

Mathematics for Computer Science:

Homework 6

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Special Problem 1

Answer: (a):

$$a_n - a_{n-1} = 3(a_{n-1} - a_{n-2}) + 3n + 1$$

Let $b_n = a_n - a_{n-1}$, so $b_n = 3b_{n-1} + 3n + 1$, $b_1 = 1, b_2 = 10$.

$$b_n = 3b_{n-1} + 3n + 1 \quad (1)$$

$$b_{n-1} = 3b_{n-2} + 3n - 2 \quad (2)$$

$$(1) - (2) : b_n - b_{n-1} = 3(b_{n-1} - b_{n-2}) + 3 \quad (3)$$

Let $c_n = b_n - b_{n-1} + \frac{3}{2}$, $c_n = 3c_{n-1}$. $c_2 = \frac{21}{2}$.

So $c_n = \frac{7}{2} \times 3^{n-1}$.

$$b_n = b_1 + \sum_{i=2}^n (c_i - \frac{3}{2}) = 1 + \frac{7(3^n - 3)}{4} - \frac{3(n-1)}{2} = \frac{7 \times 3^n - 11}{4} - \frac{3n}{2}.$$

$$a_n = a_0 + \sum_{i=1}^n b_i = \frac{7(3^{n+1} - 3)}{8} - \frac{11n}{4} - \frac{3n(n+1)}{4} + 1 = \frac{7 \times 3^{n+1} - 6n^2 - 28n - 13}{8}.$$

(b):

$$\frac{1}{a_n} = \frac{1}{a_{n-1}} + 3$$

$$\frac{1}{a_n} = 1 + 3n$$

$$a_n = \frac{1}{3n+1}$$

Special Problem 2

Answer:

(a): Suppose $a_{2k-1} = 1$ and $a_{2k} = 0$, on this condition the number of legal paying patterns is d_{k-1} (the left sequence) times d_{n-k} (the right sequence). So we get $d_n = \sum_{k=1}^n d_{k-1}d_{n-k}$.

(b): From (a), $d_{n+1} = \sum_{k=0}^n d_k d_{n-k}$. So $[x^{n+1}]D(x) = [x^n]D^2(x)$, $D(x) = 1 + xD^2(x)$. Solve the equation we get $D(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. Because $\lim_{x \rightarrow 0^+} D(x) =$

$$d_0 = 1, D(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

(c):

$$\begin{aligned} \sqrt{1-4x} &= 1 + \sum_{n=1}^{+\infty} \frac{\prod_{i=0}^{n-1} (\frac{1}{2} - i)}{n!} (-4x)^n \\ &= 1 + \sum_{n=1}^{+\infty} \frac{\prod_{i=0}^{n-1} (1 - 2i)}{2^n n!} (-4x)^n \\ &= 1 + \sum_{n=1}^{+\infty} \frac{(-1)^{2n} 2^{2n} \prod_{i=0}^{n-1} (2i - 1)}{2^n n!} x^n \\ &= 1 - \sum_{n=1}^{+\infty} \frac{2^n \prod_{i=1}^{n-1} (2i - 1)}{n!} x^n \end{aligned}$$

$$\begin{aligned} 1 - \sqrt{1-4x} &= \sum_{n=1}^{+\infty} \frac{2^n \prod_{i=1}^{n-1} (2i - 1)}{n!} x^n \\ \frac{1 - \sqrt{1-4x}}{2x} &= \sum_{n=1}^{+\infty} \frac{2^{n-1} \prod_{i=1}^{n-1} (2i - 1)}{n!} x^{n-1} \\ &= \sum_{n=0}^{+\infty} \frac{\prod_{i=1}^n (4i - 2)}{(n+1)!} x^n \end{aligned}$$

$$\begin{aligned} \text{So } d_n &= \frac{\prod_{i=1}^n (4i-2)}{(n+1)!} = \frac{\prod_{i=1}^n (n+i)}{(n+1)!} = \frac{(2n)!}{n!n!(n+1)} = (1 - \frac{n}{n+1}) \frac{(2n)!}{n!n!} = \frac{(2n)!}{n!n!} - \\ &\frac{(2n)!}{(n+1)!(n-1)!} = \binom{2n}{n} - \binom{2n}{n+1}. \end{aligned}$$

Special Problem 3

Answer: Like the phase 1, we also have the delay of one packet is at most $|S|$ (proved in Homework5). Suppose packet v_i will not leave the randomized $\sigma(i)$ until every v_i arrive at $\sigma(i)$. On this condition, the max steps T must be greater than the steps v_i leave $\sigma(i)$ immediately. Let $\rho(i)$ denote the path from $\sigma(i)$ to

$d(i)$ and define $H_{ij} = \begin{cases} 1 & \text{if } j \neq i \text{ and } \rho(i) \cap \rho(j) \neq \emptyset \text{ or with a probability of } p \text{ when } j = i \\ 0 & \text{with a probability of } 1-p \text{ and otherwise} \end{cases}$

For an edge e , let $T(e)$ denote the number of the routes passing e . Let $\rho(i) = (e_1, e_2, \dots, e_k)$, $k \leq n$ and D_i denote the delay times of v_i .

So $E(D_i) \leq E(\sum_{j=1}^N H_{ij}) \leq E(\sum_{l=1}^k T(e_l)) = \sum_{l=1}^k E(T(e_l)) = kE(T(e_1))$.

$E(\sum_{i=1}^N |\rho(i)|) = \sum_{i=1}^N E(|\rho(i)|) = \frac{Nn}{2} = \sum_e T(e) = NnT(e_1)$, so $T(e_1) = \frac{1}{2}$.

So $E(D_i) \leq \frac{k}{2} \leq \frac{n}{2}$, $E(T_i) = E(|\rho(i)|) + E(|D_i|) \leq n$.

$T_i = |\rho(i)| + D_i \leq |\rho(i)| + \sum_{j=1}^N H_{ij}$. Let $X_i = |\rho(i)| + \sum_{j=1}^N H_{ij}$. The $|\rho(i)|$ depends on the bits of $\sigma(i)$ and $d(i)$ and they are independent from bit to bit. For different j , H_{ij} are also independent because for a certain i H_{ij} only depends on $\sigma(j)$ and $d(j)$ and they are independent. For a certain j , H_{ij} can only be chosen from 0, 1, so as a certain bit of $\sigma(i)$. So X_i is subject to poisson distribution, and we can use the Chernoff bound to get: $Pr\{X_i > 7n\} \leq e^{-12n}$.

So $Pr\{T_i > 7n\} \leq e^{-12n}$.

Using the union bound, $Pr\{T > 7n\} \leq \sum_{i=1}^N Pr\{T_i > 7n\} \leq 2^n e^{-12n} \leq 2^{-11n}$.

Similarly, we can get another c and c' pairs.

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