a_{31}^2) + (f_2')^2 (a_{12}^2 + a_{22}^2 + a_{32}^2) + (f_3')^2 (a_{13}^2 + a_{23}^2 + a_{33}^2)$$

$$+ 2f_1' f_2' (a_{11} a_{21} + a_{12} a_{22} + a_{13} a_{32}) + 2f_1' f_3' (a_{11} a_{31} + a_{12} a_{32} + a_{13} a_{33})$$

$$+ 2f_2' f_3' (a_{21} a_{31} + a_{22} a_{32} + a_{23} a_{33})$$

$$= (f_1')^2 + (f_2')^2 + (f_3')^2, \quad \text{证毕} \quad \Delta$$

(科目:) 数 学 作 业 纸

编号:

班级:

姓名:

第

页

11. 证明:
$$f(x, y) = \begin{cases} x - y + \frac{xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

$f(x, y) = g(x, y) + h(x, y)$, 其中 $g(x, y) = x - y$

$$h(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

易知 $g(x, y) = x - y$ 在 $(0, 0)$ 处连续.

$|h(x, y) - h(0, 0)| = \left| \frac{xy^2}{x^2 + y^2} \right| \leq 2 \cdot 3^{-\frac{2}{3}} \cdot (x^2 + y^2)^{\frac{1}{2}}$

$\therefore \lim_{(x, y) \rightarrow (0, 0)} h(x, y) = h(0, 0).$

$\therefore g(x, y)$ 与 $h(x, y)$ 均在 $(0, 0)$ 处连续, 故 $f(x, y)$ 亦在 $(0, 0)$ 处连续.

$\frac{\partial f}{\partial x}(0, 0) = \lim_{x \rightarrow 0} \frac{x}{x} = 1$; $\frac{\partial f}{\partial y}(0, 0) = \lim_{y \rightarrow 0} \frac{-y}{y} = -1.$

考虑方向 $\vec{r}(\cos t_0, \sin t_0)$, 及沿该方向一增量 d .

$$\frac{\partial f}{\partial \vec{r}}(0, 0) = \lim_{d \rightarrow 0} \frac{d \cos t_0 - d \sin t_0 + \frac{d^3 \cos t_0 \sin t_0}{d^2}}{d}$$

$= \cos t_0 - \sin t_0 + \cos t_0 \sin t_0$, 由 t 任意性

故 $f(x, y)$ 沿任意方向方向导数存在.

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{f(x, y) - f(0, 0) - \frac{\partial f}{\partial x}(0, 0)x - \frac{\partial f}{\partial y}(0, 0)y}{\sqrt{x^2 + y^2}}$$

$$= \lim_{\rho \rightarrow 0} \left(\frac{x^2 y^2}{(x^2 + y^2)^{\frac{3}{2}}} \right)^{\frac{1}{2}} = \lim_{\rho \rightarrow 0} \left(\frac{1}{\left(\frac{x}{y}\right)^{\frac{2}{3}} + \left(\frac{y}{x}\right)^{\frac{2}{3}}} \right)^{\frac{3}{2}}$$

当 $\frac{x}{y} = k$ 时, k 不同则极限不同. 故

$f(x, y)$ 不可微. 证毕.

(科目:)

数学作业纸

编号:

班级:

姓名:

第

页

$$12. \text{证: } \frac{\partial z}{\partial t} = \frac{\partial z_0}{\partial t} = z'_0 \cdot a + z''_0 \cdot b$$

$$a \cdot \frac{\partial z}{\partial x} = a \cdot \frac{\partial z_0}{\partial x} = a \cdot z'_0$$

$$b \cdot \frac{\partial z}{\partial y} = b \cdot \frac{\partial z_0}{\partial y} = b \cdot z''_0$$

$$\therefore \frac{\partial z}{\partial t} = a \cdot \frac{\partial z}{\partial x} + b \cdot \frac{\partial z}{\partial y}$$

$$\text{又 } z(x, y, 0) = z_0(x + a \cdot 0, y + b \cdot 0) = z_0(x, y).$$

$$\text{综上所述 } \begin{cases} \frac{\partial z}{\partial t} = a \cdot \frac{\partial z}{\partial x} + b \cdot \frac{\partial z}{\partial y} \\ z(x, y, 0) = z_0(x, y) \end{cases} \text{ 的解为 } z = z_0(x + at, y + bt).$$

我仅验证了12题中

$z = z_0(x + at, y + bt)$ 是波动方程的解, 但不能确定其为通解.

$$13. (1) \text{证明: } f, g \in C^2, u(x, y, z, t) = \frac{1}{r} (f(t+r) - g(t-r))$$

$$\frac{\partial u}{\partial t} = \frac{1}{r} (f' - g')$$

$$\frac{\partial^2 u}{\partial i^2} = \left(\frac{1}{r}\right)'' \cdot (f - g) + \left(\frac{1}{r}\right)' \cdot (f' r'_i + g' r'_i) + \left(\frac{1}{r}\right)' \cdot (f' r'_i + g' r'_i) + \frac{1}{r} [f'' (r'_i)^2 + r''_i f' + g'' (r'_i)^2 + r''_i g']$$

$$\sum \frac{\partial^2 u}{\partial i^2} = 0 + 2 \sum \left[\left(\frac{1}{r}\right)' \cdot r'_i \right] \cdot (f' + g') + \frac{1}{r} (f'' - g'') + \frac{2}{r^2} (f' + g')$$

$$\text{注意到 } 2 \sum \left[\left(\frac{1}{r}\right)' \cdot r'_i \right] = -\frac{2}{r^2}$$

$$\therefore \sum \frac{\partial^2 u}{\partial i^2} = \frac{1}{r} (f'' - g'')$$

$$\text{即 } \frac{\partial u}{\partial t^2} = \frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} + \frac{\partial u}{\partial z^2}, \text{ 证毕.}$$

$$(2) \text{ 设 } F(t, x, y, z) = f(t+x, t+y, t+z), \text{ 则 } \frac{\partial F}{\partial t} = \frac{\partial f}{\partial t} = x \cdot \frac{\partial f}{\partial (t+x)} + y \cdot \frac{\partial f}{\partial (t+y)} + z \cdot \frac{\partial f}{\partial (t+z)}.$$

$$\text{又 } \frac{\partial f}{\partial (t+x)} = \frac{\partial (t^k f(x, y, z))}{\partial (t+x)} = t^{k-1} \cdot \frac{\partial f}{\partial x}, \frac{\partial f}{\partial (t+y)} = t^{k-1} \cdot \frac{\partial f}{\partial y}, \frac{\partial f}{\partial (t+z)} = t^{k-1} \cdot \frac{\partial f}{\partial z}.$$

$$\text{且 } \frac{\partial F}{\partial t} = \frac{\partial [t^k f(x, y, z)]}{\partial t} = k \cdot f(x, y, z) \cdot t^{k-1}$$

$$\therefore \text{ 结合两种计算 } F \text{ 对 } t \text{ 偏导数的方法, } t^{k-1} \cdot (x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z}) = k \cdot f(x, y, z) \cdot t^{k-1}$$

$$\text{即 } x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z} = k f(x, y, z), \text{ 证毕.}$$

$$(3) \text{ 证明: } \frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} + \frac{\partial u}{\partial z^2} = 0, \rightarrow f'' + \frac{2}{r} \cdot f' = 0, f = f(r).$$

$$\text{令 } t = f', \text{ 分离变量解出 } t = \frac{k}{r^2}, \text{ 再分离变量得 } f = \frac{a}{r} + b. \triangle$$

(科目:) 数 学 作 业 纸

编号:

班级:

姓名:

第

页

故其通解为 $f(\sqrt{x^2+y^2+z^2}) = u(x, y, z) = \frac{a}{\sqrt{x^2+y^2+z^2}} + b$. 证毕.

(4) 设 $g(x^2+y^2) = g(r) = f(x, y)$, $r = x^2+y^2$.

$\forall r_1, r_2 \in [0, +\infty)$

$$\begin{aligned} g(r_1) \cdot g(r_2) &= \left[\varphi\left(\sqrt{\frac{r_1}{2}}\right) \right]^2 \cdot \left[\varphi\left(\sqrt{\frac{r_2}{2}}\right) \right]^2 \\ &= \left[\varphi\left(\sqrt{\frac{r_1}{2}}\right) \varphi\left(\sqrt{\frac{r_2}{2}}\right) \right]^2 \\ &= \left[\varphi\left(\sqrt{\frac{r_1+r_2}{2}}\right) \varphi(0) \right]^2 \\ &= \left[\varphi\left(\sqrt{\frac{r_1+r_2}{2}}\right) \right]^2 \quad (\varphi(0) = f(0, 0) = 1) \\ &= g(r_1 + r_2) \end{aligned}$$

$$\therefore \ln[g(r_1)] + \ln[g(r_2)] = \ln[g(r_1 + r_2)] \dots\dots (1)$$

由 $f \in C$ 知 $\ln[g(x)]$ 在 $[0, +\infty)$ 上应为连续函数.

$$(f(x, y) = g(x^2+y^2) = f\left(\sqrt{\frac{x^2+y^2}{2}}, \sqrt{\frac{x^2+y^2}{2}}\right) = \left[\varphi\left(\sqrt{\frac{x^2+y^2}{2}}\right)\right]^2 \geq 0)$$

假设 $\exists X \in [0, +\infty)$, $g(X) = 0$, 则 $\because X \neq 0$ 不为极值点

故 $g'(X) = 0$, $\because g \in C^1$ 且 $g(x)$ 恒非负.

那么 X 点左右两侧显然不能异号, 否则
显然 X 为 $[0, +\infty)$ 内点

$$g(x) = 0 \Leftrightarrow \varphi\left(\sqrt{\frac{x}{2}}\right) = 0$$

$$\times \quad \varphi\left(\sqrt{\frac{x}{2}}\right) \varphi(0) = g\left(\frac{x}{2}\right) = 0$$

故 $x, \frac{x}{2}, \frac{x}{2^2}, \dots, \frac{x}{2^n}, \dots$
均为 g 的零点.

$$0 = \lim_{n \rightarrow +\infty} g\left(\frac{x}{2^n}\right) = g\left(\lim_{n \rightarrow +\infty} \frac{x}{2^n}\right) = g(0) = 1 \text{ 矛盾!}$$

因此 $\forall x \in [0, +\infty)$, $g(x) \neq 0$.

$\therefore \ln[g(x)]$ 确实在 $[0, +\infty)$ 上连续.

由柯西方法知 $\forall x \in [0, +\infty)$, (1)式成立时可导出 $\ln[g(x)] = X$

$$\therefore f(x, y) = g(x^2+y^2) = e^{x^2+y^2}.$$

(科目:) 数 学 作 业 纸

编号:

班级:

姓名:

第

页

注: 题中给出 $g \in C^1$, 即 g, g' 均连续, 但我[△]证明过程中并未用
到 g' 及其连续性非证明题, 关系大吗?
未解.

清华大学

14. 证:
$$\frac{D(u,v)}{D(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial s} f'(x) & \frac{\partial u}{\partial t} g'(x) \\ \frac{\partial v}{\partial s} f'(x) & \frac{\partial v}{\partial t} g'(x) \end{vmatrix}$$

$$= f'(x) g'(x) \begin{vmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{vmatrix} = f'(x) g'(x) \frac{D(u,v)}{D(s,t)}$$

15. (1)
$$\frac{\partial z}{\partial x} = e^{\frac{x}{y}} + x \cdot e^{\frac{x}{y}} \cdot \frac{1}{y}$$

$$\frac{\partial z}{\partial y} = x e^{\frac{x}{y}} \cdot -\frac{x}{y^2}$$

设 (x_0, y_0) 为曲面上一点

则该点切平面为 $e^{\frac{x_0}{y_0}} (1 + \frac{x_0}{y_0})(x - x_0) - \frac{x_0^2}{y_0^2} e^{\frac{x_0}{y_0}} (y - y_0) - (z - z_0) = 0$
 将 $(0, 0, 0)$ 代入发现恒成立, 即切平面恒过原点

(2) 设切点为 (x_0, y_0, z_0)

$$\frac{\partial F}{\partial x} = 2x - 1 \quad \frac{\partial F}{\partial y} = 2y \quad \frac{\partial F}{\partial z} = 2z$$

则切点的法向量为 $\vec{n} = (2x_0 - 1, 2y_0, 2z_0)$

已知平面的法向量为 $\vec{n}_1 = (1, -1, -1)$ $\vec{n}_2 = (1, -1, -\frac{3}{2})$

$$\text{则} \begin{cases} 2x_0 - 1 - 2y_0 - 2z_0 = 0 \\ 2x_0 - 1 - 2y_0 - 3z_0 = 0 \end{cases}$$

$$\text{又 } x_0^2 + y_0^2 + z_0^2 = x_0$$

$$\text{解得} \begin{cases} x_0 = \frac{2+\sqrt{2}}{4} \\ y_0 = \frac{\sqrt{2}}{4} \\ z_0 = 0 \end{cases}$$

$$\text{或} \begin{cases} x_0 = \frac{2-\sqrt{2}}{4} \\ y_0 = -\frac{\sqrt{2}}{4} \\ z_0 = 0 \end{cases}$$

故切平面为 $\frac{\sqrt{2}}{2}(x - \frac{2+\sqrt{2}}{4}) + \frac{\sqrt{2}}{2}(y - \frac{\sqrt{2}}{4}) = 0$

$$\text{或} -\frac{\sqrt{2}}{2}(x - \frac{2-\sqrt{2}}{4}) - \frac{\sqrt{2}}{2}(y + \frac{\sqrt{2}}{4}) = 0$$

清华大学

(3) 设 (x_0, y_0, z_0) 位于 F_1, F_2 的交线上

$$\text{即有 } \begin{cases} F_1(x_0, y_0, z_0) = 0 \\ F_2(x_0, y_0, z_0) = 0 \end{cases}$$

$$\text{又 } \vec{n}_1 = \left(\frac{\partial F_1}{\partial x}, \frac{\partial F_1}{\partial y}, \frac{\partial F_1}{\partial z} \right) \quad \vec{n}_2 = \left(\frac{\partial F_2}{\partial x}, \frac{\partial F_2}{\partial y}, \frac{\partial F_2}{\partial z} \right)$$

$$\text{即有 } \left. \frac{\partial F_1}{\partial x} \frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y} \frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial z} \frac{\partial F_2}{\partial z} \right|_{(x_0, y_0, z_0)} = 0$$

(4) ~~显然~~ 显然对于中心在原点的椭球面，其半长轴即为原点到椭球面的最长距离

$$\text{构造 } L = x^2 + y^2 + z^2 - \lambda(a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz - 1)$$

$$\frac{\partial L}{\partial x} = 2x - \lambda(2a_{11}x + 2a_{12}y + 2a_{13}z)$$

$$\frac{\partial L}{\partial y} = 2y - \lambda(2a_{12}x + 2a_{22}y + 2a_{23}z)$$

$$\frac{\partial L}{\partial z} = 2z - \lambda(2a_{13}x + 2a_{23}y + 2a_{33}z)$$

$$\frac{\partial L}{\partial \lambda} = -(a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz - 1) = 0$$

$$\text{令 } \frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial z} = 0 \quad \text{得 } \begin{bmatrix} \frac{1}{\lambda} - a_{11} & -a_{12} & -a_{13} \\ -a_{12} & \frac{1}{\lambda} - a_{22} & -a_{23} \\ -a_{13} & -a_{23} & \frac{1}{\lambda} - a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

令 $X = (x, y, z)^T$ 即 $(\frac{1}{\lambda}I - A)X = 0$ $\frac{1}{\lambda}$ 为 A 的特征值, X 为其特征向量

$$\text{又 } X^T A X = 1 \quad \text{即 } \lambda X^T X = 1$$

$$\text{又 } a = \sqrt{x^2 + y^2 + z^2} = (\sqrt{X^T X})_{\max}$$

故 $a = \sqrt{\frac{1}{\lambda_{\min}}}$ ~~为~~ $\lambda_{\min}$ 为 A 的最小特征值

清华大学

16.

17. 解: 构造 $L(x, y, \lambda) = \frac{1}{p}x^p + \frac{1}{q}y^q - \lambda(xy-1)$

$$\frac{\partial L}{\partial x} = x^{p-1} - y\lambda$$

$$\frac{\partial L}{\partial y} = y^{q-1} - \lambda x$$

$$\frac{\partial L}{\partial \lambda} = -(xy-1)$$

$$\text{令 } \frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0 \quad \text{得 } \lambda = 1 \quad x = y = 1$$

由其唯一性, 必为极小值 $f(x, y)_{\min} = \frac{1}{p} + \frac{1}{q} = 1$

令 $xy = a$ 同上构造 $L(x, y, \lambda) = \frac{1}{p}x^p + \frac{1}{q}y^q - \lambda(xy-a)$

$$\text{令 } \frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0 \quad \text{得 } \lambda = a \quad x = \sqrt[p]{a} \quad y = \sqrt[q]{a}$$

$$\text{则 } \frac{1}{p}x^p + \frac{1}{q}y^q \geq \frac{1}{p}a + \frac{1}{q}a = a = xy$$

证毕

清华大学

18. 证: 若 $|PQ|$ 为两条曲线的距离, 则必有 P, Q 处两切线平行

$$\text{对于 } f: \frac{dy}{dx} = -\frac{f'_1}{f'_2} \quad \text{对于 } g: \frac{dy}{dx} = -\frac{g'_1}{g'_2}$$

$$\overrightarrow{PQ} = (a-c, b-d) \quad f, g \text{ 的切向量分别为 } (1, -\frac{f'_1}{f'_2}) \text{ 和 } (1, -\frac{g'_1}{g'_2})$$

$$\text{显然有 } \overrightarrow{PQ} \perp \mathbf{v}_1 \quad \text{即} \quad a-c = (b-d) \cdot \frac{f'_1}{f'_2}$$

$$\text{故} \quad \frac{a-c}{b-d} = \frac{f'_1}{f'_2} = \frac{g'_1}{g'_2}$$

$$\text{设 } P(x_1, y_1) \quad Q(x_2, y_2)$$

$$\text{由 } \frac{x_1 - x_2}{y_1 - y_2} = \frac{x_1 + y_1}{x_1 + y_1 - 4} = -1 \quad \text{解得 } x_1 = 2 \pm \sqrt{2}$$

$$x_1 = 2 + \sqrt{2} \text{ 时 } y_1 = -2 - 3\sqrt{2} \quad x_2 = 4 - \sqrt{2} \quad y_2 = -\sqrt{2} - 4$$

$$d = \sqrt{(2\sqrt{2}-2)^2 + (2-2\sqrt{2})^2} = 4 - 2\sqrt{2}$$

$$x_1 = 2 - \sqrt{2} \text{ 时 } y_1 = -2 + 3\sqrt{2} \quad x_2 = 4 + \sqrt{2} \quad y_2 = \sqrt{2} - 4$$

$$d = \sqrt{(2+2\sqrt{2})^2 + (2\sqrt{2}+2)^2} = 4 + 2\sqrt{2}$$

故最小距离为 $4 - 2\sqrt{2}$

$$19. \text{解: } \frac{\partial f}{\partial x} = \sin x \cdot (\cos x \sin(x+y) + \sin x \cos(x+y)) = \sin y \sin(2x+y)$$

$$\frac{\partial f}{\partial y} = \sin x \cdot \sin(2y+x)$$

$$\text{令 } \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \quad \text{得} \quad \begin{cases} x=0 \\ y=0 \end{cases} \quad \begin{cases} x=0 \\ y=\pi \end{cases} \quad \begin{cases} x=\pi \\ y=0 \end{cases} \quad \begin{cases} x=\pi \\ y=\pi \end{cases} \quad \begin{cases} x=\frac{\pi}{2} \\ y=\frac{\pi}{2} \end{cases}$$

$$\text{考虑边界} \quad x=0 \quad f(x, y) \equiv 0$$

$$y=0 \quad f(x, y) \equiv 0$$

$$x+y=\pi \quad f(x, y) \equiv 0$$

$$\text{又 } f(0, 0) = 0 \quad f(0, \pi) = 0 \quad f(\pi, 0) = 0 \quad f(\pi, \pi) = 0 \quad f(\frac{\pi}{2}, \frac{\pi}{2}) = \frac{3\sqrt{3}}{8}$$

$$\text{故 } f(x, y)_{\max} = \frac{3\sqrt{3}}{8} \quad f(x, y)_{\min} = 0$$