

Homework 5

①

P1

a) $\Rightarrow$ Suppose f and g have a common factor $h \in F[x]$ i.e. $f = f_1 \cdot h$, $g = g_1 \cdot h$.
 Since h is non-constant, then $\deg f_1 < m$ and $\deg g_1 < m$, then set $s = f_1$ and $t = -g_1$, then $sg + tf = 0$.

$\Leftarrow$ Suppose $\exists s, t$ as in the theorem, but f and g have no ^(nonconstant) common factor, so $\gcd(f, g) = 1$, then $\exists h_1, h_2 \in F[x]$ s.t. $h_1 f + h_2 g = 1$.
 then ~~we~~ multiply by t and use $tf = -sg$:

$$t(h_1 f + h_2 g) = t = -h_1 s g + t h_2 g = (t h_2 - h_1 s) g$$

~~Let's~~ Comparing degrees:

$$\deg t = \deg g + \deg(t h_2 - h_1 s) \geq m$$

b)

$$\begin{aligned} sg + tf &= \left(\sum_{i=0}^{m-1} c_i x^i \right) \left(\sum_{i=0}^m b_i x^i \right) + \left(\sum_{i=0}^{m-1} d_i x^i \right) \left(\sum_{i=0}^m a_i x^i \right) \\ &= \sum_{i+j=0}^{m+m-1} c_i b_j x^{i+j} + \sum_{i+j=0}^{m+m-1} d_i a_j x^{i+j} = \sum_{l=0}^{m+m-1} p_l x^l \end{aligned}$$

So ~~we~~ where $p_l = \sum_{i+j=l} (c_i b_j + d_i a_j)$

e.g. $p_0 = c_0 b_0 + d_0 a_0$

$p_{m+m-1} = ~~b_m~~ c_{m-1} + d_{m-1} a_m$

$p_{m+m-2} = b_m c_{m-2} + b_{m-1} c_{m-1} + d_{m-2} a_m + d_{m-1} a_{m-1}$

$\vdots$

c) Consider $\overset{\text{wlog}}{m \geq n}$ and define $v = (c_0, \dots, c_{m-1}, d_0, \dots, d_{m-1})$
 then, the eq. for the x^i coefficient will take the form:

$$\begin{bmatrix} b_i & b_{i-1} & \dots & b_{i-m+1} & | & a_i & a_{i-1} & \dots & a_{i-m+1} \end{bmatrix} \cdot v^T = 0$$

where some coefficients are identically 0, namely

$$b_i = 0, \text{ for } i > m$$

$$a_i = 0 \text{ for } i > m$$

So, we rather divide the eqs. on coeff. of x^i for $0 \leq i \leq m-1$, $m \leq i \leq m$
 and $m+1 \leq i \leq m+m-1$:

$$i = \begin{cases} 0 \\ \vdots \\ m-1 \end{cases} \quad x^i \quad \begin{bmatrix} \overbrace{b_0 \ 0 \ \dots \ 0}^{m-1} & \overbrace{a_0 \ 0 \ \dots \ 0}^{m-1} & \overbrace{0 \ 0 \ \dots \ 0}^{m-m} \\ \vdots & \vdots & \vdots \\ b_{m-1} \ b_{m-2} & b_0 \ a_{m-1} \ a_{m-2} & a_0 \ 0 \ \dots \ 0 \end{bmatrix} \cdot v^T = 0$$

$$i = \begin{cases} m \\ \vdots \\ m \end{cases} \quad x^i \quad \begin{bmatrix} b_m \ b_{m-1} \ \dots \ b_1 & \overbrace{a_m \ a_{m-1} \ \dots \ a_0 \ 0 \ \dots \ 0}^{m-m-1} \\ \vdots & \vdots \\ b_{m+1} \ b_m & 0 \ a_m \ a_{m-1} \ \dots \ a_0 \ 0 \ \dots \ 0 \\ \vdots & \vdots \\ b_m \ b_{m-1} & \vdots \\ b_{m-m+1} & \overbrace{0 \ \dots \ 0 \ a_m \ \dots \ a_1}^{m-m-1} \end{bmatrix} \cdot v^T = 0$$

$$i = \begin{cases} m+1 \\ \vdots \\ m+m-1 \end{cases} \quad x^i \quad \begin{bmatrix} \overbrace{0 \ b_m \ \dots \ b_{m-m+2}}^{m-m} & \overbrace{0 \ \dots \ 0 \ a_m \ a_{m-1} \ \dots \ a_2}^{m-m+1} \\ \vdots & \vdots \\ 0 \ b_m \ b_{m-1} \ \dots \ b_{m-m+3} & 0 \ \dots \ 0 \ 0 \ a_m \ \dots \ a_3 \\ \vdots & \vdots \\ 0 \ \dots \ 0 \ b_m & 0 \ \dots \ 0 \ a_m \end{bmatrix} \cdot v^T = 0$$

So, we obtain the matrix:

$$\begin{array}{c}
 \begin{array}{c} m \\ \left\{ \begin{array}{c} m \\ \left\{ \begin{array}{c} b_0 \dots b_{m-1} \\ \vdots \\ b_m \end{array} \right\} \end{array} \right\} \\ m \\ \left\{ \begin{array}{c} m \\ \left\{ \begin{array}{c} a_0 \dots a_{m-1} \\ \vdots \\ a_m \end{array} \right\} \end{array} \right\} \end{array}
 \end{array}
 = \text{Syl}(f, g)$$

e.g. take $m=2$, $m=3$, then we get a 5×5 matrix

$$\begin{array}{c|ccc}
 b_0 & 0 & a_0 & 0 & 0 \\
 b_1 & b_0 & a_1 & a_0 & 0 \\
 \hline
 b_2 & b_1 & a_2 & a_1 & a_0 \\
 b_3 & b_2 & 0 & a_2 & a_1 \\
 0 & b_3 & 0 & 0 & a_2
 \end{array}$$

d) $f=3x^2+7x+6$ $g=x^2+1$ so we expect a 4×4 matrix:

$$\text{Syl}(f, g) = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 7 & 6 & 0 & 1 \\ 3 & 7 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

so ~~Res(f, g) = 58~~ $\text{Res}(f, g) = 58 \neq 0 \therefore f$ and g have no nonconstant common factor.

P2

a) First, for $s=1$, $f(x_1) \in \mathbb{C}[x_1]$, is clear that we can choose any x_1 that is not a root of f and since the set of roots is finite, we have infinitely many $x_1 \in \mathbb{C} \setminus \{\text{roots of } f\}$ to choose from.

Assume $\exists$ such $z_1, \dots, z_{s-1}$ for $f(x_1, \dots, x_{s-1}) \in \mathbb{C}[x_1, \dots, x_{s-1}]$, then consider $h(x_1, \dots, x_s) \in \mathbb{C}[x_1, \dots, x_s]$

we can write:

$$h(x_1, \dots, x_s) = \sum_{i=1}^m P_i(x_1, \dots, x_{s-1}) x_s^i \quad \text{where } P_i \in \mathbb{C}[x_1, \dots, x_{s-1}] \text{ for } i=1, \dots, m$$

by induction hypothesis we know that $\exists z_1, \dots, z_{s-1} \in \mathbb{C}$, ~~such that~~ such that $P_m(z_1, \dots, z_{s-1}) \neq 0$

then $h(z_1, \dots, z_{s-1}, x_s) = \sum_{i=1}^m P_i(z_1, \dots, z_{s-1}) x_s^i \in \mathbb{C}[x_s]$ and

so it has finitely many roots, so we can pick $z_s \in \mathbb{C} \setminus \{\text{roots of } h(z_1, \dots, z_{s-1}, x_s)\}$
 $\Rightarrow h(z_1, \dots, z_s) \neq 0$

b) we want to evaluate $f(x_1, \dots, x_m)$ in

$$f(y_1 + \lambda_1 y_s, \dots, y_{s-1} + \lambda_{s-1} y_s, y_s)$$

Consider the evaluation of an arbitrary monomial $m = c \prod_{i=1}^s x_i^{d_i}$ in f :

$$m(y_1 + \lambda_1 y_s, \dots, y_{s-1} + \lambda_{s-1} y_s, y_s) = c \left(\prod_{i=1}^{s-1} (y_i + \lambda_i y_s)^{d_i} \right) y_s^{d_s}$$

$$= c \cdot \underbrace{y_s^{\sum_{i=1}^s d_i}}_{\substack{\parallel \\ y_s^d}} \prod_{i=1}^{s-1} \lambda_i^{d_i} + (\text{lower degree in } y_s) = y_s^d \underbrace{\tilde{m}(\lambda_1, \dots, \lambda_{s-1})}_{\text{some monomial}} + (\text{lower degree in } y_s)$$

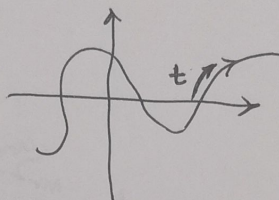
using total degree = d

then f will take the form

$$f = C(\lambda_1, \dots, \lambda_{s-1}) \cdot y_s^d + (\text{lower degree in } y_s)$$

then the coefficient of y_s^d is a polynomial $C(x_2, \dots, x_{s-1}) \in \mathbb{C}[x_2, \dots, x_{s-1}]$ evaluated at $(\lambda_2, \dots, \lambda_{s-1})$. By the part a) we know that $\exists (\lambda_2, \dots, \lambda_{s-1})$ s.t. $C(\lambda_2, \dots, \lambda_{s-1}) \neq 0$ hence we can take $\alpha = \frac{1}{C(\lambda_2, \dots, \lambda_{s-1})}$ and we get the result (3)

c) The idea is that we have a curve $(x = \frac{f_1(t)}{g_1(t)}, y = \frac{f_2(t)}{g_2(t)})$ embedded in $\mathbb{R}^2$



and we want to find its implicit eq., i.e. a polynomial $P(x, y)$ satisfying

$$P\left(\frac{f_1}{g_1}, \frac{f_2}{g_2}\right) = 0$$

we know $xg_1 - f_1 = 0$, if we see the LHS as a polynomial in $\mathbb{R}[t]$
 $yg_2 - f_2 = 0$ where $R = \mathbb{R}[x, y]$, then both eqs.

vanishing simultaneously means $xg_1 - f_1, yg_2 - f_2 \in R[t]$ have a common factor, hence

$$\text{Res}(xg_1 - f_1, yg_2 - f_2) = 0$$

$$(x = t^2, y = t^2(t+1)) \Rightarrow \begin{cases} x - t^2 = 0 \\ y - t^3 - t = 0 \end{cases} \Rightarrow \text{we want to compute}$$

~~$\det \begin{pmatrix} x & y & 1 & 0 & 0 \\ 0 & x & y & 1 & 0 \\ 0 & 0 & x & y & 1 \\ 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & 0 & x \end{pmatrix}$~~

$$\det \begin{pmatrix} y & 0 & x & 0 & 0 \\ 0 & y & 0 & x & 0 \\ -1 & 0 & -1 & 0 & x \\ -1 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 \end{pmatrix} = x^3 - x^2 + 2xy - y^2$$

and one can check $x^3 - x^2 + 2xy - y^2 \Big|_{\substack{x=t^2 \\ y=t^2(t+1)}} = 0$

$$(x = \frac{t-1}{t^2}, y = t-1) \Rightarrow$$

$$xt^2 - t + 1 = 0$$

$$y - t + 1 = 0$$

so we want $\det \begin{pmatrix} x & 1+y & 0 \\ -1 & -1 & 1+y \\ x & 0 & -1 \end{pmatrix} = xy^2 + 2xy + x - y$

and $xy^2 + 2xy + x - y \Big|_{\substack{x = \frac{t-1}{t^2} \\ y = t-1}} = 0$

P3)

a) first $\Rightarrow$, if $\text{Res}(f, g) \neq 0$, then f and g have ~~no~~ ^{nonconstant} common factor is

$$s = hf, t = hg, \text{gcd}(f, g) = 1$$

$\Leftarrow$ If $\text{gcd}(f, g) = 1$ then they have no nonconstant common factor
 $\Rightarrow \text{Res}(f, g) \neq 0$

~~If $\text{Res}(f, g) = 0$, then there is a common factor between f and g , we can write $f = h_1 g_1$ and $g = h_2 g_1$ and just take $s = g_1, t = -f_1$~~

take $s = t = 0$

If $\text{Res}(f, g) \neq 0$, then $\text{gcd}(f, g) = 1$ and so $\exists \tilde{s}, \tilde{t}$ s.t.

$$\tilde{s}f + \tilde{t}g = 1, \text{ then just take } s = \text{Res}(f, g) \cdot \tilde{s}, t = \text{Res}(f, g) \cdot \tilde{t}$$

b) $Q(x_1, \dots, x_m, y) = \sum_{i=2}^s y^{i-2} f_i(x_1, \dots, x_m)$

we can write $f_1(x_1, \dots, x_m) = \sum_{j=0}^{r_1} f_1^{(j)}(x_1, \dots, x_{m-1}) \cdot x_m^j$ for some pos. integer r_1

hence we can see Q and f_1 as polynomials in $R[x_m]$ where

$$R = \mathbb{C}[x_1, \dots, x_{m-1}, y]$$

We also have the resultant of $f, Q \in R[x_m]$

(4)

$$\text{Res}(Q, f_1; x_m) = \sum_{i=0}^d P_i(x_1, \dots, x_{m-1}) y^i$$

$\Rightarrow$ ~~Sup~~ Say $f_1 = \dots = f_s = 0$ and suppose $P_0 = \dots = P_d = 0$ has a solution

Call this solution $(x_1, \dots, x_{m-1}) = \vec{a}$ so $P_\ell(\vec{a}) = 0 \forall \ell \in \{0, \dots, d\}$
this also means there exist $\vec{a}$ s.t.

$$\text{Res}(Q(\vec{a}, y, x_m), f_1(\vec{a}, x_m); x_m) = 0$$

making it very explicit

$\therefore f_1(\vec{a}, x_m)$ and $Q(\vec{a}, x_m, y)$ have a common factor i.e., we can write them as

$$f_1(\vec{a}, x_m) = h(x_m) \cdot g_1(x_m)$$

$$Q(\vec{a}, y, x_m) = h(x_m) \cdot g_2(y, x_m) \quad \text{with } h \text{ nonconstant}$$

Note: at this stage $\vec{a} \in \mathbb{C}^{m-1}$ is just a number, and h cannot depend on y , since f_1 doesn't depend on y .

Since $h \in \mathbb{C}[x_m]$, then h has a root (by FTA) $\therefore f_1(\vec{a}, x_m)$ and $Q(\vec{a}, y, x_m)$ have a common root β (again, β doesn't depend on y), so $\exists \beta$ s.t.

$$f_1(\vec{a}, \beta) = 0$$

$$Q(\vec{a}, y, \beta) = \sum_{i=0}^d y^{i-2} f_i(\vec{a}, \beta) = 0 \quad \text{for all } y \in \mathbb{C}$$

Since y was arbitrary, this implies $f_i(\vec{a}, \beta) = 0$ for $i=2, \dots, s$, so we have found $(x_1, \dots, x_m) = (\vec{a}, \beta) \in \mathbb{C}^m$ root of $f_1 = \dots = f_s = 0$ which is a contradiction.

$\Leftarrow$ Suppose $\nexists (x_1, \dots, x_{m-1})$ s.t. $P_0 = \dots = P_d = 0$, and suppose $\exists \vec{b} (x_1, \dots, x_m) = \vec{b}$ satisfying $f_1 = \dots = f_s = 0$, so this means $Q(\vec{b}, y) = 0$ for all y and in particular $f_1(\vec{b}) = 0$

therefore $Q(x_1, \dots, x_m, y)$ and $f_1(x_1, \dots, x_m)$ have a common root for all y , this root being $\vec{b}$, hence they have a common factor, as polynomials in $\mathbb{C}[y] \subset R[x_m]$

(this is not hard to see, just set $x_i = b_i$ for $i=1, \dots, m-1$, then b_m is the common root for $Q, f_1 \in R[x_m]$)

$$\therefore \text{Res}(Q(b_1, \dots, b_{m-1}, x_m, y), f_1(b_1, \dots, b_{m-1}, x_m); x_m) = 0$$

again, since this is valid for all y , it must hold that

$$P_0(\vec{b}) = \dots = P_d(\vec{b}) = 0$$

$$(\text{here, obviously } P_e(\vec{b}) = P_e(b_1, \dots, b_{m-1}))$$

c) Case $m=1$, is proven in HW 3, so we can assume it holds for $f_i \in \mathbb{C}[x_1, \dots, x_{m-1}]$ $i=1, \dots, s$ and prove it for $\mathbb{C}[x_1, \dots, x_m]$.

$\Rightarrow$ 1 Suppose $\nexists$ a sol. to $f_1 = \dots = f_s = 0$ for $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_m]$

By we regard $Q, f_i \in R[x_m]$ as before and, by part a), we

$$(\text{w/ } R = \mathbb{C}[x_1, \dots, x_{m-1}, y])$$

have that $\exists A, B$ in $R[x_m]$ s.t. $A \cdot f_1 + B \cdot Q = \text{Res}(f_1, Q; x_m)$

$$= \sum_{l=0}^d P_l(x_1, \dots, x_m) y^l$$

and note, the LHS can be written as

$$A(x_1, \dots, x_m, y) f_1 + \sum_{i=2}^s B(x_1, \dots, x_m) y^{i-2} \cdot f_i(x_1, \dots, x_m) \stackrel{(*)}{=} \sum_{l=0}^d K_l(x_1, \dots, x_m) y^l$$

the last step, by the equality w/ the RHS, we must have

$$K_l = P_l \text{ and } K_l = \sum_{a=1}^s \beta_{la} f_a(x_1, \dots, x_m) \text{ this last}$$

equality just by ~~comparing~~ (*).

So, if A a sol. of $f_1 = \dots = f_s = 0$, by part b) A a sol. of $K_0 = \dots = K_d = 0 \Leftrightarrow P_0 = \dots = P_d = 0$, so since $P_0, \dots, P_d \in \mathbb{C}[x_1, \dots, x_{m-1}]$ we can use the inductive hypothesis $\Rightarrow \exists \alpha_0, \dots, \alpha_d \in \mathbb{C}[x_1, \dots, x_{m-1}]$

s.t. $\sum_{l=0}^d \alpha_l \cdot P_l = 1$, by using the expressions of $K_l = P_l$ as expressions in f_i 's, we have the result.

$\Leftarrow$ this is easier, since, we assume $\exists \alpha_1, \dots, \alpha_s \in \mathbb{C}[x_1, \dots, x_m]$ s.t.

$$\sum_{i=1}^s \alpha_i \cdot f_i = 1, \text{ then if } \exists \text{ a common root, we just evaluate } \sum_{i=1}^s \alpha_i f_i$$

at that root and we get $0 = 1$ which is a contradiction.

P4

a) If we expand

$$(x + \varepsilon i \sqrt{1-x^2})^m = \sum_{k=0}^m x^{m-k} i^k \varepsilon^k (1-x^2)^{k/2} \binom{m}{k}, \quad \varepsilon \in \{+1, -1\}$$

our problem are the terms w/ k odd but note that $\varepsilon^{2l+1} = -1$

Better way: define $z_m = (x + i\sqrt{1-x^2})^m$, then

$$T_m = \frac{1}{2} [z_m + \bar{z}_m] = \operatorname{Re}(z_m)$$

$$z_m = \sum_{k=0}^m x^{m-k} i^k (1-x^2)^{k/2} \binom{m}{k}$$

$i^k \in \mathbb{R} \Leftrightarrow k$ is even, so ~~we only~~

$$\operatorname{Re}(z_m) = \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} x^{m-2l} i^{2l} (1-x^2)^l \binom{m}{2l} \in \mathbb{R}[x]$$

Since $\binom{m}{2\ell} \in \mathbb{Z}$ for $2\ell \leq m$, then

$$\operatorname{Re}(z_m) \in \mathbb{Z}[x]$$

b) Using same notation as before

$$\frac{dz_m}{dx} = m(x + i\sqrt{1-x^2})^{m-1} \cdot \left(1 + i \frac{x}{\sqrt{1-x^2}}\right)$$

Note that $T_m = \frac{1}{2}(z_m + \bar{z}_m)$

So since the differential eq. is linear, let's compute how it acts on z_m first. To shorten notation, define $L = \sqrt{1-x^2}$, so

$$\frac{dz_m}{dx} = m(x + iL)^{m-1} \left(1 - \frac{ix}{L}\right)$$

$$\frac{d^2 z_m}{dx^2} = m(m-1)(x + iL)^{m-2} \left(1 - \frac{ix}{L}\right)^2 + m(x + iL)^{m-1} \left(-\frac{i}{L} - \frac{ix^2}{L^3}\right)$$

$$\begin{aligned} \text{So } L^2 \frac{d^2 z_m}{dx^2} &= m(m-1)(x + iL)^{m-2} \underbrace{\left(L - ix\right)^2}_{= \frac{(L-ix)^2}{(x+iL)^2}} + m(x + iL)^{m-1} \underbrace{\left(-\frac{i}{L} - \frac{ix^2}{L^3}\right)}_{= -\frac{i}{L}} \\ &= (x + iL)^m \frac{(L - ix)^2}{(x + iL)^2} = -(x + iL)^m \end{aligned}$$

$$= -m(m-1)z_m + m(x + iL)^{m-1} \left(-\frac{i}{L}\right)$$

So

$$L^2 \frac{d^2 z_m}{dx^2} = x \frac{dz_m}{dx} = -m(m-1)z_m + (x + iL)^{m-1} \underbrace{\left(-\frac{im}{L} - xm + \frac{imx^2}{L}\right)}_{= -m(x + iL)}$$

$$= -m^2 z_m$$

so $L^2 \frac{d^2 z_m}{dx^2} - x \frac{dz_m}{dx} + m^2 z_m = 0$, hence, z_m satisfies the differential eq.

so does $\bar{z}_m$ (since $x \in \mathbb{R}$) and then also $T_m = \frac{1}{2}(z_m + \bar{z}_m)$ does.

c) Take $x = \cos t$ then $1 - x^2 = \sin^2 t$ and $L(\cos t) = \sin t$ (6)

$$\Rightarrow T_m(\cos t) = \frac{1}{2} \left[(\cos t + i \sin t)^m + (\cos t - i \sin t)^m \right]$$

$$= \frac{1}{2} (e^{imt} + e^{-imt}) = \operatorname{Re}(e^{imt}) = \cos mt$$

$$d) \int_{-1}^1 dx \frac{T_l(x) T_j(x)}{\sqrt{1-x^2}} = - \int_{\pi}^0 \frac{\sin t}{\sin t} T_l(\cos t) T_j(\cos t) dt = \int_0^{\pi} \cos(lt) \cos(jt) dt$$

$x = \cos t$
 $dx = -\sin t dt$

$$= \begin{cases} \frac{\pi}{2} & l=j \neq 0 \\ 0 & l \neq j \\ \pi & l=j=0 \end{cases}$$

So they are orthogonal w.r.t. the measure $\frac{dx}{\sqrt{1-x^2}}$, in the interval

$[-1, 1]$