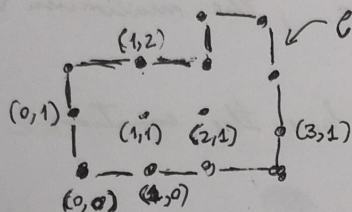


P.1

For each point $(i,j) \in \Omega$, we write the equation: (Remember: $\phi_{ij} \geq 0 \forall (i,j)$)

$$\phi_{ij} = \frac{1}{4} (\phi_{i+1,j} + \phi_{i,j-1} + \phi_{i-1,j} + \phi_{i,j+1}) \quad (*)$$

e.g.



here C has 2 interior points, hence, Two eqs.:

$$\phi_{1,1} = \frac{1}{4} (\phi_{2,1} + \phi_{1,2}^b + \phi_{0,1}^b + \phi_{1,0}^b) \quad (1)$$

$$\phi_{2,1} = \frac{1}{4} (\phi_{1,1} + \phi_{2,0}^b + \phi_{2,2}^b + \phi_{3,1}^b)$$

all the ϕ_{ij}^b 's are boundary values, given as initial conditions. we can re-write (1) as

$$\phi_{1,1} - \frac{1}{4} \phi_{2,1} = \frac{1}{4} (\phi_{1,2}^b + \phi_{0,1}^b + \phi_{1,0}^b)$$

$$\phi_{2,1} - \frac{1}{4} \phi_{1,1} = \frac{1}{4} (\phi_{2,0}^b + \phi_{2,2}^b + \phi_{3,1}^b)$$

$$\Leftrightarrow A \cdot \begin{pmatrix} \phi_{1,1} \\ \phi_{2,1} \end{pmatrix} = \vec{b} \quad , \quad A = \begin{pmatrix} 1 & -\frac{1}{4} \\ -\frac{1}{4} & 1 \end{pmatrix} \quad , \quad \vec{b} = \frac{1}{4} \begin{pmatrix} \phi_{1,2}^b + \phi_{0,1}^b + \phi_{1,0}^b \\ \phi_{2,0}^b + \phi_{2,2}^b + \phi_{3,1}^b \end{pmatrix}$$

we can check here $\det A \neq 0$, hence the sol. is unique (and indeed, the entries of $A^{-1} \vec{b}$ are all positive). For the general case, say we have N interior points, then we have N eqs. of the form (*), which we can always write s.t. all the unknowns are on the left hand side (LHS) and ϕ_{ij}^b 's in the RHS. Label the interior points as $\vec{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$, then we get an eq. of the form:

$$A \cdot \vec{\phi} = \vec{b} \quad A \in M_{N \times N}(\mathbb{Q})$$

and $\vec{b}$ contains only boundary values ϕ_{ij}^b . We know, from Linear Algebra 1, that $A \vec{\phi} = \vec{b}$ has a unique solution iff $A \vec{\phi} = 0$ has $\vec{\phi} = 0$ as its only solution.

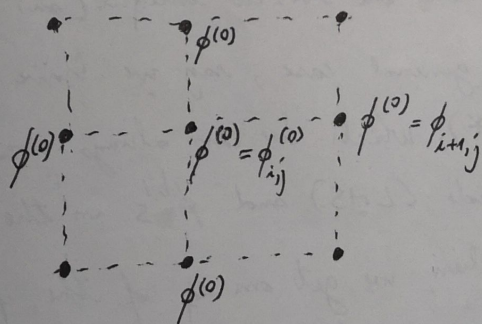
So, we look at $A\vec{\phi} = 0$ eq., this is equivalent to solve the system w/ all boundary values set to zero ($\vec{b} = 0 \Rightarrow$ all positive combinations of ϕ^b 's equal to zero, hence since $\phi_{ij}^b \geq 0 \Rightarrow \phi_{ij}^b = 0 \forall (i,j) \in C$).

Let's assume $\exists$ a sol. $\vec{\phi} \neq 0$, to $A\vec{\phi} = 0$. Then, take $\phi_{ij}^{(0)}$ with $(i,j) \in \Omega$ such that $\phi_{ij}^{(0)} \geq \phi_{i'j'}$ $\forall (i',j') \in \Omega$ i.e., the maximum value between all $\{\phi_{ij}\}_{(i,j) \in \Omega}$. Then, we must have the equation

$$\phi_{ij}^{(0)} = \frac{1}{4} (\phi_{i,j-1} + \phi_{i-1,j} + \phi_{i+1,j} + \phi_{i,j+1}) \leq M \quad (2)$$

where $M = \max(\phi_{i,j-1}, \phi_{i-1,j}, \phi_{i+1,j}, \phi_{i,j+1}) \leq \phi_{ij}^{(0)} \Rightarrow$ we must have $M = \phi_{ij}^{(0)}$ or there is no sol. for (2)

hence if any of the ϕ 's in the RHS of eq. (2) is a boundary value $\phi^b = 0$, then we get a contradiction, because the only way to attain the maximum M is to have all $\phi_{i,j-1} = \phi_{i-1,j} = \phi_{i+1,j} = \phi_{i,j+1} = \phi_{ij}^{(0)}$, so all the ϕ 's in the RHS of (2) must be in Ω . So, we fixed the values of all nearest neighbours of $\phi_{ij}^{(0)}$, namely, if we define $\phi^{(0)} := \phi_{ij}^{(0)}$, then, pictorially we have



Next, we take the equation of $\phi_{i+1,j}$, whose value was fixed to be $\phi^{(0)}$. By the same reasoning, $\phi_{i+2,j} = \phi_{i+1,j+1} = \phi_{i,j} = \phi_{i+1,j-1} = \phi^{(0)}$ (and, none can be on the boundary, or we have a contradiction). In particular, this fixes $\phi_{i+2,j} = \phi^{(0)}$. Since the area C is bounded, we can keep going on $i, i+1, i+2$, etc. but at some point there will be $i_0 > i$ such that $(i_0, j) \in C$

therefore we will have

$$\phi^{(0)} = \phi_{i_0-1, j} = \frac{1}{4}(\phi_{i_0, j}^{(1)} + \phi_{i_0-2, j} + \phi_{i_0, j+1}^{(1)} + \phi_{i_0, j-1}^{(1)}) < \phi^{(0)}$$

(2)

hence we have a contradiction, therefore a sol. $\vec{\phi} \neq 0$ doesn't exist. Then, we showed that the only sol. for $A\vec{\phi} = 0$, is $\vec{\phi} = 0 \Rightarrow A\vec{\phi} = \vec{b}$ has a unique sol. for all $\vec{b}$.

P.21

⇐ Set $b=c=0$, then

$$T(p) = (3p(4) + 5p'(6), \int_{-1}^2 x^3 p(x) dx)$$

we want to check $T(p+q) = T(p) + T(q)$ and $T(\lambda p) = \lambda T(p)$ for all $p, q \in \mathbb{R}[x]$ and $\lambda \in \mathbb{R}$

$$T(p+q) = (3(p+q)(4) + 5(p+q)'(6), \int_{-1}^2 x^3 (p+q)(x) dx)$$

by definition of $p+q$ and properties of $\frac{d}{dx}$ (which is a linear operator ^{over $\mathbb{R}[x]$} on itself), we get

$$\begin{aligned} (p+q)(4) &= p(4) + q(4) \\ (p+q)'(6) &= p'(6) + q'(6) \end{aligned}$$

$$\text{integration is also a linear operator, so } \int_{-1}^2 x^3 (p+q)(x) dx = \int_{-1}^2 x^3 p(x) dx + \int_{-1}^2 x^3 q(x) dx$$

$$\text{hence we have } T(p+q) = T(p) + T(q)$$

Since $\frac{d}{dx} \lambda = 0$ and $(\lambda \cdot p)(x) = \lambda p(x)$ (special case of multiplication of polynomials)

$$\text{we have } 3(\lambda p)(4) = 3\lambda p(4) \text{ and } 5 \frac{d}{dx}(\lambda p)(6) = 5\lambda p'(6)$$

and, finally, since $\int dx$ is also a linear map, when acting on $\mathbb{R}[x]$, we have

$$\int_{-1}^2 x^3 (\lambda p)(x) dx = \lambda \int_{-1}^2 x^3 p(x) dx \quad \therefore T(\lambda p) = \lambda T(p)$$

⇒ we only need to take a well chosen pair of polynomials, for example, we can take the constant polynomials $p = \frac{\pi}{2}$, $q = \frac{\pi}{2}$, then is easy to check that $T(\frac{\pi}{2} + \frac{\pi}{2}) = T(\pi) = T(\frac{\pi}{2}) + T(\frac{\pi}{2}) = 2T(\frac{\pi}{2})$ is satisfied only if $b=c=0$

P3 This problem is just Gaussian elimination, from linear algebra 1.
 is easy to show that via — " —, that we have the equivalent system

$$A' \vec{x} = \vec{b}' \text{ with}$$

$$A' = \begin{bmatrix} -\alpha & 2 & 0 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & \beta+2 \end{bmatrix} \quad \vec{b}' = \begin{bmatrix} 1 \\ -1 \\ 0 \\ \alpha+\beta+2 \end{bmatrix}$$

So,

- 1) Unique sol. $\Leftrightarrow \alpha \neq 0$ and $\beta \neq -2$
- 2) Infinite solutions $\Leftrightarrow \alpha = 0$ and $\beta = -2$
- 3) No solution $\Leftrightarrow (\alpha \neq 0 \text{ and } \beta = -2) \text{ or } (\alpha = 0 \text{ and } \beta \neq -2)$

For $\alpha = 1, \beta = -1$

$$A^{-1} = \begin{bmatrix} 29 & 16 & -4 & 9 \\ 13 & 7 & -2 & 4 \\ 7 & 4 & -1 & 2 \\ 4 & 2 & 0 & 1 \end{bmatrix} \text{ and sol. is } \vec{x} = \begin{pmatrix} 19 \\ 9 \\ 4 \\ 2 \end{pmatrix}$$

P4 Remember $\overset{\text{vector space}}{U \subset V}$ is a subspace of V iff $0 \in U$, U is closed under addition and closed under scalar multiplication

- 1) Clearly $0 \in \mathcal{H}$. Consider $h, p \in M_{mm}(\mathbb{R})$ such that $h, p \in \mathcal{H}$
 then $h_{ij} = p_{ij} = 0 \quad \forall i > j+1 \quad \therefore (\lambda h + \mu p)_{ij} = \lambda h_{ij} + \mu p_{ij} \Rightarrow \lambda h + \mu p \in \mathcal{H}$
 for $\lambda, \mu \in \mathbb{R}$

- 2) $t \in M_{mm}(\mathbb{R})$ is upper triangular $\Leftrightarrow t_{ij} = 0 \quad \forall i > j$, then take $h \in \mathcal{H}$

$$(t \cdot h)_{ij} = \sum_{k=1}^m t_{ik} h_{kj} = \sum_{k=1}^{j+1} t_{ik} h_{kj} + \sum_{k=j+2}^m t_{ik} h_{kj}$$

just separate the sums

take $i > j+1$, then in the first sum t_{ik} with $k \leq j+1 < i \Rightarrow k < i$ hence $t_{ik} = 0$. In the second sum $k \geq j+2$ so h_{kj} with $k > j+1$ so $h_{kj} = 0$

$$\Rightarrow (t \cdot h)_{ij} = 0 \text{ for } i > j+1 \quad \therefore t \cdot h \in \mathcal{H}$$

- 3) We need to show $A \cdot A = 1$.

$$A \cdot A = (1 - 2uu^t)(1 - 2uu^t) = 1 - 2uu^t - 2uu^t + 4(uu^t) \cdot (uu^t)$$

Since product of matrices is associative:

$$(uu^t) \cdot (uu^t) = u \underbrace{(u^t u)}_{\|u\|^2} u^t = \|u\|^2 \cdot uu^t = uu^t$$

then $A \cdot A = 1$

(3)

P5 Here we just apply the division algorithm to $x^5 + ax^2 + b : x^3 + x^2 + cx + 1$
after some computation, we get the remainder

$$r = (a + 2c - 2)x^2 + [1 - c(1 - c)]x + b + c - 1$$

we want $r = 0$, hence

$$\begin{aligned} a + 2c - 2 &= 0 \\ 1 - c(1 - c) &= 0 \\ b + c - 1 &= 0 \end{aligned}$$

we have the following solutions:

$$\begin{aligned} a &= 1 \mp i\sqrt{3} \\ b &= \frac{1}{2} \mp i\frac{\sqrt{3}}{2} \\ c &= \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \end{aligned}$$

P6 1) - Closure: $p = a_1x + a_2x^2, q = b_1x + b_2x^2 \in J$
 $p \Delta q = a_1b_1x + (a_1b_2 + a_2b_1^2)x^2$ if $a_1 \neq 0$ and $b_1 \neq 0$, then $a_1b_1 \neq 0$
 $\therefore p \Delta q \in J$

- Associativity: $h = c_1x + c_2x^2 \in J$

$$\begin{aligned} h \Delta (p \Delta q) &= h \Delta (a_1b_1x + (a_1b_2 + a_2b_1^2)x^2) \\ &= a_1b_1c_1x + (a_1b_2c_1 + a_2b_1^2c_1 + a_1^2b_1^2c_2)x^2 \end{aligned}$$

$$\begin{aligned} (h \Delta p) \Delta q &= (c_1a_1x + (c_1a_2 + c_2a_1^2)x^2) \Delta q \\ &= a_1b_1c_1x + (a_1b_2c_1 + a_2b_1^2c_1 + a_1^2b_1^2c_2)x^2 = h \Delta (p \Delta q) \end{aligned}$$

- Identity: take $e = x$ then $x \Delta q = b_1x + b_2x^2 = q$
 $q \Delta x = b_1x + b_2x^2 = q$

- Inverse: $p \Delta (\frac{1}{a_1}x - \frac{a_2}{a_1^3}x^2) = x$ and $(\frac{1}{a_1}x - \frac{a_2}{a_1^3}x^2) \Delta p = x$

So $(\frac{1}{a_1}x + \frac{a_2}{a_1^3}x^2)$ is the inverse of P

So, we showed (J, Δ) is a group. Take e.g. $P = a_1x$ and $q = b_1x + x^2$

then $P \Delta q = a_1b_1x + a_1x^2$

and $q \Delta P = a_1b_1x + a_1^2x^2 \neq P \Delta q$ if $a_1 \neq 1$ hence (J, Δ) is nonabelian.

2) $f: J \rightarrow \mathbb{R}$ since $a_1 \neq 0$ for all $p \in J$, clearly f is surjective as a map
 $(a_1x + a_2x^2) \mapsto a_1$
 $f: J \rightarrow \mathbb{R} \setminus \{0\}$

to check if f is a homomorphism, we just need to check:

$$f(P \Delta q) = f(P) \cdot f(q) \text{ for all } p, q \in J, \text{ take } q = b_1x + b_2x^2$$

$$f(P \Delta q) = a_1b_1 = f(P) \cdot f(q)$$

3) if $H = \{P \in J \mid a_2 = 0\}$ then $P \in H$ means $P = a_1x$, then, take $q = b_1x$

clearly $H \subset J$. ~~Group~~ by definition. Now compute, for $p, q \in H$

$P \Delta q = a_1b_1x = q \Delta P$; closure, associativity and inverse, identity are
 satisfied, is abelian.

P71 $P = a_0 + a_1x + a_2x^2 + \overset{\text{monic}}{1}x^3$ and we know $(x-1) \mid P$ so $P(x) = (x-1)(b_0 + b_1x + x^2)$

moreover we also know that $P(x) = (x-j) \cdot q_j + r$ $j=2,3,4$ and r is a constant
 (since $\deg(r) < \deg(x-j)$) then $P(3) = P(2) = P(4) = r$, this gives us two equations

$$P(3) = P(2)$$

$$P(3) = P(4)$$

from which we can determine $b_0 = 18, b_1 = -8$, hence

$$P = (x-1)(x^2 - 8x + 18)$$

then the roots are $x = 1, 4 + i\sqrt{2}$ and $4 - i\sqrt{2}$

P8/

This is just for practicing the division algorithm. The answers are

(4)

$$d = x^5 \Rightarrow q = 1, r = -2x^2 + 1$$

$$d = x^2 - 2 \Rightarrow q = x^3 + 2x - 2, r = 4x - 3$$

$$d = x^3 \Rightarrow q = x^2, r = -2x^2 + 1$$

$$d = x^2 - 3x + 1 \Rightarrow q = x^3 + 3x^2 + 8x + 19, r = 49x - 18$$

$$d = x - 1 \Rightarrow q = x^4 + x^3 + x^2 - x - 1, r = 0$$

P9/

We know $a \in \mathbb{R}$ is a root, so $p(a) = 0$, then $\operatorname{Re}(p(a)) = \operatorname{Im}(p(a)) = 0$

$$\Rightarrow a = \frac{1}{2}$$

P10/

First we write the equation

$$\underbrace{\left(\sum_{i=0}^{m-1} b_i x^i \right)}_{q(x)} (x-c) + r = \sum_{i=0}^m a_i x^i = p(x)$$

expand

$$\sum_{i=1}^m b_{i-1} x^i - \sum_{i=0}^{m-1} c b_i x^i + r = \sum_{i=0}^m a_i x^i$$

evaluate at $x=0$ to obtain $r = c b_0 + a_0$, next compare the coefficient of x^i

for $i=1, \dots, m$, we get

$$b_{i-1} - c b_i = a_i \quad (1)$$

and $b_m = 0$, so, for example $b_{m-1} = a_m$, etc. Then, (1) can be

rewritten as

$$b_{m-l} = c b_{m-l+1} + a_{m-l+1} \quad l=1, \dots, m$$

just by changing the index from i to $m-l+1$. This shows that b 's are given by Ruffini's rule. In particular, note (1) are m eqs. for m coeffs.

and is easy to show they are linearly independent, so b_0 (and hence r) is completely determined by them.