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相对误差 $er(x) = \frac{x-x^*}{x^*}$
又有 P 位有效数字 $\rightarrow |er(x)| < \frac{1}{2} 10^{-P}$
保留 P 位 $\rightarrow |er(x)| < \frac{1}{2} 10^{-P}$
 $|er(x)| \leq \frac{1}{2(10^P)} 10^{-P} \rightarrow$ 保留 P 位 $x = x^*$
或 x 具有 P 位正确的有效数字

有 P 位 $\rightarrow \frac{1}{2} 10^{-P}$
保留 P 位 $\rightarrow \frac{1}{2} 10^{-P}$
保留 P 位 $\leftarrow \frac{1}{2(10^P)} 10^{-P}$
 $\frac{1}{2} 10^{-P}$

牛顿法反问题每步收敛性 ≥ 0.1
 $= \left| \frac{f(x) - f(x^*)}{f(x^*)} \right| \approx \left| \frac{f'(x^*)}{f(x^*)} \right| |x - x^*|$
下证 $UFL = \beta^L$, $OFL = \beta^{U+1} (1 - \beta^{-P})$ 上证
总误差 $= (\beta - 1) \beta^P (U - L + 1) + 1$

$\sum_{match} = \frac{1}{2} \beta^{1-P} \Rightarrow 2^{-P}$, $0 < UFL < \epsilon < OFL$

二. $f(x) = y$, $cond = \left| \frac{dy}{dx} \right| \approx \frac{1}{|f'(x)|}$

全局收敛: $\forall x \in [a, b], a \leq \psi(x) \leq b$ & $|\psi'(x)| < 1$

局部收敛: $|\psi'(x^*)| < 1$

PPR收敛: $\psi' = \dots = \psi^{(P-1)}(x^*) = 0, \psi^{(P)}(x^*) \neq 0$

牛顿: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$, 一般 $P=2$

弦截: $x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$

割线: $x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$

阻尼 $x_{k+1} = x_k - \lambda \frac{f(x_k)}{f'(x_k)}$

三. 1-范数 $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$

2-范数 $\|A\|_2 = \sqrt{\lambda_{\max}(ATA)}$

无穷范数 $\|A\|_{\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$

范数条件: $\|x\| \geq 0, x=0 \Leftrightarrow \|x\|=0$
 $\|kx\| = |k| \|x\|$
 $\|x+y\| \leq \|x\| + \|y\|$

矩阵: $\|AB\| \leq \|A\| \|B\|$

矩阵条件数 $cond = \|A\| \|A^{-1}\|$

$L = \begin{bmatrix} * & & \\ * & * & \\ * & * & * \end{bmatrix}$ $U = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$

对称正定矩阵不一定要奇

存在LU分解: 各阶顺序主子式 A_k 秩 $= A_k$ 前 k 列秩

LU分解唯一: A 各阶顺序主子式 > 0
 $A = LU = LDU = LDL^T$ if A, A^T
Cholesky: $A = LD^2 D^T, L^T = LL^T$ if A 对称
 A^{mn} 右列秩 $\geq$ 左列秩 $\Rightarrow$ LU分解存在

四. $x = Bx + f$ 收敛 $\Leftrightarrow$

① $I-B$ 非奇 ② $\rho(B) < 1$

$\rho(A) = \max(|\lambda_i|)$. 若 $AT=A \Rightarrow \rho(A) = \|A\|_2$

一般: $A = D - L - U = M - N$

$Ax = b \Rightarrow x = M^{-1}Nx + M^{-1}b$

Jac: $M = D, N = L+U, x^{p+1} = D^{-1}(L+U)x^p + D^{-1}b$

GS: $M = D-L, N = U, x^{p+1} = (D-L)^{-1}Ux^p + (D-L)^{-1}b$

SOR: $M = \omega D - L, N = (\omega - 1)D + U$

$x^{p+1} = (\omega D - L)^{-1}[(\omega - 1)D + U]x^p + (\omega D - L)^{-1}b$

收敛条件:

Jac: A 对称 D 的正定或④

GS: $\|B\|_{\infty} < 1$ 或 $\|B\|_1 < 1$ 或 A 对称正定或④

SOR: $0 < \omega < 1$ 或对称正定 $0 < \omega < 2$ ④

④: A 严格对角占优/不可约弱对角占优

五. $cond(A)_2 = \frac{\lambda_{\max}(ATA)}{\lambda_{\min}(ATA)}$

图: $|\lambda - a_k| \leq \sum_{k \neq j} |a_{kj}|$

范法: $v = A \cdot norm(Av)$, 最大 $\lambda = \max(v)$

反范法: $v = norm(A^{-1}v)$, 最小 $\lambda = \max(v)$

Householder, Givens $G = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} C = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$
 $S = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$

六. 无穷范数 $\|p(x) - f(x)\|_{\infty} = \max_{x \in [a, b]} |p(x) - f(x)|$

1-范数 $\|p(x) - f(x)\|_1 = \int_a^b |p(x) - f(x)| dx$

2-范数 $\|p(x) - f(x)\|_2 = \left(\int_a^b (p(x) - f(x))^2 dx \right)^{1/2}$

法方程: $\sum_{j=1}^n x_j \langle \phi_j, \phi_k \rangle = \langle f, \phi_k \rangle (k=1 \sim n)$

$S^T x^* \phi_1 + x_2^* \phi_2 + \dots + x_n^* \phi_n \leq S^T f, S^T \geq 0$
 $\langle S^T, \phi_k \rangle = \langle f, \phi_k \rangle \Rightarrow \langle S^T f, \phi_k \rangle = 0$

$\forall S \in \Phi, \|S - f\|_2 \leq \|S^* - f\|_2 \Rightarrow 0$

最佳平方逼近多项式 $[0, 1]$ 区间

$\begin{bmatrix} 1 & \frac{1}{2} & \dots & \frac{1}{n} \\ 1 & \frac{1}{3} & \dots & \frac{1}{n^2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1}{n} & \dots & \frac{1}{n^n} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \int_0^1 f(x) dx \\ \int_0^1 x f(x) dx \\ \vdots \\ \int_0^1 x^{n-1} f(x) dx \end{bmatrix}$

$S^T(x) = x_0 + x_1 x + \dots + x_n x^n$
误差 $\|S^T(x) - f(x)\|_2 = \left(\int_0^1 (f(x) - S^T(x))^2 dx \right)^{1/2}$

勒让德多项式 $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1)$

$\langle P_j(x), P_k(x) \rangle = [j=k] \cdot \frac{2}{2k+1}$

$(k+1)P_{k+1}(x) = (2k+1)xP_k(x) - kP_{k-1}(x)$

区间 $[-1, 1]$ $P_k(x) = P_k\left(\frac{x-a-b}{b-a}\right)$ $A_2^2[1]$

最小二乘 $Ax \approx f \Rightarrow A^T Ax = A^T f$ $m \geq n$

QR: $f - Ax \approx Q(Q^T f - Rx)$ $Q: m \times n$
 $Q = [Q_1 Q_2]$ $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} n \times n$ $R: m \times n$

$R_1 x = Q_1^T f$ 解 x

拉格朗日: $L_n(x) = f(x) - L_n(x) \frac{f(x)}{(x-x_0) \dots (x-x_n)}$

牛顿误差 $R_n(x) = f(x, x_0, \dots, x_n) - L_n(x)$

$L_n(x) = (x-x_0)(x-x_1) \dots (x-x_n)$

分段线性 $\frac{M_2}{2} h^2, M_2 = \max |f''(x)|, h = \max(x_i - x_{i-1})$

埃尔米特 $\frac{f^{(2m+2)}(\xi)}{(2m+2)!} \omega_{m+1}^2(x)$

保尔分片: $\gamma = 0 \Rightarrow \frac{w_{k+1} w_k}{f_k} = \frac{w_{k+1}}{f_k} + \frac{w_k}{f_k}$

七. 代数精度: $\leq m \vee m+1$ $x \Rightarrow m+1$

NC $\frac{1}{2} \frac{1}{2}$ 余项 $-\frac{f^{(n)}(b-a)}{n!} (b-a)^n$

辛普森 $\frac{1}{8} \frac{1}{8} \frac{1}{8} \frac{1}{8}$ $-\frac{f^{(4)}(\xi)}{80} (b-a)^5$

$n=4$ $\frac{7}{80} \frac{16}{80} \frac{2}{80} \frac{16}{80} \frac{7}{80}$ $Q=5 + \frac{5}{15}$

中矩形 余项 $-\frac{f''(\xi)(b-a)^3}{12}$ $\uparrow$

复合 Simpson: $-\frac{h^4(b-a)}{2880} f^{(4)}(\eta) = O(h^4)$

高斯求积: $\frac{f^{(2m+2)}(\eta)}{(2m+2)!} \int_a^b \omega_{m+1}^2(x) p(x) dx$

高斯-勒让德 ± 1 $2m+1$ 次精度

n x_k A_k

0 0 2

1 ± 0.5773503 5/9

2 ± 0.7745967 8/9

3 ± 0.8812639 0.6708578

± 0.3399810 0.6521452

八. Lipschitz 范数 $\|f(t, y) - f(t, g)\| \leq L \|y - g\|, L > 0$

欧拉法 $y_{n+1} = y_n + h f(t_n, y_n)$

$y'(t_n) = f(t_n, y(t_n))$ 误差 $\left| \frac{dy}{dt}(t_n, y_n) \right| \leq 1$

改进欧拉法 $y_{n+1} = y_n + h f(t_n, y_n)$

$y_{n+1} = y_n + h \lambda y_n \Rightarrow |1 - h \lambda| \geq 1$ 误差

梯形法 $y_{n+1} = y_n + \frac{1}{2} h [f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$

$R_k \left\{ \begin{aligned} y_{n+1} &= y_n + h \sum_{i=1}^k c_i k_i \\ k_1 &= f(t_n, y_n) \\ k_2 &= f(t_n + \lambda_2 h, y_n + \lambda_2 h k_1) \\ k_i &= f(t_n + \lambda_i h, y_n + \lambda_i h \sum_{j=1}^i k_j) \end{aligned} \right. (i=3 \sim m)$

Cholesky

for j=1~n
 for k=1~j-1
 $a_{ij} = a_{ij} - a_{jk} \bar{a}_{ik}$
 $a_{ij} = \sqrt{a_{ij}}$ ($\bar{a}_{ij} = \sqrt{a_{ij} - \sum_{k=1}^{j-1} a_{jk} \bar{a}_{ik}}$)
 for i=j+1~n
 for k=1~j-1
 $a_{ij} = a_{ij} - a_{ik} \bar{a}_{jk}$
 $a_{ij} = a_{ij}$ ($\bar{a}_{ij} = \frac{1}{a_{jj}} (a_{ij} - \sum_{k=1}^{j-1} a_{jk} \bar{a}_{ik})$)

Householder

for k=1~n
 $\sigma_k = \text{sign}(a_{kk}) \sqrt{\sum_{j=k}^n a_{kj}^2}$ (选轴技巧 $=0$)
 if $\sigma_k = a_{kk}$ then continue (第k列)
 $v_k \in [0, 0, \dots, 0, a_{kk}, \dots, a_{nk}]^T + \sigma_k e_k$
 $p_k = v_k^T v_k$
 for j=k~n
 $v_j = v_j \bar{a}_{kj}$
 $a_{ij} = a_{ij} - (2 \bar{\sigma}_j / p_k) v_k$

$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$, $\sigma_1 = \|\begin{bmatrix} 1 \\ -1 \end{bmatrix}\|_2 = \sqrt{2}$
 $v_1 = a_1 + \sigma_1 e_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1+\sqrt{2} \\ -1 \end{bmatrix}$
 $w_1 = \frac{v_1}{\|v_1\|_2}$, $H_1 = I - 2w_1 w_1^T$
 $H_1 A = A - 2 \frac{v_1 v_1^T}{v_1^T v_1}$ (每列)
 $\sigma_2 = \|\begin{bmatrix} 0 \\ 0.7887 \\ -0.2113 \end{bmatrix}\|_2 = 1.6330$
 $v_2 = \begin{bmatrix} 0 \\ 0.7887 \\ -0.2113 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.6330 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2.4217 \\ -0.2113 \end{bmatrix}$
 $H_2 H_1 A = \begin{bmatrix} -1 & -\sqrt{2} & 0.5770 & 0.5770 \\ 0.5770 & -1.6330 & 0.8165 & 0.8165 \\ 0.6332 & 0.7231 & 0.6899 & 0.6899 \end{bmatrix}$
 $\sigma_3 = \|\begin{bmatrix} 0.6332 \\ 0.7231 \\ 0.6899 \end{bmatrix}\|_2 = 1.4142$
 $H_3 H_2 H_1 A = \begin{bmatrix} -1.7321 & 0.5770 & 0.5770 \\ -1.6330 & 0.8165 & 0.8165 \\ -1.4142 & 0.6899 & 0.6899 \end{bmatrix}$

LU分解: $A \rightarrow 2D - A^2 A$

充分: $D = D^+ D^{\frac{1}{2}}$, $B = D^{-1}(L+U) = D^{-1}(D-A) = D^{-\frac{1}{2}}(D^{\frac{1}{2}} - D^{-\frac{1}{2}} A) = D^{\frac{1}{2}}(I - D^{-\frac{1}{2}} A D^{\frac{1}{2}}) D^{\frac{1}{2}}$
 说明 $I - D^{-\frac{1}{2}} A D^{\frac{1}{2}}$ 与 B 相似, 有相同 λ . A 实对称正定, $D^{\frac{1}{2}} A D^{\frac{1}{2}}$ 也是, $\lambda > 0$
 $\therefore \lambda(I - \dots) < 1 \Rightarrow B$ 的 $\lambda_i < 1$.
 $B = D^{-1}(D-A) = D^{-1}[(2D-A)-D] = D^{-\frac{1}{2}}[D^{\frac{1}{2}}(2D-A)D^{\frac{1}{2}} - I] D^{\frac{1}{2}}$ $\therefore B$ 的 $\lambda_i > -1$.
 $\therefore \rho(B) < 1$.

A 对称正定占优, GS 收敛

$B = (D-L)^{-1}U$, $\det(\lambda I - B) = \det((D-L)^{-1}) \det(\lambda(D-L) - U) = 0$
 $\therefore A$ 对称正定 $\therefore \lambda_i > 0$
 设 $C = \lambda(D-L)U = \begin{bmatrix} \lambda a & a \\ \lambda c & \lambda a \\ \lambda c & \lambda c \end{bmatrix}$ $\forall \lambda, \det(C) = 0$
 假设 GS 不收敛, 至少有 1 个 $|\lambda_i| \geq 1$. 若 A 严格对角占优, 则 C 中 $|\lambda a_i| > |\lambda| (\sum_{j=1, j \neq i}^n |a_{ij}|) \geq \sum_{j=1, j \neq i}^n |\lambda a_{ij}| + \sum_{j=i+1}^n |a_{ij}|$. C 也是 $\uparrow$, 对称正定 $\Rightarrow \det C \neq 0$!!
 (若 A 严格对角占优, 或至少给弱对角占优, 则 A 非奇异)

若SOR收敛则 $0 < w < 2$

$B = (D - wL)^{-1}[(1-w)D + wU]$, $|\det(B)| = |\lambda_1 \lambda_2 \dots \lambda_n| \leq [\rho(B)]^n$
 $\therefore \rho(B) < 1 \therefore |\det(B)| < 1$.
 $D - wL$ 下三角, $(1-w)D + wU$ 上三角. 行列式 = 对角元素 $\prod$.
 $\det(B) = \det((D - wL)^{-1}) \det((1-w)D + wU) = (1-w)^n$
 $\Rightarrow |(1-w)^n| < 1$, $0 < w < 2$.

数据截断舍入

稳定: 递推 Gauss, Cholesky, QR, 3次样条, 高斯积分
 不稳定: 追赶, 拉格朗日, 平方多项式法方程, 高阶 Newton-Cotes.
 递推: $T_{k+1}^{(n)} = \frac{4^k T_k^{(n+1)} - T_R^{(n)}}{4^k - 1}$
 $T_1^{(n)} = \frac{4 T_0^{(n+1)} - T_0^{(n)}}{3}$, $T_2^{(n)} = \frac{16 T_1^{(n+1)} - T_1^{(n)}}{15}$
 泰勒 $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \dots + \frac{h^n}{n!} f^{(n)}(x)$
 复合梯形 $T(h) = I(f) + a_1 h^2 + a_2 h^4 + \dots + a_n h^{2l}$

T_0^0				
T_0^1	T_1^0			
T_0^2	T_1^1	T_2^0		
T_0^3	T_1^2	T_2^1	T_3^0	
T_0^4	T_1^3	T_2^2	T_3^1	T_4^0

复合梯形

2阶 RK: $y_{n+1} = y_n + \frac{h}{6} (f(t_n, y_n) + 4f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1) + f(t_n + h, y_n + h k_1 + \frac{1}{2} h k_2))$
 Kutta
 Ralston: $y_{n+1} = y_n + \frac{h}{9} (2f(t_n, y_n) + 3f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1) + 4f(t_n + \frac{3h}{4}, y_n + \frac{3h}{4} k_1))$
 4阶经典: $y_{n+1} = y_n + \frac{h}{6} (f(t_n, y_n) + 2f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1) + 2f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_2) + f(t_n + h, y_n + h k_1))$