

1. 设 $T: V \rightarrow V$ 是一个幂零变换,
设 $v \neq 0 \in V$, $C_v = \text{Span}\{v, T(v), \dots,$
 $T^k(v), \dots\}$. 证明:

C_v 是极大循环子空间, 即不存在
 $w \in V$, $C_v \subsetneq C_w \iff v \notin \text{Im } T$.

2. 设 $A \in M_n(\mathbb{C})$, 证明: A 和 A^T
相似. (提示: A 和 A^T 有相同的
Jordan 标准型)

(2) 设 $J = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}_{n \times n}$ 求可逆阵 P

使得 $P^{-1}JP = J^T$

3. 设 $a \neq 0$, $A = \begin{pmatrix} a & a & a & a \\ & a & a & a \\ & & a & a \\ & & & a \end{pmatrix}$

求 A 的 Jordan 标准型.

4. 设 $A \in M_n(\mathbb{C})$, $\text{rank}(A) = 1$.

求 A 的 Jordan 标准型.

5. 求下列方阵 Jordan 标准型 J
并求复方阵 P , 使得 $P^{-1}AP = J$

(1)

$$\begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$$

(2)

$$\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}$$

6. 设 $A \in M_n(\mathbb{C})$, $|\lambda I_n - A| = (\lambda + 2)^4 (\lambda - 1)^2$, 则 A 的 Jordan 标准型有几种可能? (不计子块顺序)

7. 设 $A \in M_3(\mathbb{C})$, 且 A 的任意特征值 λ 满足 $|\lambda| < 1$.

证明: $\lim_{m \rightarrow +\infty} A^m = 0$

(提示: 只需证明一个 Jordan 块

$J = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}_{K \times K}$ 满足 $|\lambda| < 1$, 则 $\lim_{n \rightarrow +\infty} J^n = 0$,

$J = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}_{K \times K} + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}_{K \times K} = N \quad (K \leq 3)$

$$J^n = (\Lambda + N)^n = \sum_{l=0}^n C_n^l \Lambda^{n-l} N^l$$

$$= \Lambda^3 + C_n^1 \Lambda^2 N + C_n^2 \Lambda N^2$$

8. 设 $A = \begin{pmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{pmatrix}$ 求 A^m ($m \geq 1$)

解答:

1. 设 $T(v) \neq 0$, $T^{k+1}(v) = 0$

则 C_v 有基: $T^k(v), \dots, T(v), v$

若 $v \in \text{Im } T$, 则存在 $w \in V$, $T(w) = v$

$C_v \subsetneq C_w$, C_v 不是极大的

反之, 若存在 $w \in V$, $C_v \subsetneq C_w$

设 $\dim_{\mathbb{C}} C_v = k+1$, $\dim_{\mathbb{C}} C_w = l+1$

$k < l$

且 C_w 有基: $T^l(w), \dots, T(w), w$.

令 $v = C_0 w + C_1 T(w) + \dots + C_l T^l(w)$

$$T^{k+1}(v) = 0 = C_0 T^{k+1}(w) + \dots + C_{l-k-1} T^l(w)$$

$$\Rightarrow C_0 = \dots = C_{l-k-1} = 0$$

$$\begin{aligned} \text{E.p. } v &= C_{l-k} T^{l-k}(w) + \dots + C_l T^l(w) \\ &= T^{l-k} [C_{l-k} w + \dots + C_l T^k(w)] \end{aligned}$$

$$\Rightarrow v \in \text{Im } T.$$

$$2. (1) \operatorname{rank}(A - \lambda_i I)^k = \operatorname{rank}(A^T - \lambda_i I)^k \quad \forall k \in \mathbb{N}$$

$$(2) \text{ 设 } P^{-1}JP = J^T, \text{ 令 } P = (\alpha_1, \dots, \alpha_n)$$

$$JP = PJ^T, \text{ 比较两边列, 得}$$

$$J\alpha_1 = \lambda\alpha_1 + \alpha_2, J\alpha_2 = \lambda\alpha_2 + \alpha_3, \dots, J\alpha_n = \lambda\alpha_n$$

$$(J - \lambda I_n)\alpha_1 = \alpha_2, (J - \lambda I_n)\alpha_2 = \alpha_3, \dots, (J - \lambda I_n)\alpha_n = 0$$

$$\text{即 } \alpha_2 = (J - \lambda I_n)\alpha_1, \alpha_3 = (J - \lambda I_n)^2\alpha_1, \dots, \alpha_n = (J - \lambda I_n)^{n-1}\alpha_1$$

$$(J - \lambda I_n)x = 0 \Rightarrow x = ce_1, \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{因此 } P = (e_n, e_{n-1}, \dots, e_1)$$

$$3. |\lambda I_4 - A| = (\lambda - a)^4$$

$$\operatorname{rank}(aI_4 - A) = 3$$

$$\Rightarrow \dim N(aI_4 - A) = 1$$

$\Rightarrow$ 只有一个 Jordan 块.
A 相似于 $\begin{pmatrix} a & 1 & & \\ & a & 1 & \\ & & a & 1 \\ & & & a \end{pmatrix}$

4. 设 $P^{-1}AP = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_t \end{pmatrix} = J$

$J_1, \dots, J_t$ Jordan 块

因为 $r(A) = r(J) = 1$

$\Rightarrow J_1, \dots, J_t$ 只有一个非零元.

两种情形:

(1) 非零元在对角线

$$J = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \\ & & & \lambda \end{pmatrix} \quad \lambda \neq 0.$$

(2) 非零元在次对角线

$$J = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix},$$

5. (1)

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 3 & 6 & 10 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$(2) \quad J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

6. 对应特征值 $\lambda_1 = -2$ 的 Jordan 块

有 5 种可能:

① 4 个 1 阶块.

② 2 个 1 阶块, 1 个 2 阶块

③ 2 个 2 阶块

④ 一个 1 阶块, 1 个 3 阶块

⑤ 一个 4 阶块.

对应 $\lambda_2 = 1$ 的 Jordan 块有 2 种可能:

- ① 2 个 1 阶块;
- ② 1 阶 2 阶块.

$\Rightarrow$ 共 $5 \times 2 = 10$ 种可能.

7. 设 $J_{\lambda, k} = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}_{k \times k} \quad k \leq 3$

$$J_{\lambda, k} = \Lambda + N \quad |\lambda| < 1$$

其中 $\Lambda = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$

$$J_{\lambda, k}^m = \sum_{l=0}^m C_m^l \Lambda^{m-l} N^l$$

$$= \Lambda^m + C_m^1 \Lambda^{m-1} N + C_m^2 \Lambda^{m-2} N^2.$$

$$\lim_{m \rightarrow +\infty} J_{\lambda, k}^m = \lim_{m \rightarrow +\infty} \Lambda^m +$$

$$\lim_{m \rightarrow +\infty} m \Lambda^{m-1} N + \lim_{m \rightarrow +\infty} \frac{m(m-1)}{2} \Lambda^{m-2} N^2$$

检查 $\lim_{m \rightarrow +\infty} m \lambda^{m-1} = 0$

$$\left[\text{令 } a_m = m |\lambda|^m, \quad \frac{a_{m+1}}{a_m} = \frac{m+1}{m} \cdot |\lambda| \right]$$

当 m 足够大以后, $0 < a_{m+1} < a_m$

$\Rightarrow$ 极限存在, 若 $\lim_{m \rightarrow +\infty} a_m \neq 0$,

$$\text{则 } \frac{a_{m+1}}{a_m} = \frac{m+1}{m} \cdot |\lambda| \Rightarrow \lim_{m \rightarrow +\infty} a_m$$

$$= |\lambda| \cdot \lim_{m \rightarrow +\infty} a_m \quad \text{矛盾!)$$

$$\text{同理 } \frac{m(m-1)}{2} \lambda^{m-2} \rightarrow 0 \quad m \rightarrow +\infty$$

$$\text{因此 } \lim_{m \rightarrow +\infty} J_{\lambda, k}^m = 0$$

$$\text{设 } A \in M_3(\mathbb{C}), \text{ 则 } P^{-1}AP = J$$

$$\begin{aligned}\lim_{m \rightarrow +\infty} A^m &= \lim_{m \rightarrow +\infty} (PJP^{-1})^m \\ &= \lim_{m \rightarrow +\infty} PJ^m P^{-1} = 0\end{aligned}$$

$$8. |\lambda I_3 - A| = (\lambda + 1)^3.$$

$$\text{rank}(A + I) = 1$$

$$\Rightarrow \dim_{\mathbb{C}} N(A + I) = 2$$

$\Rightarrow$ 有两个 Jordan 块, $1\beta_1$ 和 $2\beta_2$

$$\text{求解 } (A + I)^2 X = 0 \text{ 且 } (A + I)X \neq 0$$

$$\Rightarrow X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(A+I) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{求 } (A+I)x=0, \text{ 且 } x \notin C(A+I)$$

$$\Rightarrow x = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{令 } P = \begin{pmatrix} -2 & 3 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\text{则 } P^{-1}AP = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} = J$$

$$A^m = P J^m P^{-1}$$

$$= (-1)^{m-1} \begin{pmatrix} 3m-1 & 6m & 75m \\ m & 2m-1 & 5m \\ m & 2m & 5m-1 \end{pmatrix}$$