

第五周作业及答案



1. 设 $T: \mathbb{C}^n \rightarrow \mathbb{C}^n$, 求 $\mathbb{C}^n$ 的
 $x \mapsto Ax$

广义特征子空间分解, 求可逆
阵 P , 使得 $P^{-1}AP$ 是分块对角阵,
且每一块是上三角阵.

① $n=3$, $A = \begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$

② $n=3$, $A = \begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}$

$$(3) n=4, \quad A = \begin{pmatrix} 0 & -2 & 3 & 2 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$

2. 设 $T: \mathbb{C}^n \rightarrow \mathbb{C}^n$ 线性变换,
 $x \mapsto Ax$

λ_0 是一个特征值. 令 $k \geq 1$.

$$N_{\lambda_0, k} = \{x \in \mathbb{C}^n \mid (A - \lambda_0 I_n)^k x = 0\}$$

证明: $\exists k_0 \geq 1$, 使得

$$N_{\lambda_0, k_0} = N_{\lambda_0, k_0+1} = N_{\lambda_0, k_0+2} = \dots = G_{\lambda_0}$$

3. 设 $T: \mathbb{C}^n \rightarrow \mathbb{C}^n$ 是一个线性变换.
$$x \mapsto Ax$$

展示: ① $\mathbb{C}^n \supseteq C(A) \supseteq C(A^2) \supseteq C(A^3) \supseteq \dots$

② 存在 $k_0 \geq 1$, $C(A^{k_0}) = C(A^{k_0+1})$

③ 若 $C(A^{k_0}) = C(A^{k_0+1})$, 则 $C(A^{k_0+1}) = C(A^{k_0+2})$

4. 设 $T: \mathbb{C}^5 \rightarrow \mathbb{C}^5$ 满足 $C(A^4) \neq C(A^5)$,
$$x \mapsto Ax$$

则 A 是幂零阵 (即存在 $t \in \mathbb{N}$, $A^t = 0$)

5. 设 $T: \mathbb{C}^n \rightarrow \mathbb{C}^n$ 是一个线性变换,

$x \mapsto Ax$, 令 $G_{\lambda_0}(T)$ 是 T 关于 λ_0 的广义特征子空间.

证明: (1) 若 A 可逆, 则 $G_{\lambda_0}(T) = G_{\lambda_0^{-1}}(T^{-1})$

(2) 若对于任意特征值 λ .

$$G_{\lambda}(T) = V_{\lambda}(T) \text{ (特征子空间)}$$

则 T 可对角化;

(3) 应用 (2), 证明若 $A^2 = A$,

则 T 可对角化.

6. 设 $T: \mathbb{C}^n \rightarrow \mathbb{C}^n$ 是一个线性变换, A 不零, 证明:

变换, A 不零, 证明:

$$\mathbb{C}^n = N(A^{n-1}) \oplus C(A^{n-1}).$$

(参考 8.5 证明, 提示, 因为 $\dim N(A^n) < n$

$$N(A^{n-1}) = N(A^n).$$

$$\begin{aligned} \text{则 } \mathbb{C}^n &= N(A^n) + C(A^n) \\ &= N(A^{n-1}) \oplus C(A^n) \end{aligned}$$

但 $C(A^n) \subseteq C(A^{n-1})$, 检查

$$\begin{aligned} \dim N(A^{n-1}) + \dim C(A^{n-1}) &= \dim N(A^n) + \dim C(A^n) \\ &= n. \end{aligned}$$

答案

$$1. (1) \quad |\lambda I_3 - A| = \lambda^2(\lambda - 1) \quad \begin{matrix} \mu_1 = 0, \\ \mu_2 = 1 \end{matrix}$$

$$G_{\mu_1} = \left\{ x \mid A^2 x = 0 \right\} = \left\{ c_1 \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \mid c_1, c_2 \in \mathbb{C} \right\}$$

$$G_{\mu_2} = \left\{ x \mid (A - I)x = 0 \right\} = \left\{ c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid c \in \mathbb{C} \right\}$$

$$\Rightarrow \mathbb{C}^3 = G_{\mu_1} \oplus G_{\mu_2}$$

$$\text{令 } \beta_1 = \alpha_1 - 2\alpha_2, \quad \beta_2 = \alpha_1, \quad \beta_3 = \alpha_3$$

$$\text{则 } P = (\beta_1, \beta_2, \beta_3)$$

$$P^{-1}AP = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(2) \quad |\lambda I_3 - A| = (\lambda + 1)^2 (\lambda - 3)$$

$$G_{\mu_1 = -1} = \left\{ x \mid (A + I)^2 x = 0 \right\}$$

$$= \left\{ c_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}^{=\alpha_1} + c_2 \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}^{=\alpha_2} \mid c_1, c_2 \in \mathbb{C} \right\}$$

$$G_{\mu_2 = 3} = \left\{ c \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}^{=\alpha_3} \mid c \in \mathbb{C} \right\}$$

$$\mathbb{C}^3 = G_{\mu_1} \oplus G_{\mu_2}$$

$$\hat{\sim} P = (\alpha_1, \alpha_2, \alpha_3)$$

$$P^{-1}AP = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

$$(3) |\lambda I_4 - A| = \lambda^2 (\lambda - 2)^2$$

$$\mu_1 = 0, \mu_2 = 2$$

$$G_{\mu_1} = \left\{ c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mid c_1, c_2 \in \mathbb{C} \right\}$$

$$G_{\mu_2} = \left\{ c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \mid c_1, c_2 \in \mathbb{C} \right\}$$

$$G_{\mu_1} \oplus G_{\mu_2} = \mathbb{C}^4$$

$$P = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix}$$

2. 略

3. 因为 $\mathbb{C}^n \supseteq C(A^k) \supseteq C(A^{k+1}) \supseteq \dots \supseteq \{0\}$

则存在 k_0 , $C(A^{k_0}) = C(A^{k_0+1})$

$\forall v \in C(A^{k_0+1})$, 则 $\exists w \in \mathbb{C}^n$

$$v = A^{k_0+1} w = A(A^{k_0} w)$$

因为 $A^{k_0} w \in C(A^{k_0}) = C(A^{k_0+1})$

$$\exists u \in \mathbb{C}^n, A^{k_0} w = A^{k_0+1} u.$$

$$\Rightarrow v = A^{k_0+2} u \in C(A^{k_0+2})$$

4. 由 3.

$$\begin{array}{c} C(A) \subsetneq C(A^2) \subsetneq C(A^3) \subsetneq C(A^4) \\ \cap \\ \mathbb{C}^5 \end{array} \quad \begin{array}{c} \\ \cup \\ C(A^5) \end{array}$$

$C(A) \neq \mathbb{C}^5$ 否则 A 可逆.

$$\Rightarrow \dim_{\mathbb{C}} C(A) \leq 4$$

$$\Rightarrow \dim_{\mathbb{C}} C(A^5) = 0 \Rightarrow A^5 = 0.$$

5. (17, (2) 17)

(3) $A^2 = A \Rightarrow A$ 的特征值是 0, 1.

(因为 $A\alpha = \lambda\alpha \Rightarrow A^2\alpha = \lambda^2\alpha = \lambda\alpha$
 $\Rightarrow \lambda^2 = \lambda$) $\mu_1 = 1, \mu_2 = 0$

$$G_{\mu_1} = \{x \mid (A - I)^k x = 0, \exists k \geq 1\}$$

$$G_{\mu_2} = \{x \mid A^k x = 0, \exists k \geq 1\}$$

因为 $A^2 = A$, $G_{\mu_2} = V_{\mu_2}$

$$(A - I)^2 = (I - A)^2 = I - 2A + A^2 = I - A$$

$$\Rightarrow G_{\mu_1} = V_{\mu_1}$$

6. 证明: 8.5 已证:

$$\mathbb{C}^n = N(A^n) \oplus C(A^n)$$

因为 A 不是零, $\dim N(A^n) < n$,

$$\Rightarrow N(A^{n-1}) = N(A^n)$$

$$\text{则 } \mathbb{C}^n = N(A^{n-1}) \oplus C(A^n)$$

$$\text{但 } C(A^n) \subseteq C(A^{n-1}).$$

$$\Rightarrow \mathbb{C}^n = N(A^{n-1}) + C(A^{n-1})$$

$$\text{但 } \dim N(A^{n-1}) + \dim C(A^{n-1}) = n$$

$\Rightarrow$ 以上的和是直和.