

(a): Suppose $a_{2k-1} = 1$ and $a_{2k} = 0$, on this condition the number of legal paying patterns is d_{k-1} (the left sequence) times d_{n-k} (the right sequence). So we get $d_n = \sum_{k=1}^n d_{k-1}d_{n-k}$.

(b): From (a), $d_{n+1} = \sum_{k=0}^n d_k d_{n-k}$. So $[x^{n+1}]D(x) = [x^n]D^2(x)$, $D(x) = 1 + xD^2(x)$. Solve the equation we get $D(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. Because $\lim_{x \rightarrow 0^+} D(x) =$

$d_0 = 1$, $D(x) = \frac{1 - \sqrt{1-4x}}{2x}$.

(c):

$$\begin{aligned}\sqrt{1-4x} &= 1 + \sum_{n=1}^{+\infty} \frac{\prod_{i=0}^{n-1} (\frac{1}{2} - i)}{n!} (-4x)^n \\ &= 1 + \sum_{n=1}^{+\infty} \frac{\prod_{i=0}^{n-1} (1 - 2i)}{2^n n!} (-4x)^n \\ &= 1 + \sum_{n=1}^{+\infty} \frac{(-1)^{2n} 2^{2n} \prod_{i=0}^{n-1} (2i - 1)}{2^n n!} x^n \\ &= 1 - \sum_{n=1}^{+\infty} \frac{2^n \prod_{i=1}^{n-1} (2i - 1)}{n!} x^n\end{aligned}$$

$$\begin{aligned}1 - \sqrt{1-4x} &= \sum_{n=1}^{+\infty} \frac{2^n \prod_{i=1}^{n-1} (2i - 1)}{n!} x^n \\ \frac{1 - \sqrt{1-4x}}{2x} &= \sum_{n=1}^{+\infty} \frac{2^{n-1} \prod_{i=1}^{n-1} (2i - 1)}{n!} x^{n-1} \\ &= \sum_{n=0}^{+\infty} \frac{\prod_{i=1}^n (4i - 2)}{(n+1)!} x^n\end{aligned}$$

$$\begin{aligned}\text{So } d_n &= \frac{\prod_{i=1}^n (4i-2)}{(n+1)!} = \frac{\prod_{i=1}^n (n+i)}{(n+1)!} = \frac{(2n)!}{n!n!(n+1)} = (1 - \frac{n}{n+1}) \frac{(2n)!}{n!n!} = \frac{(2n)!}{n!n!} - \\ &\frac{(2n)!}{(n+1)!(n-1)!} = \binom{2n}{n} - \binom{2n}{n+1}.\end{aligned}$$

Special Problem 3

Answer: Like the phase 1, we also have the delay of one packet is at most $|S|$ (proved in Homework5). Suppose packet v_i will not leave the randomized $\sigma(i)$ until every v_i arrive at $\sigma(i)$. On this condition, the max steps T must be greater than the steps v_i leave $\sigma(i)$ immediately. Let $\rho(i)$ denote the path from $\sigma(i)$ to