

Homework 2

(1)

P1 we know $x^m - 1 = \prod_{l=0}^{m-1} (x - \zeta^l)$ because of factorization of polynomials over $\mathbb{C}$, since $\zeta^l, l=0, \dots, m-1$ are precisely the roots of $x^m - 1 = 0$.

we also can write

$$\frac{x^m - 1}{x - 1} = x^{m-1} + x^{m-2} + \dots + x + 1 = \prod_{l=1}^{m-1} (x - \zeta^l)$$

$$\text{so } \prod_{l=1}^{m-1} |x - \zeta^l| = \frac{|x^m - 1|}{|x - 1|} = |x^{m-1} + \dots + x + 1| \quad \text{eval at } x=1:$$

$$\prod_{l=1}^{m-1} |1 - \zeta^l| = m$$

P2 Note first all roots of $x^m - 1$ are simple i.e. there are no repeated roots, so $x^m - 1$ have m distinct roots and a factorization

$$x^m - 1 = \prod_{l=0}^{m-1} (x - \zeta^l) \quad \zeta = e^{2\pi i/m}$$

on the other hand, say $e^{2\pi i k/d} = \omega_d$ is a root of $\Phi_d(x)$ (so $\gcd(k, d) = 1$) and $\omega_{d'} = e^{2\pi i k'/d'}$ is a root of $\Phi_{d'}(x)$ (so $\gcd(k', d') = 1$), Suppose $\omega_{d'} = \omega_d \Rightarrow \omega_{d'}^d = 1 = e^{2\pi i k' d/d'}$

so we must have $d' | d$, likewise we must have $\omega_d^{d'} = 1$ so

$d | d' \therefore d = d'$. We have to show that $\Phi_d(x)$ and $\Phi_{d'}(x)$

do not share any root if $d \neq d'$

So, all roots in the product $\prod_{d|m} \Phi_d(x)$ are simple as well

now, if z is a root of $x^m - 1 \Rightarrow z = e^{2\pi i k/m}$, if $k \nmid m$, then z is a root of $\Phi_m(x)$, which is in the product $\prod_{d|m} \Phi_d(x)$, since $m|m$

If $k|m$, so $m = s \cdot k$ then $z = e^{2\pi i/s}$ and, by definition $s|m$

so z is a root of $\Phi_s(x)$ and since $s|m$, this polynomial is also in the product $\prod_{d|m} \Phi_d(x)$. We have shown then that if z is a root of $x^m - 1$, so is a root of $\prod_{d|m} \Phi_d(x)$ as well.

Now we show the opposite, if ω is a root of $\prod_{d|m} \Phi_d(x) \Rightarrow \omega^m = 1$

So is also a root of $x^m - 1$.

then both share the same roots and since Φ_d 's are all monic

$$\prod_{d|m} \Phi_d(x) = \prod_{\ell=0}^{m-1} (x - \zeta^\ell) = x^m - 1$$

P3

$$(1+x)^{2m} = \left[(1+x)^m \right]^2 = \left[\sum_{k=0}^m \binom{m}{k} x^k \right]^2$$

$$\text{so, take } x=1 \Rightarrow \left[\sum_{k=0}^m \binom{m}{k} \right]^2 = 4^m$$

P4]

(2)

$$(x-1)^m (x+1)^m = \sum_{k=0}^{2m} \left(\sum_{i+j=k} (-1)^{m-i} \binom{m}{i} \binom{m}{j} \right) x^k$$

just by
taking the
product of both
binomial expansions

on the other hand, we can also expand

$$(x-1)^m (x+1)^m = (x^2-1)^m = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} x^{2i}$$

Compare the coeffs. of x^m if m even

C_m ← the value we want to compute

$$(-1)^m \sum_{i+j=m} (-1)^i \binom{m}{i} \binom{m}{j} = (-1)^m \left(\binom{m}{0} \binom{m}{m} - \binom{m}{1} \binom{m}{m-1} + \dots + \binom{m}{m} \binom{m}{0} \right)$$

$$= (-1)^{m-\frac{m}{2}} \binom{m}{m/2}$$

$$\Rightarrow C_m = (-1)^{m/2} \binom{m}{m/2}$$

P5]

we ~~now~~ showed that all roots of p are real if λ is a root of $p \in \mathbb{R}[x]$, $\bar{\lambda}$ is also a root. So, if p has odd degree, if all roots are complex, they come in pairs, so at least one root must satisfy $\lambda = \bar{\lambda}$ i.e., is real.

P6/

$$\text{Arg}(x) = \text{Arg}(1 + i \tan \theta) = \arctan(\tan \theta) = \theta \quad |\theta| < \frac{\pi}{2}$$

$$\text{so } \text{Arg}(x^m) = m\theta$$

$$\text{and } x^m = (1 + i \tan \theta)^m = \sum_{\ell=0}^m \binom{m}{\ell} i^\ell (\tan \theta)^\ell = \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell} (-1)^\ell (\tan \theta)^{2\ell}$$

$$+ i \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell+1} (-1)^\ell (\tan \theta)^{2\ell+1}$$

$$\text{since } \text{Arg}(x^m) = m\theta \Rightarrow \tan(m\theta) = \frac{\text{Im } x^m}{\text{Re } x^m} = \frac{\sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell+1} (-1)^\ell (\tan \theta)^{2\ell+1}}{\sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2\ell} (-1)^\ell (\tan \theta)^{2\ell}}$$

P7/ $q(x) = (x+1)p(x) - x$, $\deg q = m+1$ and $x=0, \dots, m$ are roots of $q(x)$

$$\text{So } q = c \cdot \prod_{\ell=0}^m (x-\ell), \text{ by unique factorization}$$

$$\text{note that } q(-1) = 1, \text{ so } 1 = c \prod_{\ell=0}^m (-1-\ell) = \cancel{(-1)^{m+1}} c \cdot (-1)^{m+1} \prod_{\ell=0}^m (1+\ell)$$

$$= c \cdot (-1)^{m+1} (m+1)! \Rightarrow c = \frac{(-1)^{m+1}}{(m+1)!}$$

$$\text{so } (x+1)p(x) - x = \frac{(-1)^{m+1}}{(m+1)!} \prod_{\ell=0}^m (x-\ell), \text{ take } x=m+1$$

$$(m+2)p(m+1) - (m+1) = \frac{(-1)^{m+1}}{(m+1)!} \prod_{\ell=0}^m (m+1-\ell) = (-1)^{m+1}$$

$$\text{so } p(m+1) = \frac{(m+1 + (-1)^{m+1})}{m+2}$$

P8/

(3)

So, using unique factorization

$$P = \underset{\substack{\uparrow \\ \text{monic}}}{1} \cdot (x - \lambda_1)(x - \bar{\lambda}_1)(x - \lambda_2)(x - \bar{\lambda}_2)$$

and we know $\lambda_1 = 2e^{i\alpha}$

By expanding and comparing with $x^4 - 4x^3 + 10x^2 - 12x + 8$, we get the eqs.

coeff. of $x^0 \rightarrow |\lambda_2|^2 = 2$

coeff. of $x^3 \rightarrow 4\cos\alpha + 2\operatorname{Re}(\lambda_2) = 4$

coeff. of $x^2 \rightarrow 8\cos\alpha \cdot \operatorname{Re}(\lambda_2) + 6 = 10$

$$\Rightarrow \begin{aligned} \lambda_1 &= 2e^{i\pi/3} \\ \lambda_2 &= \sqrt{2}e^{i\pi/4} \end{aligned}$$

P9/

a) $x^2 + x + 1 = (x - \lambda)(x - \bar{\lambda}) \Rightarrow |\lambda|^2 = 1 \Rightarrow \lambda = e^{i\alpha}$ and $\operatorname{Re}(\lambda) = -\frac{1}{2} \Rightarrow \alpha = \frac{2\pi}{3}$

So the roots are $\lambda = e^{2\pi i/3}$ and $\bar{\lambda} = e^{-2\pi i/3} = e^{4\pi i/3}$ which are roots of 1.

b) $p(\lambda) = 1 + \lambda + \lambda^2 = 0$

$\Rightarrow p(\bar{\lambda}) = 0$ as well

so λ and $\bar{\lambda}$ are roots of $p \therefore p = (x - \lambda)(x - \bar{\lambda})h(x) = q(x)h(x)$, for any value

of m_1, m_2, m_3

P10/

a) $\det(A_m)(x_1)$ is a polynomial of degree m in x_1 . ~~P10~~ we know $\det(A_m) = 0$

when rows are linearly dependent, this happens if $x_1 = x_2, x_1 = x_3, \dots, x_1 = x_m$

So there are $m-1$ roots of $\det(A_m)(x_1)$ and we know there are the maximum number of roots, so they are all of them.

b) For $m=2$, we have $\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} \Rightarrow \det A_2 = x_2 - x_1$

Assume it holds for m and we show it for $m+1$

$$\text{So } \det A_m = \prod_{1 \leq i < j \leq m} (x_j - x_i)$$

and we want $\det A_{m+1}$ where $A_{m+1} = \begin{pmatrix} 1 & x_1 & \dots & x_1^{m-1} & x_1^m \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & x_m & \dots & x_m^{m-1} & x_m^m \\ 1 & x_{m+1} & \dots & x_{m+1}^{m-1} & x_{m+1}^m \end{pmatrix}$

$\det(A_{m+1})(x_{m+1})$ has roots ~~x_{m+1}~~ $x_1, \dots, x_m$ and is of degree m in x_{m+1} , so it has a factorization of the form:

$$\det(A_{m+1})(x_{m+1}) = C \cdot \prod_{l=1}^m (x_{m+1} - x_l)$$

where C is a ^(polynomial) function of $x_1, \dots, x_m$, and $\det(A_{m+1})(x_{m+1}) = C x_{m+1}^m + O(x_{m+1}^{m-1})$

So C is the coefficient of x_{m+1}^m . On the other hand, if we compute

$\det A_{m+1}$ using the lowest row, we get

$$\det A_{m+1} = (\det A_m) \cdot x_{m+1}^m + \dots$$

$$\text{So } C = \det A_m = \prod_{1 \leq i < j \leq m} (x_j - x_i)$$

$$\Rightarrow \det A_{m+1} = \prod_{1 \leq i < j \leq m+1} (x_j - x_i)$$