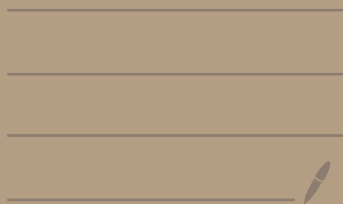


复变函数

2023 秋

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by ubec



$$(x, y) \xrightarrow{\text{复数}} x+iy = z$$

$$\mathbb{C} = \{z \mid z = x+iy\}, \mathbb{C} \cong \mathbb{R}^2$$

1° 表示

$$\begin{aligned} z &= x+iy \\ e^{i\theta} &= \cos\theta + i\sin\theta, r = \sqrt{x^2+y^2} \\ z &= r e^{i\theta} \end{aligned}$$

$$\text{Arg} z = \arg z + 2k\pi, k \in \mathbb{Z}$$

↪ 辐角主值, $\in (-\pi, \pi]$

$$\text{极坐标公式 } (r e^{i\theta})^n = r^n e^{in\theta}$$

$$\bar{z} = \overline{x+iy} = x-iy$$

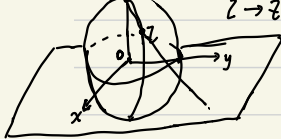
$$\text{Arg}(z_1 z_2) = \text{Arg} z_1 + \text{Arg} z_2$$

2° 扩充复平面 $\bar{\mathbb{C}}$

$$\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

对极投影 P

$$S^1 \rightarrow \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \quad \begin{matrix} N \rightarrow \infty \\ Z \rightarrow z \end{matrix}$$



有界闭域中 $f(z)$ 有界
(模的意义)

2. 区域和曲线

1° 区域: 连通开集

Jordan 区域

$$2^\circ \text{ 曲线 } C \subseteq \mathbb{R}^2 \quad \begin{cases} x = x(t) \\ y = y(t) \end{cases} t \in [a, b]$$

$$C \subseteq \mathbb{C}: z = x(t) + iy(t), t \in [a, b]$$

$$(1) \text{ 连续曲线: } x(t), y(t) \in C[a, b]$$

$$(2) \text{ 光滑曲线: } x(t), y(t) \in C^1[a, b], \text{ 且}$$

$$|z'(t)| = |x'(t) + iy'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2} \neq 0, \forall t \in [a, b]$$

3° 分段光滑曲线

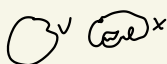
有限段光滑曲线首尾连接可得

$$\begin{cases} x(t) = t^2 \\ y(t) = t^3 \end{cases} t \in \mathbb{R}$$

4° 简单曲线

没有自交点

(端点)



1° 定义

Def: 设 $G \subseteq \mathbb{C}$, 给从 G 到 $\mathbb{C}$ 的一个对应规则 f

s.t. 对 $\forall z = x+iy \in G$, 都有一个或多个 $w = u+iv \in \mathbb{C}$ 与之对应
则称 w 是关于 z 的复变函数, 记作 $w = f(z)$ (复变函数)

$$f: G \rightarrow \mathbb{C}$$

(1) $\forall z \in G$, 有唯一 w 与之对应, 称为单值函数

(2) $\exists z \in G$, s.t. 至少有两个 w 与之对应, 称为多值函数

$$\begin{aligned} \text{Ex: (1) } w &= f(z) = \text{Arg} z, z \neq 0 \\ &= \arg z + 2k\pi, k \in \mathbb{Z} \end{aligned}$$

辐角函数 (无穷多值, 由于 k)

$$k=0, w_0 = \arg z = f_0(z) \text{ 主辐角函数}$$

$$\arg z \in (-\pi, \pi]$$

$$(2) w = f(z) = z^k = \sqrt[k]{z} = \begin{cases} 0, & z=0 \\ |z|^{1/k} e^{i \frac{\arg z + 2k\pi}{k}}, & z \neq 0 \end{cases}$$

$$\text{反函数 } f^{-1}: G^* \rightarrow G \quad G^* = \{w = f(z) \mid z \in G\}$$

Rem. 若 f, f^{-1} 均是单值的, 则 f 是一一映射

$$w = f(z) = z^k \Leftrightarrow z = f^{-1}(w) = w^{1/k}$$

2° 函数极限 无特殊说明均为单值函数

$$\text{Def: 设 } w = f(z) \text{ 在 } z_0 \text{ 的邻域 } B_r^*(z_0) = \{z \mid 0 < |z - z_0| < r\}$$

内有意义, 若 $\exists A \in \mathbb{C}$, $\forall \epsilon > 0$, 都 $\exists \delta = \delta(\epsilon) > 0$, 当 $0 < |z - z_0| < \delta$ 时, 有 $|f(z) - A| < \epsilon$
则称当 $z \rightarrow z_0$ 时, $f(z)$ 以 A 为极限, 记作 $\lim_{z \rightarrow z_0} f(z) = A$ or $f(z) \rightarrow A, z \rightarrow z_0$

复数是无序集, 不可随便比较大小

$$\text{THM: } \lim_{z \rightarrow z_0} f(z) = A = \alpha + i\beta \in \mathbb{C} \Leftrightarrow \begin{cases} \lim_{x \rightarrow x_0} u(x, y) = \alpha \\ \lim_{x \rightarrow x_0} v(x, y) = \beta \end{cases}$$

判定是否有极限

3° 函数连续性

Def: 若 $\lim_{z \rightarrow z_0} f(z) = f(z_0)$, 称 $f(z)$ 在 z_0 连续

若对 $\forall z \in D$, $f(z)$ 在 z 连续, 称 f 是 D 上连续函数

$$\text{Ex (1) } w = f(z) = \frac{z}{z} = \frac{x-iy}{x+iy} = u(x, y) + iv(x, y) \text{ 无极限}$$

$$(2) w = \begin{cases} 0 & z=0 \\ \frac{\text{Re } z \cdot \text{Im } z}{|z|^2} & z \neq 0 \end{cases} \quad \text{连续是可微的必要条件}$$

$$f(z_0) \text{ 连续} \Leftrightarrow u(z_0), v(z_0) \text{ 皆连续}$$

Chap 2 解析函数

§1. 导数和解析函数

1° 导数 Def: 设 $w=f(z)$ 在 D 上有定义, $z_0 \in D$.

若 $\lim_{\Delta z \rightarrow 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = A \in \mathbb{C}$ ($\Delta z = \Delta x + i\Delta y$)

称 $f(z)$ 在 z_0 可导, A 为 $f(z)$ 在 z_0 导数, 记作 $f'(z_0)$ 或 $\frac{dw}{dz} \Big|_{z=z_0}$

2° 可微与微分

Def: 设 $w=f(z)$ 在 $z_0 = x_0 + iy_0$ 可导是指存在某个邻域内有表达式

$$f(z_0+\Delta z) - f(z_0) = A\Delta z + \rho(\Delta z)\Delta z$$

其中 $A \in \mathbb{C}$, $\lim_{\Delta z \rightarrow 0} \rho(\Delta z) = 0$

$$w=f(z) = z\bar{z} = |z|^2 = x^2 + y^2$$

$$\frac{\Delta f}{\Delta z} = \frac{(x_0+\Delta x)(x_0+\Delta x + i(y_0+\Delta y)) - x_0^2 - y_0^2}{\Delta x + i\Delta y}$$

$$= \bar{z}_0 + \bar{\Delta z} + \frac{\Delta \bar{z}}{\Delta z} \Delta z \rightarrow \begin{cases} D, \Delta z \rightarrow 0 \\ x, \Delta z \rightarrow 0 \end{cases}$$

用归纳

3° 解析函数 解析不必满足 C_1 一个函数解析 其 D 内解析

Def: $U \subset \mathbb{C}$, 若 $w=f(z)$ 在某邻域内处处可导

则称 $f(z)$ 在 U 解析, 称 U 为解析域

否则在 z_0 不解析, 称 z_0 为奇点 不解析的点

奇点分类: (1) 无定义 (2) 不连续 (3) 不可导 (4) 可导但不解析

Ex: (1) $f(z) = a_1 z^2 + a_2 + a_0$ 在 $\mathbb{C}$ 上解析

Def: 在整 $\mathbb{C}$ 上解析的函数称为整函数 绝大多数整函数由级数构造

(2) $\frac{f(z)}{g(z)}$ 在 $\mathbb{C} \setminus \{z \mid g(z)=0\}$ 上解析

§2 函数可导与解析的充要条件

首先设 $w=f(z)=u+iv$ 在 $z=x+iy$ 可导 ($\Leftrightarrow$ 可微)

$$\Rightarrow \Delta f = \Delta u + i\Delta v = f(z_0+\Delta z) - f(z_0) = f'(z)\Delta z + \rho(\Delta z)\Delta z$$

$$f'(z) = \alpha + i\beta$$

$$\Rightarrow \Delta f = (\alpha + i\beta)(\Delta x + i\Delta y) + \rho(\Delta z)(\Delta x + i\Delta y)$$

$$\Rightarrow \begin{cases} \Delta u = \alpha \Delta x - \beta \Delta y + \rho_1 \Delta x - \rho_2 \Delta y \\ \Delta v = \beta \Delta x + \alpha \Delta y + \rho_2 \Delta x + \rho_1 \Delta y \end{cases}$$

$\Rightarrow$ I. u, v 在 (x, y) 可微

$$\text{II. } \begin{cases} u_x = \alpha & u_y = \beta \\ v_x = \beta & v_y = \alpha \end{cases} \Rightarrow \begin{cases} u_x = v_y \\ v_x = -u_y \end{cases} \quad (1)$$

Cauchy-Riemann

$$\text{III } f'(z) = \alpha + i\beta = u_x + iv_x = u_x - iv_y = \dots \quad (2)$$

$$\text{反方向} \Leftrightarrow \begin{cases} \Delta u = u_x \Delta x + u_y \Delta y + o(\sqrt{\Delta x^2 + \Delta y^2}) \\ \Delta v = v_x \Delta x + v_y \Delta y + o(\sqrt{\Delta x^2 + \Delta y^2}) \end{cases} \text{ @ } \mathbb{R}$$

$$\begin{aligned} \Delta f &= \Delta u + i\Delta v = u_x \Delta x + u_y \Delta y + o(|\Delta z|) + i(v_x \Delta x + v_y \Delta y + o(|\Delta z|)) \\ &= (u_x + iv_x)\Delta x + (u_y + iv_y)\Delta y + o(|\Delta z|) \\ &= (u_x + iv_x)(\Delta x + i\Delta y) + \rho(\Delta z)\Delta z, \text{ 此即可微形式 (A)} \end{aligned}$$

THM: $w=f(z)=u+iv$ 在 $z=x+iy$ 可导 $\Leftrightarrow$

u, v 在 (x, y) 可微且满足 CR

THM $w=f(z)=u+iv$ 在 D 上解析 $\Leftrightarrow$

u, v 在 D 上可微且满足 CR

推论: 设 $u, v \in C^1(U)$ 且在 D 上满足 CR

则 $w=f(z)=u+iv$ 在 D 上解析

反过来成立吗?

$$\text{Ex: (1) } f(z) \equiv 0 \Leftrightarrow f(z) = \text{const}$$

$$\Rightarrow f(z) = u_x + iv_x = u_y - iv_y = 0$$

$$\Rightarrow u_x = u_y = v_x = v_y = 0 \Rightarrow u, v = \text{const}$$

$$(2) f(z) = z = x+iy$$

$$(3) f(z) = \bar{z} = x-iy$$

$$(4) w=f(z) \equiv e^x(\cos y + i\sin y) = e^x e^{iy} = e^z$$

$$u = e^x \cos y \quad v = e^x \sin y$$

[补] 形式导数 $u=u(x, y)$

$$\bar{z} = x-iy \quad \bar{z} = x-iy$$

$\bar{z}$ 是形式变量

$$\Rightarrow \begin{cases} x = \frac{z+\bar{z}}{2} \\ y = \frac{z-\bar{z}}{2i} \end{cases} \Rightarrow \begin{cases} x_0 = x_0 = \frac{1}{2} \\ y_0 = -y_0 = \frac{1}{2i} \end{cases} \quad \begin{vmatrix} x_0 & x_0 \\ y_0 & y_0 \end{vmatrix} \neq 0$$

$$u_x = u_x x + u_y y = \frac{1}{2}u_x + \frac{1}{2i}u_y$$

$$u_y = u_x x + u_y y = \frac{1}{2}u_x - \frac{1}{2i}u_y$$

V 同理

$$\text{eg. } w=f(z)=u(x, y)+iv(x, y)$$

$$\begin{aligned} f_z &= \frac{\partial f}{\partial z} = u_x + iv_x = \frac{1}{2}u_x - \frac{1}{2i}u_y + \frac{1}{2}(u_x - iv_y) \\ &= \frac{1}{2}(u_x + v_y) + \frac{1}{2i}(u_x - u_y) \end{aligned}$$

$$\text{Claim: } f_z = 0 \Leftrightarrow \text{CR} \Rightarrow f_z = \frac{\partial f}{\partial z} = u_x + iv_x$$

$$\text{Ex (1) } f(z) = \bar{z}, f_z = 0$$

$$(2) f(z) = \bar{z}, f_z = 1$$

$$(3) f(z) = \bar{z}, f_z = \bar{z}$$

$$(4) |z|^2 = z\bar{z}, f_z = \bar{z}$$

是否可解析

§3 初等函数

1° 指数函数 π^e 怎么定义

找到 $w = f(z)$ s.t. $(1) f(z)$ 是整函数

$$(2) f(z) = f'(z)$$

$$(3) \operatorname{Im} z = y = 0, f(z) = e^x$$

存在性 $w = f(z) = e^x (\cos y + i \sin y)$

$$= e^x \cdot e^{iy} = e^{x+iy}$$

$$f'(z) = u_x - i u_y = f(z) = u + i v$$

$$\text{唯一性} \Rightarrow \begin{cases} u_x = u_y = u \\ u_y = -u_x = -v \end{cases} \Rightarrow \text{方程组只有唯一解}$$

$$u(x, 0) = e^x, v(x, 0) = 0$$

Def. $\exp z = e^x e^{iy}$ 展指数函数

prop: (1) $\exp z$ 是整函数且 $\exp' z = \exp z$

$$(2) \exp(b_1 + i a_1) = \exp b_1 \cdot \exp i a_1$$

$$(3) \exp(z + T) = \exp z \quad T = 2k\pi i, k \in \mathbb{Z}, k \neq 0$$

$$(4) \exp z \neq 0, \forall z \in \mathbb{C}$$

$$(5) \sum_{n=0}^{\infty} \exp z \text{ 不存在}$$

方便起见, 以 e^{iy} 代表 $\exp z$ ($e^{iy} = \cos y + i \sin y$)

2° 对数函数: 定义为指数函数的反函数

即满足方程 $e^w = z (z \neq 0)$ 的函数 $w = f(z) = u + i v$

$$\Rightarrow e^{u+iv} = z = |z| e^{i \arg z}$$

$$\Rightarrow \begin{cases} e^u = |z| \Rightarrow u = \ln |z| \\ v = \operatorname{Arg} z \end{cases}$$

$$\Rightarrow \operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z = \ln |z| + i \arg z + 2\pi i k$$

$$k=0, w_0 = \ln |z| + i \arg z \in \mathbb{C}$$

prop: (1) $\operatorname{Ln} z$ 在 $\mathbb{C} \setminus (-\infty, 0]$ 上解析
 $\frac{d(\operatorname{Ln} z)}{dz} = \frac{d \ln |z|}{dz} + i \frac{d \arg z}{dz} = \frac{1}{z} \frac{dz}{dz} = \frac{1}{z}$

$$(2) \operatorname{Ln}(z_1 z_2) = \operatorname{Ln} z_1 + \operatorname{Ln} z_2 \quad (X) \quad \arg z_1 + \arg z_2 \text{ 超出 } (-\pi, \pi]$$

$$\text{修正 } \operatorname{Ln}(z_1 z_2) = \operatorname{Ln} z_1 + \operatorname{Ln} z_2 + \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

$$(3) \operatorname{Ln} z^n = n \operatorname{Ln} z \quad (n \in \mathbb{Z}) \quad X$$

$$\operatorname{Ln} z^n = n \operatorname{Ln} z \quad (n \in \mathbb{Z}) \quad X \quad \text{右侧 } 2k\pi i \rightarrow 2kn\pi i$$

$$\operatorname{Ln} z^n = \operatorname{Ln} z + \dots + \operatorname{Ln} z$$

$$(4) \sqrt[n]{z} = \text{有 } n \text{ 个多值} \sim \text{单值}(X)$$

$$\operatorname{Ln} \sqrt[n]{z} = \frac{1}{n} \operatorname{Ln} z \quad \checkmark, \text{书上错了! } \sqrt[n]{z} \text{ 本身多值, 多值对多值!}$$

$$\text{Ex. } \operatorname{Ln}(-1) = \ln|-1| + i \operatorname{Arg}(-1) = i(2k+1)\pi, k \in \mathbb{Z}, \ln(-1) = i\pi \Leftrightarrow e^{i\pi} = -1$$

$$\Leftrightarrow e^{i\pi} + 1 = 0$$

3° 幂函数

$$\text{Def: } (1) z^b \triangleq e^{b \operatorname{Ln} z}$$

$$= e^{b \ln |z| + 2b k \pi i}, k \in \mathbb{Z}$$

$$= e^{b \ln |z|} \cdot e^{2b k \pi i} \quad \text{考虑多值性}$$

Case 1 $b \in \mathbb{R} \quad (1) b \in \mathbb{Q}$

$$(2) b \in \mathbb{Z} \quad e^{2b k \pi i} = 1 \quad \text{— 单值}$$

$$(3) b \in \mathbb{Q} \setminus \mathbb{Z} \quad b = \frac{p}{q}, n \geq 2, (n, m) = 1$$

$$e^{2b k \pi i} = e^{2 \frac{p}{q} k \pi i} \quad n \text{ 值}$$

$$(4) b \in \mathbb{R} \setminus \mathbb{Q}, \text{ 即无理数}$$

$$e^{2b k \pi i} = e^{2k \pi i (b k)} \quad \text{无规律}$$

Case 2 $b \in \mathbb{C} \setminus \mathbb{R} \quad b = a + i \beta (\beta \neq 0)$

$$e^{2(a + i \beta) k \pi i} = e^{-2\beta k \pi} \cdot e^{2a k \pi i} \quad \text{无规律}$$

$$\text{Ex. } i^i = e^{i \operatorname{Ln} i} = e^{i(\ln |i| + i \arg i)} = e^{-\ln |i|} = e^{-\ln 1} = 1$$

$$e^{i \pi} = \cos \pi + i \sin \pi = -1$$

$$e^{i \pi} = \cos \pi + i \sin \pi = -1$$

4° 三角函数

$$\text{Def } \begin{cases} \cos z = \frac{e^{iz} + e^{-iz}}{2} \\ \sin z = \frac{e^{iz} - e^{-iz}}{2i} \end{cases} \quad z \in \mathbb{C} \quad \tan z = \frac{\sin z}{\cos z} \quad (T = \pi)$$

prop: (1) $\sin z, \cos z$ 整函数

$$\sin' z = \cos z$$

$$(2) \sin(-z) = -\sin z \quad \cos(-z) = \cos z$$

$$(3) \sin(z + T) = \sin z \quad \cos(z + T) = \cos z \quad T = 2k\pi$$

$$(4) \sin z \text{ 周期 } 2\pi, \cos z \text{ 也是, 可取任意实数}$$

$$(5) \sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2 \quad \checkmark$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \quad \checkmark$$

$$(6) e^{iz} = \cos z + i \sin z \quad \text{Euler 公式}$$

5.3.2 双曲函数

$$\text{Def } \begin{cases} \cosh z = \frac{e^z + e^{-z}}{2} \\ \sinh z = \frac{e^z - e^{-z}}{2} \end{cases}$$

$$\tanh z = \frac{\sinh z}{\cosh z}$$

Prop: (1) $\cosh z, \sinh z$ 是整函数且 $\cosh' z = \sinh z, \sinh' z = \cosh z$

$$(2) \cosh(-z) = \cosh z, \sinh(-z) = -\sinh z$$

$$(3) \cosh(z+T) = \cosh z, \sinh(z+T) = \sinh z, T = 2k\pi i$$

$$(4) \cosh, \sinh \text{ 无解}$$

$$(5) \sinh(z \pm w) = \sinh z \cosh w \pm \cosh z \sinh w$$

$$\cosh(z \pm w) = \cosh z \cosh w \pm \sinh z \sinh w$$

$$(6) \text{ Yao Formula } e^z = \cosh z + \sinh z$$

Chap 3 复变函数积分学

§1 积分

1° 积分定义

Def: 有向曲线: 右半平面是子弧的并 (单连通) 曲线



Def: 设 $w=f(z)$ 在 D 上有意义, $C \subset D$ 为一条有向曲线

将 C 分为 n 个弧段, 长度为 $z_k = A, z_1, \dots, z_n = B$

每个弧段 $z_{k-1} z_k$ ($k=1, \dots, n$) 上取点 $\zeta_k = \xi_k + i\eta_k$ 和 $\bar{\zeta}_k$

$$I_n = \sum_{k=1}^n f(\zeta_k) \Delta z_k = \sum_{k=1}^n f(\xi_k + i\eta_k) (\alpha_k + i\beta_k)$$

令 Δz_k 乘 Δz_k 乘积 $S_n = \sum_{k=1}^n \Delta z_k$

当 $n \rightarrow \infty$ 且 $S_n \rightarrow 0$ 时, 若极限存在且 S_n 乘积为零

上有唯一极限 $I = \lim_{n \rightarrow \infty} I_n$ 称为 $f(z)$ 沿 C 的积分

若 C 为封闭曲线, 则 $I = \oint_C f(z) dz$

2° 积分存在条件及算法

设 $w=f(z) = u+iv$, 沿 C 连接

$$I_n = \sum_{k=1}^n f(\zeta_k) \Delta z_k = \sum_{k=1}^n [u(\xi_k, \eta_k) + i v(\xi_k, \eta_k)] (\alpha_k + i \beta_k)$$

$$= \sum_{k=1}^n [u(\xi_k, \eta_k) \alpha_k - v(\xi_k, \eta_k) \beta_k] + i \sum_{k=1}^n [v(\xi_k, \eta_k) \alpha_k + u(\xi_k, \eta_k) \beta_k]$$

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n u \alpha_k - v \beta_k + i \lim_{n \rightarrow \infty} \sum_{k=1}^n v \alpha_k + u \beta_k$$

$$= \int_C (u+iv) (dx+idy)$$

$$C: z = a+ib, x=a, y=b, z \in [a, b]$$

$$dz = x' dx + y' dy$$

$$I = \int_a^b f(x+iy) (x' dx + y' dy)$$

$$= \int_a^b f(x+iy) dx dy$$

设 $C = C_1 + \dots + C_n$ 为有向光滑曲线

$$I = \int_C f(z) dz = \sum_{k=1}^n \int_{C_k} f(z) dz$$

性质: 以 C 为定向的闭曲线

$$(1) \int_C f(z) dz = - \int_{-C} f(z) dz$$

$$(2) \int_C k f(z) dz = k \int_C f(z) dz \quad k \in \mathbb{C}$$

$$(3) \int_C [f(z) + g(z)] dz = \int_C f(z) dz + \int_C g(z) dz$$

$$(4) \int_C f(z) dz = \int_C f(z) dz = \int_C (u+iv) (dx+idy) = \int_C u dx - v dy + i \int_C v dx + u dy$$

(5) 若 $|f(z)| \leq M < +\infty$ ($\forall z \in C$) C 弧长为 L ,

$$\text{则 } |\int_C f(z) dz| \leq ML$$

$$\leq \int_C |f(z)| |dz| \leq \int_C M |dz| = ML$$

三角不等式

$$\text{Ex: } \int_C |dz| = |z_1 - z_2|$$

$$\int_C |dz| = C \text{ 弧长}$$

$$(2) I_n = \int_C \frac{dz}{(z-a)^n}, n \in \mathbb{Z}, |z-a|=R, R>0$$

$$f(z) = \frac{1}{(z-a)^n}$$

$$\text{令 } z-a = Re^{i\theta}, \theta \in [0, 2\pi)$$

$$dz = iRe^{i\theta} d\theta$$

$$I_n = \int_0^{2\pi} iR e^{in\theta} \frac{1}{(Re^{i\theta})^n} d\theta$$

$$= iR^n \int_0^{2\pi} e^{i(n-n)\theta} d\theta \quad \text{指数函数不能用 N-L 公式}$$

$$= \frac{i}{R^n} \int_0^{2\pi} (\cos n\theta - i \sin n\theta) d\theta \quad \text{原式成为实数形式}$$

$$= \begin{cases} 2\pi i, & n=0 \\ 0, & n \neq 0 \end{cases}$$

$$\text{直接 N-L 公式}$$

§2 Cauchy-Goursat 定理

Green 公式: 设 D 为单连通区域

$P(x,y), Q(x,y) \in C^1(D)$, $C \subset D$ 为一条 Jordan 闭曲线

$$\text{则 } \oint_C P dx + Q dy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx dy$$

设 $w=f(z) = u+iv$ 在 D 内解析

$$f'(z) dz = f'(z) u dx - v dy + i f'(z) v dx + u dy$$

$$\oint_C f'(z) dz = \oint_C (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) dx dy + i \oint_C (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) dx dy$$

$$\oint_C 0 = 0$$

Cauchy-Goursat THM 证明 CR ② 二维区域有定理

设 $w=f(z)$ 在单连通区域 D 内解析, $C \subset D$ 为任意闭曲线, 则 $\oint_C f'(z) dz = 0$

强形式 C-G THM. 设 $w=f(z)$ 在有界单连通区域 D 上解析.

在 $D = D \cup \partial D$ 上连续, 则 $\oint_{\partial D} f(z) dz = 0$

复变函数积分与多元实变函数

柯西积分定理 (就是柯西-古尔萨定理)

Jordan 曲线: 连续简单闭曲线

证明 $f(z)$ 可导 $\Leftrightarrow u, v$ 可导, $C-R$ 成立

证充分性

$$\begin{aligned} f(z+\Delta z) - f(z) &= u(x+\Delta x, y+\Delta y) - u(x, y) \\ &\quad + i(v(x+\Delta x, y+\Delta y) - v(x, y)) \\ &= \Delta u + i\Delta v \\ &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \\ &\quad + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y\right) \end{aligned}$$

且 $\lim_{\Delta z \rightarrow 0} \varepsilon_i = 0$

由 $C-R$, $f(z+\Delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y$

$\Rightarrow \frac{f(z+\Delta z) - f(z)}{\Delta z} \rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

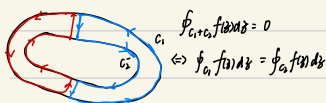
则 $f(z)$ 导数公式 $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$
 $= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

§ 复合闭路定理

1° 连续变形定理

解析函数沿闭路的积分不因闭路的连续变形

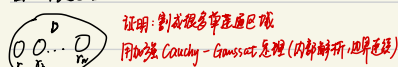
连续变形: 曲线在运动时不能碰到地窟 (非解析点)



2° 复合闭路定理

Jordan 闭曲线族 $\gamma = \gamma_1 + \dots + \gamma_n$ (如图)

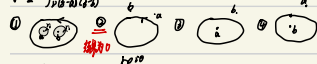
围成一个多连通区域 D $\partial D = \gamma$



THM: 设 $w=f(z)$ 在 Jordan 闭曲线族 γ (如图) 围成区域内解析, 在 $\bar{D} = D \cup \gamma$ 上连续, 则 $\oint_{\gamma} f(z)dz = \sum_{i=1}^n \oint_{\gamma_i} f(z)dz$

Ex: 设 $a \neq b$, $\gamma \subset \mathbb{C}$ 为一带不过 a, b 的 Jordan 闭曲线

求 $I = \oint_{\gamma} \frac{dz}{(z-a)(z-b)}$



case 1

$\oint_{\gamma} \frac{dz}{(z-a)(z-b)} = \frac{1}{a-b} \oint_{\gamma} \left(\frac{1}{z-a} - \frac{1}{z-b} \right) dz$
 $= \frac{1}{a-b} (2\pi i + 0 - 2\pi i) = 0$

case 3

$\oint_{\gamma} \frac{dz}{(z-a)(z-b)} = \frac{1}{a-b} \oint_{\gamma} \left(\frac{1}{z-a} - \frac{1}{z-b} \right) dz = \frac{2\pi i}{a-b}$

§ 4. 原函数/不定积分

设 $w=f(z)$ 在单连通区域 D 上解析

$F(z) := \int_{\gamma} f(z)dz$ (well-defined)

THM: 设 $f(z)$, D , $F(z)$ 如上所述

则 $F(z)$ 是解析的, 且 $F'(z) = f(z) \quad (\forall z \in D)$

proof: $F(z+\Delta z) - F(z) = \int_{\gamma} f(z)dz + \int_{\gamma'} f(z)dz$
 $= \int_{\gamma \cup \gamma'} f(z)dz = \int_{\gamma} f(z)dz + \int_{\gamma'} f(z)dz$
 $= f(z)\Delta z + o(\Delta z)$

不妨设 $\gamma' = \gamma \cup \gamma''$

其中 $\gamma'' = \frac{1}{\Delta z} \int_{\gamma} (f(z) - f(z'))dz$

$|f(z)| \leq \frac{1}{|\Delta z|} \int_{\gamma} |f(z) - f(z')| |dz|$

$\leq \frac{1}{|\Delta z|} \int_{\gamma} \varepsilon ds = \varepsilon$

而 $\forall \varepsilon > 0, |\Delta z| < \delta < 1$

$|f(z) - f(z')| < \varepsilon, \quad \forall z \in \gamma, \gamma''$

Def: 若在 D 上有 $G'(z) = f(z)$, 则称 $G(z)$ 为 f 的原函数

f 在 D 上两个原函数只相差常数

$G(z) + C$ 称作 $f(z)$ 在 D 上的不定积分, 记

$G(z) + C = \int f(z)dz$

Newton-Leibnitz 公式

设 $f(z)$ 在单连通区域 D 上解析

$G(z)$ 为 f 在 D 上的一个原函数

则对 $\forall z_1, z_2 \in D$ 有 $\int_{\gamma} f(z)dz = G(z_2) - G(z_1)$

分部积分公式: 设 $f(z), g(z)$ 在单连通区域 D 上解析

则对 $\forall z_1, z_2 \in D$ 有 $\int_{\gamma} f(z)g'(z)dz = f(z)g(z) \Big|_{z_1}^{z_2} - \int_{\gamma} f'(z)g(z)dz$

单连通区域是不可收缩的乘积

Ex: $D = \mathbb{C} \setminus \{0\}, \quad f(z) = \frac{1}{z}, \quad g(z) = z$

$\int_{\gamma} \frac{1}{z} dz = -\frac{1}{2} \int_{\gamma} \frac{1}{z} dz = -\frac{1}{2} \int_{\gamma} \frac{1}{z} dz$



Ex: $\int_{\gamma} \ln z dz$ (沿 C_1)

$= \int_{\gamma} \ln z dz = \int_{\gamma} \ln z dz$

$= \int_{\gamma} \ln z dz = \int_{\gamma} \ln z dz$



由于在实轴上不解析

[补充]

THM: 设 $f(z)$ 在 D 上解析, 则下列三者等价

(1) $f'(z) \equiv 0$ 对 D 中 z 恒同

(2) $f(z)$ 在 D 上具有任意位置无相关性

(3) $f(z)$ 在 D 上具有原函数

解析必有原函数

若上述 V 定理, 则 $N-L$ 可用

$\oint_C$ Cauchy 积分公式

THM: 设 $w=f(z)$ 在有界单连通区域 D 上解析

$\bar{D}$ 在 D 上连续, 则 V 定理

有 $f(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta) d\zeta}{\zeta - z}$ 第一平均值公式

$$\text{proof: RHS} = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta) + g(\zeta)(\zeta - z)}{\zeta - z} d\zeta$$

$$= f(z) + E(z), \quad E(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{g(\zeta)(\zeta - z)}{\zeta - z} d\zeta$$

证明 $|E(z)| < \varepsilon \Rightarrow E(z) = 0$

$$|E(z)| < \frac{1}{2\pi} \oint_{\partial D} \frac{|g(\zeta)(\zeta - z)|}{|\zeta - z|} |d\zeta|$$

$$< \frac{1}{2\pi} \oint_{\partial D} |f(\zeta) - f(z)| |d\zeta|$$

$$< \frac{1}{2\pi} \oint_{\partial D} \varepsilon |d\zeta| = \varepsilon$$

$$\forall \varepsilon > 0, \quad \exists \delta > 0, \quad \forall |z - \zeta| < \delta, \quad |f(z) - f(\zeta)| < \varepsilon, \quad \text{e.g.}$$

连续性

利用连续性

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{it}) e^{it} dr}{re^{it} - z} = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) e^{it} dt$$

$Q: z = 0, \quad R = \mathbb{R}^2$, 同乘 $r dr$ 再对 t 积分 (面积分法)

$$f(z) \int_0^{2\pi} r dr = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty f(re^{it}) e^{it} r dr dt$$

$$\frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) e^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) e^{it} dt$$

第二平均值公式

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) e^{it} dt$$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) e^{it} dt$$

$$\text{Ex: } I = \oint_{|z|=4} \left(\frac{1}{z} + \bar{z} \right) \cos z dz$$

$$= \oint_{|z|=4} \frac{1}{z} \cos z dz + \oint_{|z|=4} \bar{z} \cos z dz$$

$$= 2\pi i \cos z \Big|_{z=0} + \oint_{|z|=4} \frac{1}{z} \cos z dz = 34\pi i$$

解析

且包含边界

$\S 6$ 解析函数的高阶导数

THM 设 $w=f(z)$ 在 D 上解析, 则 $f(z)$ 在 D 上具有任意阶解析导数

且对 $\forall z \in D$ 有 $f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}$ ($n \geq 1$), 其中 $C \subset D$ 为 Jordan 闭曲线且周界

Proof: 对 n 作归纳 (设 $n=1$) 完全属于 D

$$f'(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^2}$$

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^2}$$



$$f^{(n+1)}(z) - f^{(n)}(z) = \frac{1}{2\pi i} \left[\oint_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+2}} - \oint_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}} \right]$$

$$= \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}} \left(\frac{1}{\zeta - z} - 1 \right) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}} \left(\frac{1 - (\zeta - z)}{\zeta - z} \right)$$

$$= \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}} + E(z)$$

$$E(z) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}} \left(\frac{1 - (\zeta - z)}{\zeta - z} \right)$$

$$|E(z)| \leq \frac{1}{2\pi} \oint_{\partial D} \frac{|f(\zeta)| |d\zeta|}{|\zeta - z|^{n+1}} \leq \frac{1}{2\pi} \oint_{\partial D} \frac{1}{r^{n+1}} |d\zeta|$$

$$= A |z|^{-n-1} \rightarrow 0$$

常用思路:

证明不为 0

用巨数

$$f^{(n+1)}(z) = \frac{d}{dz} f^{(n)}(z) = \frac{d}{dz} \left(\frac{n!}{2\pi i} \oint_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}} \right)$$

不能假设 $\frac{1}{z}$ 与 z 可交换

$$\text{Ex: (1) } I = \oint_{|z|=2} \frac{z - 5\bar{z}}{z^2} dz$$

$$= \frac{2\pi i}{5!} \left(\frac{5!}{2\pi i} \oint_{|z|=2} \frac{z - 5\bar{z}}{z^6} dz \right)$$

$$= \frac{2\pi i}{5!} \left(\frac{5!}{2\pi i} \right) \Big|_{z=0} = -\frac{2\pi i}{5!} = -\frac{2\pi i}{60}$$

$$(2) \text{ 求 } I = \oint_{|z|=1} \frac{5z + 2}{(z-1)^2} dz$$

$$= \oint_{|z|=1} \frac{5z + 2}{(z-1)^2} dz = \oint_{|z|=1} \frac{5z + 2}{(z-1)^2} dz$$

$$= 2\pi i \left(\frac{5z + 2}{(z-1)^2} \right) \Big|_{z=1} + 2\pi i \left(\frac{5z + 2}{(z-1)^2} \right) \Big|_{z=1}$$

$$= 2\pi i (5(1) + 2) = 14\pi i$$

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用复合回路定理

① 复合回路定理

保证每个圈里有一个奇点

② 柯西定理

$C \subset D$ (单): $f(z)$ 解析 $\Rightarrow \oint_C f(z) dz = 0$

设 $w=f(z)$ 在单连通域 D 中具有 $\oint_C f(z) dz = 0$, $\forall$ 回路 $C \subset D$

则 $f(z)$ 在 D 上解析

一般不对

$$\text{Ex } f(z) = \begin{cases} 0, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

任意解析函数极点, 不改变可积性

设 $w=f(z)$ 在 D 连续且具有:

Morera 定理

$\oint_C f(z) dz = 0, \quad \forall$ 回路 $C \subset D$

则 $f(z)$ 在 D 上解析 $f(z) \equiv \int_a^z f(\zeta) d\zeta$

$\Rightarrow f(z) \equiv 0$

沿任意曲线积分均为 0

代数基本定理: 设 $P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$
 则 $P(z)$ 在 $\mathbb{C}$ 上恰有 n 个复根 (计算重数)

Cauchy 不等式: 设 $w = f(z)$ 在 $B(z_0, R)$ 解析 ($R > 0$)

上解析, 且 $|f(z)| \leq M < +\infty$ ($\forall z \in B(z_0, R)$)

$$\text{则 } |f^{(n)}(z_0)| \leq \frac{n! M}{R^n} \quad (n \geq 1)$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z) dz}{(z-z_0)^{n+1}}$$

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \oint_{|z-z_0|=R} \frac{|f(z)| |dz|}{|z-z_0|^{n+1}} \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{M ds}{R^{n+1}} = \frac{n! M}{R^n} \xrightarrow{R \rightarrow \infty} 0$$

柯西积分公式求导

Liouville 定理: 有界函数必为常数

$$|f(z)| \leq M < +\infty \quad \forall z \in \mathbb{C}$$

$$\forall z \in \mathbb{C}, |f'(z)| \leq \frac{M}{R} \quad R \text{ 任意大} \Rightarrow \text{则 } \frac{M}{R} \rightarrow 0, |f'(z)| \rightarrow 0$$

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \oint_{|z-z_0|=R} \frac{|f(z)| |dz|}{|z-z_0|^{n+1}} \leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{M ds}{R^{n+1}} = \frac{n! M}{R^n} \xrightarrow{R \rightarrow \infty} 0$$

则代数基本定理 $\Leftrightarrow P(z)$ 在 $\mathbb{C}$ 上至少有一个零点

$$P(z) = (z-z_1)P_1(z) = (z-z_1)\dots(z-z_n)$$

$$\text{反证: } P_n(z) \neq 0 \quad \forall z \in \mathbb{C}$$

$$\text{令 } f(z) = \frac{1}{P_n(z)}, \text{ 则 } f \text{ 整}$$

$$|f(z)| = \frac{1}{|P_n(z)|} \xrightarrow{z \rightarrow \infty} 0$$

$$\Rightarrow |z| > R, \text{ 有 } |f(z)| \leq 1 \quad (\text{矛盾})$$

复变 HW 07

$f(z)$ 整, $f(0) = A, f'(0) = B, r > 0, n \in \mathbb{N}^*$

$$(1) I_1 = \int_0^{2\pi} f(re^{i\theta}) \cos \frac{\theta}{2} d\theta$$

$$(2) I_2 = \int_0^{2\pi} f(re^{i\theta}) \sin \frac{\theta}{2} d\theta$$

解: 若令 $z = re^{i\theta}$, 则 $dz = re^{i\theta} i d\theta = i z d\theta$

$$\text{则 } \begin{cases} \cos \theta + i \sin \theta = \frac{z}{r} \\ \cos \theta - i \sin \theta = \frac{\bar{z}}{r} \end{cases} \Rightarrow \cos \theta = \frac{z + \bar{z}}{2r}$$

$$I_1 + I_2 = \int_0^{2\pi} f(re^{i\theta}) d\theta = \frac{1}{i} \oint_C \frac{f(z) dz}{z}$$

$$(\text{其中 } C: |z| = r) = \frac{1}{i} \cdot 2\pi i f(z) \Big|_{z=0}$$

$$= 2\pi A$$

$$I_1 - I_2 = \int_0^{2\pi} f(re^{i\theta}) \cos \theta d\theta$$

$$= \frac{1}{2ir} \oint_C \frac{f(z)(z^2 + \bar{z}^2)}{z^2} dz$$

由于 $f(z)$ 是整函数, $f(z)(z^2 + \bar{z}^2)$ 也是

$$= \frac{1}{2ir} \cdot 2\pi i \cdot (f'(z)(z^2 + \bar{z}^2)) \Big|_{z=0}$$

$$= \frac{\pi}{r} \cdot n r^2 \cdot A^* B = \pi n r A^* B$$

$$\Rightarrow I_1 = \pi(A^* + \frac{n r A^* B}{2})$$

$$I_2 = \pi(A^* - \frac{n r A^* B}{2})$$

练习 2 $f(z)$ 在 $|z| \leq 2$ 内解析, $f(0) = A, f'(0) = B, f''(0) = C$

$$\text{求 } I = \frac{1}{2\pi i} \oint_{|z|=1} (2-z) f(z) dz$$

$$= \frac{1}{2\pi i} \oint_{|z|=1} (2-z) f(z) dz$$

$$\text{而 } z = e^{i\theta}, dz = e^{i\theta} i d\theta$$

$$\bar{z} = e^{-i\theta}, d\bar{z} = e^{-i\theta} (-i) d\theta \Rightarrow d\bar{z} = -\frac{dz}{z^2}$$

$$\text{故 } I = -\frac{1}{2\pi i} \oint_{|z|=1} \frac{2-z}{z^2} f(z) \cdot (-\frac{dz}{z^2})$$

$$= +\frac{1}{2\pi i} \oint_{|z|=1} \frac{(2-z)f(z)}{z^4} dz$$

$$= +\frac{1}{2\pi i} \cdot \frac{2\pi i}{2} \cdot (2-z)f(z) \Big|_{z=0}$$

$$= \frac{1}{2} (4B - C)$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z) dz}{(z-z_0)^{n+1}}$$

$$\oint \frac{f(z) dz}{(z-z_0)^{n+1}} = \frac{2\pi i f^{(n)}(z_0)}{n!}$$

从 $\oint_{|z|=1} \rightarrow \oint_{|z|=r}$
 添更多, $\therefore$ 反向了

练习 3 $f(z)$ 在 $|z| > 1$ 解析, 有界, 给定 $z_0, |z_0| < 1, n \in \mathbb{N}^*$

$$\text{证 } \oint_{|z|=2} \frac{f(z) dz}{(z-z_0)^{n+1}} = 0$$

$$|z| > 1 \text{ 时, } |f(z)| < M$$

$$\forall r > 2 \times 2 = 4, \oint_{|z|=r} \frac{f(z) dz}{(z-z_0)^{n+1}} = \oint_{|z|=r-1} \frac{f(z) dz}{(z-z_0)^{n+1}} \quad (\text{复合回路定理})$$

$$\text{则 } \left| \oint_{|z|=2} \frac{f(z) dz}{(z-z_0)^{n+1}} \right| = \left| \oint_{|z|=2} \frac{f(z) dz}{(z-z_0)^{n+1}} \right| \leq \oint_{|z|=2} \frac{|f(z)| |dz|}{|z-z_0|^{n+1}}$$

$$= \frac{1}{r^{n+1}} \oint_{|z|=2} |f(z)| |dz| < \frac{M}{r^{n+1}} \oint_{|z|=2} ds = \frac{2\pi r M}{r^{n+1}} = \frac{2\pi M}{r^n}$$

令 $r \rightarrow \infty$, 由夹逼定理即得证

不若求直接套定理

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§ 解析函数与调和函数

Def. 设 $\varphi = \varphi(x, y) \in C^1(D)$ 且在 D 上满足 $\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$

则称 φ 是 D 上的调和函数

$\Delta_n = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ — Laplace 算子

Thm 设 $w = f(z) = u(x, y) + iv(x, y)$ 在 D 上解析

则 u, v 在 D 上调和 $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ $\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Def: 上述的 v 称作 u 在 D 上的共轭调和函数 **有无穷**

if $(u, v) = (-v, u) \Rightarrow u$ 是 $-v$ 在 D 上

给定 D 上调和函数 $u = u(x, y)$, 是否 $\exists D$ 上共轭调和函数 v ✓

i.e. $\exists D$ 上解析函数 $f(z)$ s.t. $u = \operatorname{Re} f$

① 通次微分

Thm: 设 $u = u(x, y)$ 是单连通区域 D 上的调和函数 ② 读微分 $f = u + iv$

则 $\exists D$ 上解析函数 $f(z)$, s.t. $\operatorname{Re} f = u$

$f'(z) = u_x - iv_y = u'(z)$ (在 D 上解析) $\Rightarrow f(z) = \int u'(z) dz$ (不是积分)

Ex: $u = u(x, y) = \ln \sqrt{x^2 + y^2}$, $D = \mathbb{C}^* (\Delta u = 0)$ **One**

$f'(z) = u_x - iv_y = \frac{1}{z} = \frac{1}{2} \left(\frac{2x}{x^2 + y^2} - i \frac{2y}{x^2 + y^2} \right) = \frac{1}{2} \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{2} \frac{1}{x + iy} = \frac{1}{2z}$

$\Rightarrow f(z) = \int \frac{dz}{z} = \ln z + C, C \in \mathbb{C}$
 $= \ln|z| + i \arg z + C$

Ex: 设 $u = u(x, y) = x^2 - 3xy$ ($\Delta u = 0$) 求共轭函数 v 及 $f(z)$ s.t. $u = \operatorname{Re} f$

不是取分派 $u(x, y) = f(z) = u_x - iv_y = 2x - 3y = 3z^2$

$\Rightarrow f(z) = \int u'(z) dz = \int 3z dz = \frac{3}{2} z^2 + C$

$(v(x, y) = -y^2 + 3xy + C)$

偏取分派 $u_x = -u_y = 6xy$

$u_y = u_x = 3(x^2 - y^2)$

$\Rightarrow v = v(x, y) = \int 6xy dx = 3xy + g(y)$

$u_y: 3(x^2 - y^2) = 3x^2 + g'(y) \Rightarrow g'(y) = -3y^2$

$\Rightarrow g(y) = -y^3 + C$

$\Rightarrow v = -$

先换成一元!

Ex: 设 $w = f(z) = u(x, y) + iv(x, y)$ 是复函数, 求 $f(z)$

$\Delta u = 0$ 令 $z = x - iy$

$u_x = 2x + 4y$ $u_{xx} = 2$ $u_{yy} = 4$

$v_y = -2y$ $v_{yy} = -2$ $v_{xx} = 4$

$\Rightarrow \Delta u = u_{xx} + u_{yy} = 2 + 4 = 6 \neq 0$

$\Rightarrow u_{xx} = 0$ ($x, y \neq 0$)

$\Rightarrow u = C_1 x + C_2$ $C_1, C_2 \in \mathbb{R}$

$= C_1(x^2 + y^2) + C_2$

$\Rightarrow f(z) = C_1 z^2 + C_2$

$f(z) = u + iv = u_x - iv_y$

$= C_1 \cdot 2x + i \cdot C_1 \cdot 2y$

$= 2C_1(x + iy) = 2C_1 z$

$\Rightarrow f(z) = Az^2 + B$

证: 求导公式 (省 $\frac{\partial}{\partial z}$)

$f(z) = \frac{u + iv}{z}$

$= \frac{u_x + iv_x}{z}$

$= \frac{u_x + iv_x}{z} + \frac{u_y + iv_y}{z} = \frac{u_x + iv_x}{z} + \frac{u_y + iv_y}{z}$

$= \frac{(u_x + iv_x)dx + (u_y + iv_y)dy}{z}$

$= \frac{(u_x + iv_x)dx + (u_y + iv_y)dy}{z}$

$= \frac{u_x + iv_x}{z} + \frac{u_y + iv_y}{z}$

$= \frac{u_x + iv_x}{z} + \frac{u_y + iv_y}{z}$

$= \frac{u_x + iv_x}{z} + \frac{u_y + iv_y}{z}$

$= \frac{u_x + iv_x}{z} + \frac{u_y + iv_y}{z}$

Chap 4 级数

§1 复数项级数

1° 极限 Def: 设 $\{a_n\} \subset \mathbb{C}$, $A \in \mathbb{C}$, 若对 $\forall \epsilon > 0$

都 $\exists N \in \mathbb{N}$ (s.t. $n > N$ 时, 都有 $|a_n - A| < \epsilon$

则称 $n \rightarrow \infty$ 时, a_n 以 A 为极限, 记作 $\lim_{n \rightarrow \infty} a_n = A$, 或 $a_n \rightarrow A, n \rightarrow \infty$

2° 级数, 形如 $I = \sum_{n=0}^{\infty} a_n$

部分和 $S_n = \sum_{k=0}^n a_k \leftrightarrow \{S_n\}$

Def: 若 $\lim_{n \rightarrow \infty} S_n = A \in \mathbb{C}$, 称 I 是收敛的 (CV = convergent)

否则称 I 是发散的 (DV = divergent)

① 若 $\sum_{n=0}^{\infty} |a_n|$ CV, 称 I 是绝对收敛 (AC = Absolutely Convergent)

② 若 I CV 但非 AC 称 I 是 CC 级 (CC = Conditional CV)

Thm 1: $\lim_{n \rightarrow \infty} a_n = A = \alpha + i\beta \Leftrightarrow \begin{cases} \lim_{n \rightarrow \infty} \alpha_n = \alpha \\ \lim_{n \rightarrow \infty} \beta_n = \beta \end{cases}$

Thm 2: I CV $\Leftrightarrow \sum_{n=0}^{\infty} a_n$ 收敛 CV

Thm 3: I AC $\Leftrightarrow \sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} |a_n|$ 均 AC

Thm 4: I AC $\Leftrightarrow I$ CV 且 $|I| = \sum_{n=0}^{\infty} |a_n|$

3° 敛散判易性

I. Cauchy 根式判别法

(i) 若 $\sqrt[n]{|a_n|} \leq q < 1 (n > N_0) \Rightarrow I \text{ AC}$

(ii) 若 $\sqrt[n]{|a_n|} \geq q > 1 (n > N_0) \Rightarrow I \text{ DV}$

★ (2) $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = q$, $\begin{cases} < 1 \Rightarrow AC \\ > 1 \Rightarrow DV \\ = 1 \text{ 不定形} \end{cases}$ Ex. $\sum \frac{1}{n!}$, $\sum \frac{1}{n^n}$, $\sum \frac{1}{n^n}$

II. D'Alembert 判别法

(i) 若 $|\frac{a_{n+1}}{a_n}| \leq q < 1 (n > N_0) \Rightarrow I \text{ AC}$

(ii) 若 $|\frac{a_{n+1}}{a_n}| \geq q > 1 (n > N_0) \Rightarrow I \text{ DV}$

★ (2) $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| = q$, $\begin{cases} < 1 \text{ AC} \\ > 1 \text{ DV} \\ = 1 \text{ 不定形} \end{cases}$

III. 设 $\{a_n\}$, $\{b_n\} \subset \mathbb{R}$

Dirichlet 判别法: 设 $\{a_n\}$ 单调趋于 0

$\{b_n\}$ 部分和有界, 则 $\sum a_n b_n$ CV.

Abel 判别法: ① 设 $\{a_n\}$ 有界 ② $\{b_n\}$ 收敛

则 $\sum a_n b_n$ CV

Ex: (1) $I = \sum_{n=1}^{\infty} \frac{e^{in}}{n} \xrightarrow{\text{部分和有界}} \text{CV}$

(2) $I = \sum_{n=1}^{\infty} \frac{(2023 + i2024)^n}{n!} \xrightarrow{\text{自变量}} AC$

$$\leq \frac{\sum_{n=0}^{\infty} (2023 + i2024)^n}{n!} \leq e^{(2023 + i2024)}$$

$$\Rightarrow |e^z| \leq e^{|z|}$$

5° 幂级数

$$\{f_n(x)\}_{n=0}^{\infty}$$

1° (复变) 函数项级数

$$I = \sum_{n=0}^{\infty} f_n(z)$$

部分和 $S_n(z) = \sum_{k=0}^n f_k(z) \leftrightarrow \{S_n(z)\}_{n=0}^{\infty}$

Def: $\forall \delta \in D$, 有 $\lim_{n \rightarrow \infty} S_n(z) = S(z) \in \mathbb{C}$, 称 I 在 δ 收敛

若对 $\forall \delta \in D$, I 在 $\delta \subset V$, 称 I 在 D 上 CV $\Rightarrow S(z) = \sum_{n=0}^{\infty} f_n(z)$

2° 幂级数 形如 $I = \sum_{n=0}^{\infty} a_n z^n$ 或一般地: $I = \sum_{n=0}^{\infty} a_n (z - a)^n \leftrightarrow \{a_n\}_{n=0}^{\infty}$

Abel 定理: 设 $I = \sum_{n=0}^{\infty} a_n z^n$

(1) 若 I 在 $a \subset V$, $a \neq 0$, 则 $\forall \theta, |z| < |a|$, I 在 a AC.

(2) 若 I 在 a DV, 则对 $\forall \theta: |z| > |a|$, I 在 a DV

Proof: (1) $\sum_{n=0}^{\infty} a_n \theta^n \text{ CV } (\theta \neq 0)$

$$\exists: |a| < |\theta| \quad I = \sum_{n=0}^{\infty} a_n \theta^n$$

$$I = \sum_{n=0}^{\infty} a_n \theta^n = \sum_{n=0}^{\infty} a_n \theta^n \theta^n \quad \theta = \frac{a}{\theta}, |a| < |\theta|$$

$$|I| \leq \sum_{n=0}^{\infty} |a_n| |\theta| |\theta|^n \leq \sum_{n=0}^{\infty} M |\theta|^n \quad (\text{若 } a_n \text{ 有界})$$

要证 $|a_n| \leq M < +\infty, \forall n$, 这是由 $\sum_{n=0}^{\infty} a_n \theta^n \text{ CV} \Rightarrow \lim_{n \rightarrow \infty} a_n \theta^n = 0$

3° 收敛半径与收敛圆盘

Def: $R \equiv \sup \{ |a| : I \text{ 在 } \delta \text{ CV} \}$

$$\equiv \sup \{ |a| : I \text{ 在 } \delta \text{ AC} \}$$

$$\equiv \inf \{ |a| : I \text{ 在 } \delta \text{ DV} \}$$

称为 I 的收敛半径

$$\text{Case 1: } R = 0 \quad \sum_{n=0}^{\infty} a_n \theta^n$$

$$2: R = +\infty \quad \sum_{n=0}^{\infty} a_n \theta^n (= e^z) \quad R = 1$$

$$3: 0 < R < +\infty \quad \sum_{n=0}^{\infty} a_n \theta^n \text{ 收敛圆盘 } \sum_{n=0}^{\infty} a_n (z-a)^n$$

Def: $C_R: |z-a| = R$, 或 $|z-a| < R$ 称为收敛圆(盘)

$$D_R = B_R(R): |z-a| < R \text{ 称为收敛圆(盘)}$$

Ex: $0 < R < +\infty (R=1)$ 盘上一点 $a \in C_R$, 则所有点均 AC

(1) $\sum_{n=0}^{\infty} a_n \theta^n$ 在 C_R 上 AC

(2) $\sum_{n=0}^{\infty} \frac{e^{in}}{n} \theta^n$

(3) $\sum_{n=0}^{\infty} a_n \theta^n$ 在 C_R 既有 CV 又有 DV ($\theta = 1 \text{ CV}, \theta = i \text{ DV}$)

(4) $\sum_{n=0}^{\infty} a_n \theta^n$ 在 C_R DV

4° 收敛半径求法

THM 1 (比值法) 设 $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| = \lambda$, 则 $R = \begin{cases} +\infty & \lambda = 0 \\ \frac{1}{\lambda} & \lambda \neq 0, +\infty \end{cases}$

THM 2 (根值法) $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lambda$, 则 R 同上

Proof: $\lambda \neq 0, +\infty$ 上极限引理

$$I = \sum_{n=0}^{\infty} a_n \theta^n$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n \theta^n|} = |a| |\theta| = \lambda |a| = q$$

$$\begin{cases} < 1 \text{ AC} & |a| < \frac{1}{\lambda} \\ > 1 \text{ DV} & |a| > \frac{1}{\lambda} \end{cases} \Rightarrow R = \frac{1}{\lambda}$$

$$I = \sum_{n=0}^{\infty} a_n \theta^n \quad \text{上极限存在为 } 1 \Rightarrow R = 1$$

或用 Abel 思想, 猜一个临界点 $\begin{cases} \text{AC 对} \\ \text{DV 对} \end{cases}$

$$\text{Ex } I = \sum_{n=0}^{\infty} \frac{n!}{n^n} \theta^n \quad \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n!}{n^n} = (1 + \frac{1}{n})^n \rightarrow e \Rightarrow \lambda = \frac{1}{e}$$

5° 幂级数运算

$$(1) f(z) = \sum_{n=0}^{\infty} a_n z^n, |z| < R_1, g(z) = \sum_{n=0}^{\infty} b_n z^n, |z| < R_2$$

$$f(z) \pm g(z) = \sum_{n=0}^{\infty} (a_n \pm b_n) z^n = \sum_{n=0}^{\infty} (a_n \pm b_n) z^n, |z| < R, R \geq \min(R_1, R_2)$$

$$f(z)g(z) = \sum_{n=0}^{\infty} a_n z^n \sum_{m=0}^{\infty} b_m z^m = \sum_{n=0}^{\infty} c_n z^n, |z| < R, c_n = a_n b_n + \dots + a_0 b_n, R \geq \min(R_1, R_2)$$

$$(2) f(z) = \sum_{n=0}^{\infty} a_n z^n, |z| < R$$

$$h(z) \text{ 定义于 } D, \text{ 且 } |h(z)| < R, \forall z \in D$$

$$\text{则 } f(h(z)) = \sum_{n=0}^{\infty} a_n (h(z))^n \text{ 在 } D$$

$$\text{Esp: } h(z) = \sum_{n=0}^{\infty} a_n z^n, |z| < r: |h(z)| < R, \forall z \in r$$

$$\Rightarrow f(h(z)) = \sum_{n=0}^{\infty} a_n (\sum_{m=0}^{\infty} a_m z^m)^n: |z| \in r$$

6° 幂级数和函数

THM: 设 $I = \sum_{n=0}^{\infty} a_n(x-a)^n$ 收敛半径为 R
则在收敛圆 $D_R: |x-a| < R$ 上,

① 和函数 $f(x)$ 解析

② $f'(x)$ 可通过逐项求导得到:

$$f'(x) = \left(\sum_{n=0}^{\infty} a_n(x-a)^n \right)' = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

③ $f(x)$ 可通过逐项积分得到

设圆环 $C \subset D_R$

$$\int_C f(x) dx = \int_C \left(\sum_{n=0}^{\infty} a_n(x-a)^n \right) dx = \sum_{n=0}^{\infty} \left(\int_C a_n(x-a)^n dx \right)$$

$$\text{ESP: } \int_a^b f(x) dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (b-a)^{n+1}$$

$$\text{Ex (1)} f(x) = \sum_{n=0}^{\infty} (n+1)x^n: R=1$$

$$= \sum_{n=0}^{\infty} (x^{n+1})' = \left(\sum_{n=0}^{\infty} x^{n+1} \right)' = \left(\frac{x}{1-x} \right)' = \frac{1}{(1-x)^2}$$

$$\text{或} = \left(\sum_{n=0}^{\infty} x^n \right)' = \left(\frac{1}{1-x} \right)' \text{ 或者 } n f(x) = f'(x)$$

$$(2) g(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}: R=1$$

$$g'(x) = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\Rightarrow g(x) - g(0) = g'(x) = \int_0^x \frac{dx}{1-x} = -\ln(1-x) + C$$

$$\Rightarrow \text{原式} = -\ln(1-x)$$

$$\text{Ex} \sum_{n=0}^{\infty} a_n x^n, R_1, \sum_{n=0}^{\infty} b_n x^n, R_2$$

考虑 $|x| < R_1, R_2$, 必存在 $a < R_1, R_2$, $x=a$, 否则 $a \geq R_1, R_2$

此时 $\sum_{n=0}^{\infty} |a_n a^n| = \sum_{n=0}^{\infty} |a_n a^n| \cdot |a_n a^n|$ (0)

又由 $\sum_{n=0}^{\infty} a_n a^n$ AC, 可知 $\sum_{n=0}^{\infty} a_n a^n = 0 \Rightarrow |a_n a^n|$ 必有界

$$|x| < M \sum_{n=0}^{\infty} |a_n a^n| AC,$$

这证明收敛半径 $R \geq R_1, R_2$

$$\text{Ex} \sum_{n=0}^{\infty} (1) f(x) = \sum_{n=0}^{\infty} n(n-1)x^n$$

$$\frac{n(n-1)}{n(n-1)} = 1 \Rightarrow D: |x| < 1$$

$$f(x) = x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2 \sum_{n=2}^{\infty} (n+2)(n+1)x^n$$

$$= x^2 \left(\sum_{n=2}^{\infty} (n+2)x^{n+1} \right)' = x^2 \left(\sum_{n=1}^{\infty} (n+1)x^n \right)' = x^2 \left(\frac{1}{1-x} - 1 \right)'$$

$$= \frac{2x^3}{(1-x)^3}$$

$$(2) g(x) = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} x^n$$

$$\frac{1}{n(n-1)} \cdot n(n-1) = 1 \Rightarrow R=1$$

$$g'(x) = \sum_{n=2}^{\infty} \frac{1}{(n-1)} x^{n-1} = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$$

$$= \frac{1}{1-x}$$

$$\Rightarrow g'(x) - g'(0) = -\ln(1-x) = g(x)$$

$$\Rightarrow g(x) = g(x) - g(0) = (0-1) \ln(1-x) + C$$

$$\text{Ex 4. (1)} \sum_{n=1}^{\infty} n a_n x^n \quad (2) \sum_{n=1}^{\infty} n a_n x^{n-1} \quad (3) \sum_{n=1}^{\infty} \frac{a_n}{n} x^{n-1} \quad R \text{ 相同}$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \text{ 与 } \sum_{n=1}^{\infty} n a_n x^n \text{ 收敛半径相等}$$

$$\sum_{n=1}^{\infty} n a_n x^n \text{ 收敛半径大于等于 } \sum_{n=1}^{\infty} n a_n x^n \quad (|a_n| < n|a_n| \Rightarrow \frac{1}{|a_n|} > \frac{1}{n|a_n|}, n \neq 0)$$

$$\text{故 } R_1 \geq R_2$$

$$\text{① } R=0, \text{ 显然 } R_2=0$$

$$\text{② } R_1 > 0, R \text{ 满足 } |x| < R \text{ 时 } \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ 收敛}$$

$$\text{令 } |x| < |x_0| < R, \text{ 取 } q = \frac{|x_0|}{|x|} < 1, \sum_{n=1}^{\infty} n q^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} n |a_n x^n| = \sum_{n=1}^{\infty} n |a_n x_0^n| \left(\frac{|x|}{|x_0|} \right)^n = \sum_{n=1}^{\infty} n q^n |a_n x_0^n|$$

$$< M \sum_{n=1}^{\infty} |a_n x_0^n|,$$

$$\text{因此判别法, } \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ 在 } |x| < R \text{ 时}$$

$$\Rightarrow \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ 在 } |x| < R \text{ 时 AC.}$$

Ex1 求 R

$$(1) \sum_{n=0}^{\infty} \frac{n!}{n!} z^n$$

$$\text{使用根值判别法 考虑 } \sqrt[n]{\frac{n!}{n!}} = \frac{n}{n!}$$

$$\text{而 } \lim_{n \rightarrow \infty} \frac{n}{n!} = \frac{1}{n!} \left(\ln \frac{1}{n!} + \ln \frac{n}{n!} \right)$$

$$= -\frac{1}{n!} \ln \frac{1}{n!}$$

另一方面, 由黎曼积分定义

$$\sum_{n=1}^{\infty} \frac{1}{n!} \ln \frac{1}{n!} = \int_0^1 \ln x dx = (x \ln x - x) \Big|_0^1$$

$$= -1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{n}{n!} = 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{n}{n!} = e$$

$$\text{而 } a_n = \frac{1}{n!}, n \neq 0$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{n+1}$$

$$\text{考虑 } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \right) = \frac{1}{e} = \lim_{n \rightarrow \infty} \left(\frac{1}{n!} \right)^{\frac{1}{n}} = \frac{1}{e}$$

$$\text{故收敛半径为 } \frac{1}{e}$$

$$(2) \sum_{n=1}^{\infty} (1 - \sin \frac{1}{n})^{-n} z^n$$

$$a_n = (1 - \sin \frac{1}{n})^{-n}$$

$$a_n^{\frac{1}{n}} = (1 - \sin \frac{1}{n})^{-1} = (1 + (-\sin \frac{1}{n}))^{-1} = \frac{1}{1 - \sin \frac{1}{n}}$$

$$\lim_{n \rightarrow \infty} a_n^{\frac{1}{n}} = e^{\lim_{n \rightarrow \infty} \ln a_n^{\frac{1}{n}}} = e$$

$$\Rightarrow R = \frac{1}{e}$$

$$\text{例1: } a_n = \frac{n!(-1)^n}{2}, b_n = \frac{n!(-1)^{n+1}}{2}$$

$$R_1 = R_2 = 1, R = \infty$$

§ Taylor 级数

$f(x) \in C^{\infty}(R)$, $x_0 = 0$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad |x| < \delta = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Ex. $f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x = 0 \end{cases}$ 速降函数, 用某函数

$f^{(n)}(0) = 0, \forall n \in N$. 光滑化

Taylor 展开定理: 设 $w = f(z)$ 在 D 上解析, 给定 $z_0 \in D$, $d = \inf\{|z - z_0|$

则对 $\forall z \in B_d(z_0)$, $(|z - z_0| < d)$, 有 $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, 其中 $a_n = \frac{f^{(n)}(z_0)}{n!}$

且上述展开唯一确定, 称作泰勒展开式(级数)

proof: STEP 1 z 给定 $z \in B_d(z_0)$

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z}$$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 + z_0 - z} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}$$

$$\text{其中 } h(\zeta) = \frac{z - z_0}{\zeta - z_0}, \quad \rho = \frac{|z - z_0|}{\rho} < 1, \quad \frac{1}{1 - \rho} = \sum_{n=0}^{\infty} \rho^n$$

$$\Rightarrow \frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}$$

$$\stackrel{\text{Cauchy}}{\Rightarrow} \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z} = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

$$R_n(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} (z - z_0)^n d\zeta$$

$$|R_n(z)| \leq \frac{1}{2\pi} \oint_{\gamma} \left| \frac{f(\zeta)}{(\zeta - z)^{n+1}} (z - z_0)^n \right| |d\zeta|$$

$$\leq \frac{1}{2\pi} \oint_{\gamma} \left(\frac{M \rho^n}{\rho^{n+1}} \right) ds = \frac{M \rho^n}{\rho} \rightarrow 0$$

STEP 2 假如 $f(z)$ 有 2 种展开

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} b_n (z - z_0)^n \quad B_d(z_0)$$

令 $z = z_0$, 必有 $a_0 = b_0$

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n = \sum_{n=1}^{\infty} b_n (z - z_0)^n$$

$$\Rightarrow a_1 = b_1 \Rightarrow a_2 = b_2 \Rightarrow \dots$$

$$\text{Rem: (1) } f(z) = \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n + o((z - z_0)^n)$$

$R \geq d$

$$\text{Ex } \frac{1}{z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1 = d = R(z_0 = 0)$$

$$f(z) = \begin{cases} \frac{1}{z}, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases} \rightarrow \text{非解析}$$

$$(2) f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < R \text{ 收敛半径}$$

$$(0 < R < \infty) \quad G_R(z) = R$$

Claim: $f(z)$ 在 R 上必有奇点



若在 R 上无奇点

利用有限覆盖原理反证

$\Rightarrow$ 在 R 上解析, 矛盾

原函数解析 $\Leftrightarrow$ 可展开为幂级数

$$\text{Ex: (1) } e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \quad \forall z \in C$$

$$= \sum_{n=0}^{\infty} \frac{e^0}{n!} z^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

$$(2) \cos z = \sum_{n=0}^{\infty} \frac{\cos^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

$$\text{非幂级数展开 } \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

$$(3) \frac{1}{(1-z)^2} = \left(\frac{1}{1-z} \right)' = \left(\sum_{n=0}^{\infty} z^n \right)' = \sum_{n=0}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} n z^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1) z^n$$

$$(4) f(z) = h(z) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} z^n$$

$$f(z) = \frac{1}{z} = -\sum_{n=0}^{\infty} z^n \Rightarrow f(z) \text{ 在 } z=0 \text{ 处不解析}$$

‘根号’函数零点

§4 解析函数零点

Def: 设 $w = f(z)$ 在 D 上解析, $z_0 \in D$

(1) 若 $f(z_0) = 0$, 称 z_0 为 $f(z)$ 零点

(2) 若 $f(z_0) = f'(z_0) = \dots = f^{(m-1)}(z_0) = 0$, 但 $f^{(m)}(z_0) \neq 0$, $m \geq 1$

称 z_0 为 $f(z)$ m 根(阶)零点 ($m=1$ 简单零点)

(3) 若 $f(z_0) = 0$ 但 $f(z)$ 在某点邻域中不取 0, 称 z_0 为 $f(z)$ 孤立零点

Thm: z_0 为 $f(z)$ m 根零点 ($m \geq 1$) 的充要条件是

$\exists$ 某点 $\rho(z_0)$ 且其上解析函数 $\varphi(z)$, s.t. $f(z) = (z - z_0)^m \varphi(z)$, $z \in B_{\rho}(z_0)$

其中 $\varphi(z)$ 在 $B_{\rho}(z_0)$ 上不取 0

$\Leftarrow$ 显然

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

$$= a_0 (z - z_0)^0 + a_1 (z - z_0)^1 + \dots$$

$$= (z - z_0)^m \varphi(z)$$

$$\varphi(z) = a_m + a_{m+1}(z - z_0) + \dots = B_d(z_0)$$

$$a_m = \varphi(z_0) \neq 0 \Rightarrow$$
 用连续性有 $\delta < d$, $\varphi(z) \neq 0$

Thm 设 $w = f(z)$ 在 D 上解析, 则 D 上所有零点都是孤立点, 除非 $f \equiv 0$

Lemma 设 $D \subset C$ 为区域, $J_1, J_2 \subseteq D$ 满足

(i) J_1, J_2 是开集

(ii) $J_1 \cap J_2 = \emptyset$

(iii) $J_1 \cup J_2 = D$



则 J_1, J_2 必有一个为 $\emptyset$

Thm 设 $w = f(z)$ 在 D 上解析, 则 D 上所有零点都是孤立点, 除非 $f \equiv 0$

proof: $f(z) \not\equiv 0$, 令 $J_1 = \{z \in D \mid f(z) \neq 0\}$

J_2 上恒取 0

$$J_2 = \{z_0 \in D \mid \exists \text{ 邻域 } U(z_0) \text{ 在 } U(z_0) \text{ 上恒取 } 0\}$$

Claim (i) J_1, J_2 开 $\vee$

(ii) $J_1 \cap J_2 = \emptyset$

(iii) $J_1 \cup J_2 = D$

$\forall z_0 \in D$ $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

case 1: $a_0 \neq 0, \forall n \Rightarrow z_0 \in J_1$

case 2: $a_0 = 0$, 设 m 为初 0 系数的最小指数

$$f(z) = a_m (z - z_0)^m + \dots = (z - z_0)^m \varphi(z), \quad \varphi(z) = a_m + a_{m+1}(z - z_0) + \dots = g(z)$$

$$\varphi(z_0) = a_m \neq 0$$

$$\Rightarrow z_0 \in J_2$$

D是区域

Thm(解析函数的唯一性)

设 $f(z)$, $g(z)$ 在 D 上解析, $a \in D$

设若 $\exists \{z_n\}_{n=1}^{\infty} \subset D$, $z_n \rightarrow a$

(1) $\lim_{n \rightarrow \infty} z_n = a$, (2) $f(z_n) = g(z_n)$, $\forall n$

则 $f(z) = g(z) \forall z \in D$

Ex 010

练习1 收敛半径是什么

(1) $y = \tan z$, $z_0 = \frac{\pi}{4}$

$$y' = \frac{1}{\cos^2 z} = 2$$

$$y'' = \frac{-2 \cos z (-\sin z)}{\cos^4 z}$$

$$= \frac{2 \sin z}{\cos^3 z} = 4$$

$$y''' = \frac{2(1+2 \sin^2 z)}{\cos^4 z} = 16$$

$$\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\tan z = 1 + 2(z - \frac{\pi}{4}) + 2(z - \frac{\pi}{4})^2 + \frac{8}{3}(z - \frac{\pi}{4})^3, R = \frac{\pi}{4}$$

(2) $y = e^{\frac{1}{z}}$, $z_0 = 0$

$$y' = e^{\frac{1}{z}} (-\frac{1}{z^2}) = -1$$

$$y'' = e^{\frac{1}{z}} (\frac{1}{z^3} + \frac{2}{z^4}) = -1$$

$$y''' = e^{\frac{1}{z}} (-\frac{3}{z^4} - \frac{8}{z^5} - \frac{6}{z^6} - \frac{6}{z^7})$$

$$= e^{\frac{1}{z}} (-\frac{6}{z^4} - \frac{6}{z^5} - \frac{6}{z^6} - \frac{6}{z^7}) = -1$$

$$e^{\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{2}z^2 - \frac{1}{6}z^3, R = 1$$

泰勒级数收敛半径的求法

把展开点 a 以 a 为圆心, 最近奇点的距离为半径
圆内收敛, 圆外不收敛

$$\text{练习2 } \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = S(z) = 1 + \frac{z}{1!} + \dots$$

$$S'(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = \frac{z}{1!} + \dots$$

$$S''(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+2)!} = \frac{z}{2!} + \dots$$

$$S'''(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+3)!} = \frac{z}{3!} + \dots$$

$$\Rightarrow \text{得 } z = e^{iz}$$

$$\text{则 } dz = 1 \Rightarrow dz = i, w, w'$$

$$\text{代入 } f(w) = 1, f'(w) = 0, f''(w) = 0$$

$$\text{有 } f(w) = \frac{1}{2}(e^{iz} + e^{-iz}), w = \frac{z-i}{2}$$

练习3 $\lim_{n \rightarrow \infty} (1 + \frac{z}{n})^n |_{z=0} (n \geq 1)$

$$\ln(1 + \frac{z}{n}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{z^k}{n^k}$$

$$\text{取 } z = 1, \ln(1 + \frac{1}{n}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{n^k}$$

$$\text{取 } \ln(1 + \frac{1}{n}) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{n^k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{n^k}$$

$$\text{令 } z = 0, \text{得 } \lim_{n \rightarrow \infty} (1 + \frac{z}{n})^n |_{z=0} = \frac{(-1)^{k+1}}{k} \cdot i^k (n!)!$$

练习4

$$f(z) \equiv 0 \Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n$$

在 $C_R: |z| = R, R > 0$ 上,

$$|f(z)| = |\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} z^n| \leq \sum_{n=0}^{\infty} \frac{|f^{(n)}(z_0)|}{n!} |z|^n$$

$$\leq \sum_{n=0}^{\infty} \frac{R^n}{n!} = e^R$$

$$\text{由 } S \notin \mathbb{Z}, n \neq S$$

$$\text{由 } S > n, S - n > 0$$

$$\text{令 } R \rightarrow +\infty, \text{得 } |f(z)| \rightarrow 0$$

$$\text{由 } S < n, S - n < 0$$

$$\text{令 } R \rightarrow 0, \text{得 } |f(z)| \rightarrow 0$$

$$\text{由 } \forall n, a_n = 0 \Rightarrow f(z) \equiv 0$$

§5 Laurant 展开式

1°-级数收敛, 例如 $I = \sum_{n=0}^{\infty} a_n(z-z_0)^n = \varphi(z)$

$$= I_1 + I_2, I_1 = \sum_{n=0}^{\infty} a_n(z-z_0)^n: |z-z_0| < R$$

$$I_2 = \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}: |z-z_0| > \frac{1}{r} \triangleq r$$

Case 1 $r > R, I_2$ 无公共收敛区域

Case 2 $r < R, D(z, r, R): r < |z-z_0| < R$ 环形的收敛区域

Thm 设 $I = I_1 + I_2$ 如上, 其收敛圆环域为 $D(z, r, R)$, 则在 $D(z, r, R)$ 上

(1) 它和函数 $f(z) = \varphi(z) + \psi(z)$ 的解析

(2) $f'(z)$ 可通过逐项求导得到:

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1} + \sum_{n=1}^{\infty} (-n) a_{-n} (z-z_0)^{-n-1}$$

(3) $f(z)$ 与逐项求导:

$$\text{沿曲线 } C \subset D(z, r, R), \int_C f(z) dz = \int_C (\sum_{n=0}^{\infty} a_n (z-z_0)^n) dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n dz$$

2° Laurant 展开定理

$$\text{Ex: } f(z) = \frac{1}{z(z-1)} \quad z_0 = 0$$

在 $0 < r < 1$ ① $1 < r < +\infty$ 分别解析

$$f(z) = \frac{1}{z} + \frac{1}{z-1}$$

$$0 = \frac{1}{z} + \sum_{n=0}^{\infty} z^n \quad 0 < |z| < 1$$

$$0 = \frac{1}{z} - \frac{1}{1-z} = \frac{1}{z} - \sum_{n=0}^{\infty} z^n, 1 < |z| < +\infty$$

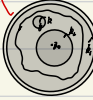
Laurant 展开定理: 设 $w = f(z)$ 在 $D(z, r, R)$ 上解析

则对 $\forall z \in D(z, r, R)$ 有 $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, 其中 $a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta) d\zeta}{(\zeta-z_0)^{n+1}}$, $C \subset D(z, r, R)$ 为环绕 z_0 的 Jordan 闭曲线

类似 n 阶导
但 n 阶导可能不存在

$$\phi_k = \phi_k + \phi_k \quad \phi_k = \phi_k + \phi_k$$

且上述展式是唯一-确切的, 称作 Laurent 展式 (级数)



Proof. Step 1 当 \$z \in D(a, R)\$

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z}$$

$$I_1 = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z}$$

$$\textcircled{2} \text{ 当 } k, \frac{1}{\zeta - z} = \frac{1}{\zeta - a + (a - z)} = \frac{1}{\zeta - a} \cdot \frac{1}{1 - \frac{z - a}{\zeta - a}}$$

$$I_2 = -\frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - a} \cdot \frac{1}{1 - \frac{z - a}{\zeta - a}}$$

Step 2 唯一性 \$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n = \sum_{n=0}^{\infty} \tilde{C}_n (z-a)^n; D(a, r, R)\$

逐项求导法则

$$\forall p \in \mathbb{Z}, \frac{f(z)}{(z-a)^p} = \sum_{n=0}^{\infty} C_n (z-a)^{n-p} = \sum_{n=0}^{\infty} \tilde{C}_n (z-a)^{n-p}$$

$$\textcircled{2} \text{ 洛朗级数 } \frac{1}{z} = \frac{1}{z} \cdot \frac{1}{1 - \frac{a}{z}}$$

Rem (1) 洛朗级数

$$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n = \sum_{n=0}^{\infty} \tilde{C}_n (z-a)^n; R' < |z-a| < R$$

$$\textcircled{2} f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n; |z-a| < R$$

$$f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n; 0 < |z-a| < R \text{ Laurent}$$

$$\tilde{C}_n = 0, n \leq -1, \tilde{C}_n = C_n, n \geq 0$$

$$\text{Ex (1)} f(z) = \frac{e^z}{z^2} \text{ 洛朗级数}$$

$$f(z) = \frac{e^z}{z^2} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{n-2}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{z^{n-2}}{n!}$$

$$n=0, n=1$$

$$\textcircled{2} f(z) = \frac{z^2 + 5}{(z^2 + 1)(z^2 + 2)}; z=0$$

$$= \frac{z^2 - 2(z^2 + 1)}{(z^2 + 1)(z^2 + 2)} = \frac{1}{z^2 + 2}$$

$$\textcircled{3} = -\frac{1}{2} \cdot \frac{1}{1 - \frac{z^2}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z^2}{2}\right)^n = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{z^{2n}}{2^n}; 0 < |z| < 1$$

$$\textcircled{4} = -\frac{1}{2} \cdot \frac{1}{1 - \frac{z^2}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z^2}{2}\right)^n = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{z^{2n}}{2^n}; 1 < |z| < 2$$

$$\textcircled{5} = \frac{1}{z} \cdot \frac{1}{1 - \frac{z^2}{2}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{z^2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{z^{2n-1}}{2^n}; 2 < |z| < +\infty$$

$$\text{应用 } C_1 = \frac{1}{2\pi i} \oint_{\gamma} f(\zeta) d\zeta$$

$$\Leftrightarrow \oint_{\gamma} f(\zeta) d\zeta = 2\pi i C_1$$

$$\text{Ex: } f(z) = \frac{ze^z}{z-1}, C: |z|=2$$

$$I = \oint_{\gamma} f(\zeta) d\zeta = \oint_{\gamma} \frac{ze^z}{z-1} d\zeta$$

$$\text{在 } 1 < |z| < \infty, \frac{ze^z}{z-1} = \frac{1}{1-\frac{1}{z}} e^z = (1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots) (1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots)$$

$$I' = \oint_{|z|=1} f(\zeta) d\zeta$$

$$\text{在 } 0 < |z| < 1, \frac{ze^z}{z-1} = \frac{-z}{1-z} e^z = -(1 + z + z^2 + \dots) (1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots)$$

$$I' = 2(2-0)2\pi$$

\$\sum C_n z^n\$ 在 \$z=0\$ 发散 \$\forall |z| > |a|\$, 级数收敛

级数 \$\sum C_n z^n\$ 收敛,

$$\sum_{n=-\infty}^{\infty} C_n z^n = \sum_{n=-\infty}^{\infty} C_n z^n \left(\frac{z}{z}\right)^n < \sum_{n=-\infty}^{\infty} M \left|\frac{z}{z}\right|^n = M \frac{1 + |z| + |z|^2 + \dots}{|z|}, \text{ 矛盾!}$$

不必证明就反面考虑

$$\oint_{\gamma} \frac{1}{(z-a)^n} d\zeta, C: |z-a|=R, z-a = Re^{i\theta}$$

$$= \int_0^{2\pi} \frac{1}{R^n} \frac{1}{e^{in\theta}} R e^{i\theta} i d\theta = e^{-ni\theta} = \cos n\theta - i \sin n\theta$$

$$= \int_0^{2\pi} \frac{i d\theta}{R^n e^{in\theta}} = \frac{1}{R^n} \int_0^{2\pi} \frac{i d\theta}{e^{in\theta}}$$

$$= \frac{1}{R^n} \int_0^{2\pi} e^{-int} dt = -\frac{1}{R^n} (e^{-2\pi i n} - e^0)$$

$$\frac{1}{2} n=0, = \frac{1 - e^{-2\pi i n}}{n} = \frac{2\pi i \cdot 1}{1} = 2\pi i$$

$$\text{若 } n \neq 0, = 0$$

练习1 (1) $\frac{1}{z(1+z^2)}: 0 < |z| < 1, 1 < |z| < +\infty$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots + (-1)^n z^n, |z| < 1$$

$$\Rightarrow \frac{1}{z(1+z^2)} = \frac{1}{z} - z + z^3 - z^5 + \dots + (-1)^n (n+1) z^n, 0 < |z| < 1$$

$$\frac{1}{z(1+z^2)} = \frac{1}{z} - z + z^3 - z^5 + \dots + (-1)^n (n+1) z^n$$

$$\frac{1}{z(1+z^2)} = \frac{1}{z} - z + z^3 - z^5 + \dots + (-1)^n (n+1) z^n$$

$$= z^{-1} - z + z^3 - z^5 + \dots + (-1)^n (n+1) z^{n+1}$$

$$(2) \frac{1}{z^2}, 1 < |z| < +\infty$$

$$\frac{1}{1+z} = -\frac{1}{z(1+\frac{1}{z})} = -\frac{1}{z} (1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots)$$

$$= -(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots)$$

$$\frac{1}{z^2} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

$$\Rightarrow \frac{1}{z^2} = \frac{1}{z} - (\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots) + (\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots)^2$$

$$= \frac{1}{z} - (\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots) + (\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots)^2$$

$$= \frac{1}{z} - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} + \dots$$

练习2 (1) $I_1 = \frac{1}{2\pi i} \oint_{|z|=1} \frac{z}{z^2-1} dz$

$$z \sin \frac{1}{z} = (z^2+1) \sin(\frac{1}{z})$$

$$= (z^2+1) (\sin(\frac{1}{z}) \cos(\frac{1}{z}) + \cos(\frac{1}{z}) \sin(\frac{1}{z}))$$

$$= \sin(\frac{1}{z}) (z^2+1) (\frac{1}{z} - \frac{1}{z^3} + \frac{1}{z^5} - \dots)$$

$$+ \cos(\frac{1}{z}) (z^2+1) (\frac{1}{z} - \frac{1}{z^3} + \frac{1}{z^5} - \dots)$$

$$\Rightarrow C_1 = \cos 1 - \frac{1}{2} \sin 1$$

$$\Rightarrow I_1 = \frac{1}{2\pi i} \cdot 2\pi i (\cos 1 - \frac{1}{2} \sin 1) = \cos 1 - \frac{1}{2} \sin 1$$

$$(2) I_2 = \frac{1}{2\pi i} \oint_{|z|=1} \frac{z \sinh \frac{z}{2}}{z^2-1} dz \quad \sinh z = z \cosh z$$

$$z \sinh \frac{z}{2} = (z^2+1) (\sinh \frac{z}{2} \cosh \frac{z}{2} + \cosh \frac{z}{2} \sinh \frac{z}{2})$$

$$= \cosh \frac{z}{2} = \cosh \frac{z}{2} = 1 + \frac{1}{2} (\frac{z}{2})^2 + \frac{1}{24} (\frac{z}{2})^4$$

$$\sinh \frac{z}{2} = \frac{1}{2} \sinh \frac{z}{2} = \frac{1}{2} (\frac{z}{2} - \frac{1}{24} (\frac{z}{2})^3 + \dots)$$

$$z \sinh \frac{z}{2} = (z^2+1) (\sinh \frac{z}{2} \cosh \frac{z}{2} + \cosh \frac{z}{2} \sinh \frac{z}{2})$$

$$\Rightarrow C_1 = \cosh 1 + \frac{1}{2} \sinh 1$$

$$\Rightarrow I_2 = \cosh 1 + \frac{1}{2} \sinh 1$$

练习3 $\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ 在 $C \setminus \{0\}$ 上和函数

$$Ch z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \Rightarrow Ch z^2 = \sum_{n=0}^{\infty} \frac{z^{4n}}{(2n)!} \Rightarrow Ch \frac{z}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

$$故 f(z) = Ch z^2$$

练习4 $A = \{z, \dots, z\}$, $f(z)$ 在 $C \setminus A$ 解析. 有界 $\Rightarrow f(z) = C$

① 取 $R = \max\{|f(z_1)|, \dots, |f(z_n)|\}$, 当 $R < |z| < +\infty$ 时

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, |a_n| = \left| \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \oint_{|z|=R} \frac{|f(z)|}{|z|^{n+1}} |dz|$$

$$\leq \frac{1}{2\pi} \oint_{|z|=R} \frac{M}{R^{n+1}} ds = \frac{M}{2\pi R^{n+1}} \cdot 2\pi R = \frac{M}{R^n}, \forall R > M$$

$$\text{令 } R \rightarrow +\infty, \text{ 则 } |a_n| \leq 0, a_n = 0$$

② 当 $n=1$ 时, 由复合闭路定理, 取足够小的 $\varepsilon, \varepsilon < |z|$, $|z-z_1| = \varepsilon$ 内有一个孤立点

$$|a_n| = \left| \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \oint_{|z|=R} \frac{M}{R^{n+1}} ds = \frac{M}{R^n}$$

$$n \geq 1, \text{ 故 } \sum_{n=1}^{\infty} M R^{-n} = M M R^{-1}, \text{ 当 } \varepsilon \rightarrow 0, a_n \rightarrow 0$$

故 $f(z) = C_0$, 又因 $f(z)$ 解析 $\Rightarrow f(z)$ 为常数

练习5 $|f(z)| \leq |z|^m \Rightarrow f(z) = k z^m, |k| \leq 1$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

① 考虑 $n > m, m-n \leq -1$

$$|a_n| = \left| \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \oint_{|z|=R} \frac{|f(z)|}{|z|^{n+1}} |dz| = \frac{1}{2\pi} \oint_{|z|=R} \frac{r^{m-n}}{r^{n+1}} ds = r^{m-n} \leq r^{-1}$$

$$\text{令 } r \rightarrow +\infty, a_n = 0$$

② 考虑 $n < m, m-n \geq 1$

$$|a_n| = r^{m-n}, \text{ 令 } r \rightarrow 0, a_n = 0$$

$$故 f(z) = a_m z^m$$

$$\text{而 } |a_m z^m| \leq |z|^m \Rightarrow |a_m| \leq 1$$

$$故 f(z) = k z^m, |k| \leq 1$$

Chap5 留数

§1 孤立奇点

Def $z_0 \in C$ 称为 $f(z)$ 的孤立奇点, 若 z_0 为 $f(z)$ 奇点, 且 $f(z)$ 在某个 $B(z_0)$ 内解析 (不能连成一片)



Ex. $\frac{\sin z}{z}, \frac{\sin z}{z^2}, e^{\frac{1}{z}}$

1° 孤立奇点分类

设 $W=f(z)$, $z_0 \in C$ 为孤立奇点

$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, 0 < |z-z_0| < \delta$

I. 可去奇点 若 Laurent 展式中不含负幂项

$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, B_\delta(z_0)$

若 $f(z) = C_0$, 若补充定义, $f(z_0) = C_0$, 则 $f(z)$ 在 $B_\delta(z_0)$ 上解析

Ex $\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots, 0 < |z| < +\infty$

补充定义, $f(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}, f(z) \in C^{\infty}(R)$

$f(z)$ 是整函数

(2) $\frac{\cos(z+z_0)}{z} = 1 - \frac{1}{2}z + \frac{1}{24}z^3 - \dots, 0 < |z| < 1$

$f(z) = \begin{cases} \frac{\cos(z+z_0)}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$ 在 $|z| < 1$ 解析

Thm: 设 $z_0 \in C$ 为 $f(z)$ 的孤立奇点, 则下列三条件等价:

(1) z_0 为 $f(z)$ 的可去奇点

(2) $\lim_{z \rightarrow z_0} f(z) = A \in C$

★★★☆☆ 考试证明

(3) $f(z)$ 在某个 $B_\delta(z_0)$ 内有界

Proof (1) $\Rightarrow$ (3) $|f(z)| < M < +\infty, \forall z \in B_\delta(z_0)$

$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, B_\delta(z_0) \Rightarrow (n \geq 1)$

$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz, n \geq 1, C: |z-z_0|=r, 0 < r < \delta$

$|a_n| \leq \frac{1}{2\pi} \oint_C \frac{|f(z)|}{|z-z_0|^{n+1}} |dz| \leq \frac{1}{2\pi} \oint_C \frac{M}{r^{n+1}} |dz| = \frac{M}{r^n} = M r^{-n}$

II. m 级 (Pole) 极点 ($m \geq 1$) ($m=1$ 简单极点), 若 Laurent 展式

中含有限个负幂项, 且最高负幂项指数为 $-m$

$f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n, (a_{-m} \neq 0)$

$= \frac{1}{(z-z_0)^m} (C_m + C_{m+1}(z-z_0) + \dots + a_0(z-z_0)^m) + \sum_{n=0}^{\infty} a_{n+m}(z-z_0)^n$

$= \frac{g(z)}{(z-z_0)^m} = \infty$

$g(z)$ 在 $B_\delta(z_0)$ 解析, $g(z_0) = C_m \neq 0$

Ex. $\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \dots, 0 < |z| < +\infty$

$\frac{z}{z^2} = \frac{1}{z}$ 是二阶奇点
一阶奇点 $1/z = 2$

Thm 设 $z_0 \in C$ 为 $f(z)$ 的孤立奇点, 则下列条件等价

(1) z_0 为 $f(z)$ 的 m 级极点

(2) 存在某个 $B_\delta(z_0)$ 内, $f(z) = \frac{g(z)}{(z-z_0)^m}$, 其中 $g(z)$ 在 $B_\delta(z_0)$ 内解析且不取 0

(3) $\lim_{z \rightarrow z_0} (z-z_0)^m f(z)$ 存在

(4) z_0 为 $f(z)$ 的 m 级极点, 在 z 平面上下

$1 \Leftrightarrow 2 \Rightarrow 3 \Leftrightarrow 4$

Proof (1) $\Rightarrow$ (4) $\lim_{z \rightarrow z_0} (z-z_0)^m f(z) = A \neq 0$

置 $g(z) = (z-z_0)^m f(z)$, 则 $\lim_{z \rightarrow z_0} g(z) = A \neq 0$

$\Rightarrow g(z)$ 至少在 $B_\delta(z_0)$ 解析

$\Rightarrow g(z)$ 的解析点 (可去奇点)

$\Rightarrow \frac{1}{f(z)} = (z-z_0)^m \cdot \frac{1}{g(z)} = (z-z_0)^m \cdot q(z), q(z)$ 在某个 $B_\delta(z_0)$ 解析且不取 0

$\Rightarrow z_0$ 为 $f(z)$ 的 m 级极点

Ex $\frac{\cos z}{(z-1)^2} = \frac{1}{(z-1)^2} - \frac{1}{2(z-1)} + \dots$

II. 本性奇点

Laurent 展式中含无穷个负幂项

$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, B_\delta(z_0)$

$\lim_{z \rightarrow z_0} f(z) = ?$

Ex $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots, z \in C^*$

补充

① Weierstrass 定理: 设 $z_0 \in C$ 为 $f(z)$ 的本性奇点

则 $\forall$ 任意 $A \in C$, 都存在 $\{z_n\} \subset B_\delta(z_0)$

s.t. (1) $z_n \rightarrow z_0$

(2) $f(z_n) \rightarrow A$

Picard THM 是 Weierstrass 强化版?

② Picard 定理: 设 $z_0 \in C$ 为 $f(z)$ 的本性奇点

则对 $\forall$ 任意 $A \in C$ (最多有一个例外值 A_0)

都存在 $\{z_n\} \subset B_\delta(z_0)$, s.t.

(1) $z_n \rightarrow z_0$

(2) $f(z_n) \rightarrow A, A \neq A_0$

例外值可等于 ∞ 或 $-\infty$

Ex $e^{\frac{1}{z}} = A \neq 0 \Leftrightarrow \frac{1}{z} = \ln A = a + i2\pi n, n \in Z$

$\Rightarrow z_n = \frac{1}{a + i2\pi n} \rightarrow 0$ 有例外值

$\cos \frac{1}{z}, \sin \frac{1}{z}$ 无例外值

2° 作为奇点

Def: 称 $f(z)$ 孤立奇点. 若 $f(z)$ 在 $0 < |z| < \rho$ 内解析

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad r < |z| < \rho$$

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^n, \quad 0 < |z| < \rho$$

Def: 称 $f(z)$ 为 $z=0$ 的 $\begin{cases} \text{I 型奇点} \\ \text{II 型奇点} \\ \text{III 本性奇点} \end{cases}$

若 $a_0 = 0$ 为 $\varphi(z)$ 的 m 重奇点

$\Rightarrow \begin{cases} \text{I 型奇点} \\ \text{II 型奇点} \\ \text{III 型奇点} \end{cases}$
 以上奇点

总结: 称 $f(z)$ 孤立奇点

则 $f(z)$ 为 $z=0$ 孤立奇点 $\Leftrightarrow \lim_{z \rightarrow 0} f(z) = A \in \mathbb{C}$

1° 极点 $\Leftrightarrow \lim_{z \rightarrow 0} f(z) = \infty$

2° 本性奇点 $\Leftrightarrow \lim_{z \rightarrow 0} f(z)$ 不存在, 亦不为 ∞

$$\text{Ex: } f(z) = e^{\frac{1}{z}} + \frac{1}{z-1} - \frac{1}{2}$$

讨论 $f(z)$ 在 $z=0$ 上所有奇点分类

1° $z=0$, 1 级极点, $\lim_{z \rightarrow 0} (f(z) - \frac{1}{z}) = -\frac{1}{2}$, 可去

2° $z=1$, 本性奇点

3° $z=\infty$, 本性奇点

4° $z=\infty$, 本性奇点

$n \rightarrow \infty$

∞ 孤立 $\Leftrightarrow$ 非 ∞ 有限个

$\rightarrow$ 极点非孤立

$T_s f(z), g(z) \neq 0, f(z_0) = g(z_0) = 0$, 证 $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \lim_{z \rightarrow z_0} \frac{f'(z)}{g'(z)}$ (或两端同为 ∞)

证: 考虑 $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = A$ 有限情况

设 z_0 是 $f(z)$ 的 m 级零点, $g(z)$ 的 n 级零点

$$\text{则 } f(z) = (z-z_0)^m \varphi(z), \quad g(z) = (z-z_0)^n \psi(z)$$

$$f(z) = m(z-z_0)^{m-1} \varphi(z) + (z-z_0)^m \varphi'(z)$$

$$g(z) = n(z-z_0)^{n-1} \psi(z) + (z-z_0)^n \psi'(z)$$

$$\Rightarrow \frac{f(z)}{g(z)} = (z-z_0)^{m-n} \frac{\varphi(z)}{\psi(z)}$$

$$\frac{f'(z)}{g'(z)} = (z-z_0)^{m-n} \frac{m\varphi(z) + (z-z_0)\varphi'(z)}{n\psi(z) + (z-z_0)\psi'(z)}$$

$$\text{其中 } \lim_{z \rightarrow z_0} \frac{\varphi(z)}{\psi(z)} \neq 0, \quad \lim_{z \rightarrow z_0} \frac{m\varphi(z) + (z-z_0)\varphi'(z)}{n\psi(z) + (z-z_0)\psi'(z)} = \frac{m}{n} \frac{\varphi(z)}{\psi(z)} \neq 0$$

$$\text{① } m=n, \quad \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{\varphi(z)}{\psi(z)} = \lim_{z \rightarrow z_0} \frac{f'(z)}{g'(z)}$$

$$\text{② } m < n, \quad \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = 0 = \lim_{z \rightarrow z_0} \frac{f'(z)}{g'(z)}$$

$$\text{③ } m > n, \quad \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \infty = \lim_{z \rightarrow z_0} \frac{f'(z)}{g'(z)}$$

$T_b \varphi(z), \psi(z)$ 以 $z=a$ 为 m, n 级极点 (零点), $z=a$ 时

$$\text{只考虑极点情形, } \varphi(z) = \frac{f(z)}{(z-a)^m}, \quad \psi(z) = \frac{g(z)}{(z-a)^n}$$

$$1) \varphi(z)\psi(z) = \frac{f(z)g(z)}{(z-a)^{m+n}}, \quad \text{为 } m+n \text{ 级极点}$$

$$2) \frac{f'(z)}{g'(z)} = (z-a)^{m-n} \frac{f'(z)}{g'(z)}, \quad \begin{cases} m > n, m-n \text{ 级极点} \\ m < n, n-m \text{ 级零点} \\ m=n, \text{可去奇点} \end{cases}$$

$$3) \varphi(z) + \psi(z) = (z-a)^{-n} f(z) + (z-a)^{-m} g(z)$$

$$\text{① } m \neq n, \text{ 为 } \max(m, n) \text{ 级极点}$$

$$\text{② } m=n, \quad \varphi(z) + \psi(z) = \frac{f(z) + g(z)}{(z-a)^m}$$

$$\text{设 } f(z) + g(z) = (z-a)^k r(z), \quad r(z_0) \neq 0$$

则 $k \geq m$, 为可去奇点

$k < m$, 为 $m-k$ 级极点

练习1

$$(1) \frac{z}{(1+iz)^3(1+iz)^3}$$

若 $1+iz=0$, $z=i$ 为根极点

若 $1+iz=0$, $z=i$

若 $z=0$

诱导考虑

① $z=i$ 为 3 级根极点

② $z=0$, 为 5 级根极点

③ $z=0$, 为非孤立奇点

$$(2) \frac{1}{z} - \frac{1}{\sin z} + \frac{\sin z}{(z-z)^2}$$

① 0 , 非孤立奇点

② $1/z$, $1/z \neq 0$ 1 级根极点

③ $z=0$, 可有根点 $[1/z - \sin z] \rightarrow 0$

④ $z=2$, $5-1=4$ 级根极点

练习2 (Weierstrass) 若 f 为 $f(z)$ 本性奇点, $\forall A \in \mathbb{C}$

$\exists$ 无穷多 $z_n \in \mathbb{C} \setminus \{z_0\}$, s.t. (1) $\lim_{n \rightarrow \infty} z_n = z_0$ (2) $\lim_{n \rightarrow \infty} f(z_n) = A$

证: 假若结论不成立

若 $A \in \mathbb{C}$, 则 $\exists b > 0$, s.t. $|f(z) - A| > b$, $\forall z \in \mathbb{C}(z_0)$

$$\text{则 } 0 < \frac{1}{|f(z) - A|} < b$$

① $f(z_0) = 0$, 若 $f(z_0) = A$ 可开奇点, 这方不是 $f(z)$ 本性奇点矛盾

因为 $f(z) - A$ 为 $f(z)$ 的 Laurent 展开无奇数项

② $f(z_0) \neq 0$, $\exists M > 0$, $|f(z)| < M$, $\forall z \in \mathbb{C}(z_0)$

则 $f(z)$ 为可开奇点, 矛盾

练习3 若 $f(z)$ 孤立奇点, $m, k \in \mathbb{Z}^+$, $m < k$

(1) $\lim_{z \rightarrow z_0} f(z) = 0 \Rightarrow$ 可开

(2) $\lim_{z \rightarrow z_0} (z-z_0)^k f(z) = A \in \mathbb{C} \setminus \{0\} \Rightarrow$ 若 $(k-m)$ 级根极点

证: (1) $f(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n + C_0 + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$

$$f^{(m)}(z) = \sum_{n=1}^{\infty} a_n n(n-1)\dots(n-m+1)(z-z_0)^{n-m} + C_m + \sum_{n=1}^{\infty} a_{-n}(-1)^m n(n-1)\dots(n-m+1)(z-z_0)^{-n-m}$$

若 $z \rightarrow z_0$, 若 $C_0, \dots, C_{m-1}$ 不全为 0

$$\sum_{n=0}^{\infty} a_n n(n-1)\dots(n-m+1)(z-z_0)^{n-m} \text{ 为 } 0 \text{ 或不存在}$$

$$\text{则 } C_0 = \dots = C_{m-1} = 0$$

则 $f(z) = \sum_{n=m}^{\infty} a_n(z-z_0)^n$, 由根度定义, 可开

(2) $\lim_{z \rightarrow z_0} (z-z_0)^k f(z) = A \neq 0$

$$(z-z_0)^k f^{(m)}(z) = \sum_{n=m}^{\infty} a_n n(n-1)\dots(n-m+1)(z-z_0)^{n-m+k} + C_{m+k} + \dots$$

乘根度, $C_{m+k} = \dots = C_{m+1} = 0$

而 $(z-z_0)^k f^{(m)}(z) = C_{m+k} \neq 0$, 且 $m < k$, 由根度

$\Rightarrow$ 若 $(k-m)$ 级根极点

§2 留数

1° 留数定义

设 z_0 为 $f(z)$ 的孤立奇点

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad B_0^+(z_0)$$

$$C_1 = \frac{1}{2\pi i} \oint_{\gamma} f(z) dz \quad (C \in B_0^+(z_0))$$

Def: 上述 C_1 称为 $f(z)$ 的留数

若 f 为 $f(z)$ 的孤立奇点, $C_1 \in \text{Res}[f(z), z_0]$ ($\text{Res}[f(z), z_0]$)

For 若 $z_0 \in \mathbb{C}$ 为 $f(z)$ 的可开奇点, 则 $\text{Res}[f(z), z_0] = 0$

留数定理: 设 $w = f(z)$ 在单连通区域 D 内除有限个孤立奇点 $z_1, \dots, z_n$ 外解析

$C \subset D$ 为一条环绕所有奇点的 Jordan 闭曲线, 则

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}[f(z), z_k]$$

$$\oint_C f(z) dz = \oint_C \sum_{k=1}^n \text{Res}[f(z), z_k] dz = 2\pi i \sum_{k=1}^n \text{Res}[f(z), z_k]$$

2° 留数计算规则

Rule I 设 $z_0 \in \mathbb{C}$ 为 $f(z)$ 的一级(简单)根点, 则 $\text{Res}[f(z), z_0] = \lim_{z \rightarrow z_0} (z-z_0)f(z)$

$$\because f(z) = \frac{g(z)}{h(z)} + a + \dots, \quad B_0^+(z_0)$$

Rule II 设 $f(z) = \frac{P(z)}{Q(z)}$, 其中 $P(z), Q(z)$ 在 $z_0 \in \mathbb{C}$ 解析, $P(z_0) \neq 0, Q(z_0) = 0$

且 $Q'(z_0) \neq 0$, 则 z_0 为简单根点, 且 $\text{Res}[f(z), z_0] = \frac{P(z_0)}{Q'(z_0)}$ P, Q 均非 0

$$f(z) = \frac{P(z)}{Q(z)}, \quad \text{Res}[f(z), z_0] = \lim_{z \rightarrow z_0} (z-z_0) \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q'(z_0)} = \frac{P(z_0)}{Q'(z_0)}$$

Rule III 设 $z_0 \in \mathbb{C}$ 为 $f(z)$ 的 $m(m \geq 1)$ 级根点, 则 $\text{Res}[f(z), z_0] = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$

$$f(z) = \frac{C_m}{(z-z_0)^m} + \dots + \frac{C_1}{(z-z_0)} + \dots \quad B_0^-(z_0)$$

$$(z-z_0)^m f(z) = C_m + C_{m-1}(z-z_0) + \dots + C_1(z-z_0)^{m-1} + \dots$$

$$\text{求 } k+1 \text{ 阶导, } \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] = (m-1)! C_1$$

$$E_2(1) f(z) = \frac{1}{z^2}$$

$$\oint_{\gamma} f(z) dz = \dots = 2\pi i \text{Res}[f(z), 0]$$

$$C_1 = \text{Res}[f(z), 0]$$

$$\textcircled{1} \frac{1}{z^2} = \frac{1}{z} (z^{-1}) = \frac{1}{z} (z^{-1} + z^{-1} + \dots)$$

$$\Rightarrow C_1 = \frac{1}{1}$$

② $m=3, k=1$

$$C_1 = \frac{1}{2!} \left(\frac{d^2}{dz^2} (z^{-3}) \right)_{z=0} = \frac{1}{2!} \left(\frac{d^2}{dz^2} \frac{1}{z^3} \right)_{z=0}$$

③ $k=6$

$$C_1 = \frac{1}{5!} \left(\frac{d^5}{dz^5} \frac{1}{z^6} \right)_{z=0} = \frac{1}{5!} (-6 \cdot 5 \cdot 4 \cdot 3 \cdot 2) \frac{1}{z^6} = -\frac{1}{1}$$

$$\textcircled{2} f(z) = \frac{\cos z}{z^2}, \quad I = \oint_{\gamma} f(z) dz$$

$$\text{原式} = 2\pi i (\text{Res}[f(z), 0]) + 2\pi i (\text{Res}[f(z), 1])$$

$$\text{Res}[f(z), 0] = 1$$

$$\text{Res}[f(z), 1] = \left(\frac{d}{dz} \frac{\cos z}{z^2} \right)_{z=1}$$

$$= \frac{d}{dz} \left(\frac{\cos z}{z^2} \right)_{z=1} = \frac{-\sin z}{z^2} - \frac{2 \cos z}{z^3} = -\frac{1}{1} - \frac{2}{1} = -\frac{3}{1}$$

$3^0 \infty$ 为孤立奇点

设 0 为 $f(z)$ 孤立奇点

$f(z) = \sum_{n=0}^{\infty} a_n z^n, 0 < |z| < \infty$

$C_1 = \frac{1}{2\pi i} \oint_{C_1} f(z) dz$



Def $\text{Res}[f(z), \infty] = -C_1$

全函数定理: 设 w 在 $f(z)$ 上除去有限个孤立奇点 (包括 ∞) 外解析

则 $f(z)$ 在所有奇点处留数之和为 0

$\oint_{C_1} f(z) dz = 2\pi i \sum_{n=1}^{\infty} \text{Res}[f(z), a_n]$

$\oint_{C_2} f(z) dz = 2\pi i \text{Res}[f(z), \infty]$



$$\text{Res}_{\infty} f = -\sum \text{Res}_{\text{other}}$$

Rule IV $\text{Res}[f(z), \infty] = \text{Res}[-\frac{1}{z}f(\frac{1}{z}), 0]$

$f(z) = \sum_{n=0}^{\infty} a_n z^n, 0 < |z| < \infty$

$\frac{1}{z}f(\frac{1}{z}) = \sum_{n=0}^{\infty} a_n z^{-n} = \sum_{n=0}^{\infty} c_n z^n, 0 < |z| < 1$

$\Rightarrow \frac{1}{z}f(\frac{1}{z}) = \sum_{n=0}^{\infty} c_n z^n \Rightarrow C_1 = \text{Res}[\frac{1}{z}f(\frac{1}{z}), 0] = \text{Res}[-\frac{1}{z}f(\frac{1}{z}), 0] = \text{Res}[f(z), \infty]$

Ex: (1) $I = \oint_{|z|=1} \frac{z dz}{z^2 - 1}, n=2, f(z) = \frac{z}{z^2 - 1} \Rightarrow a_1, \dots, a_n$

(1) $\text{Res} = 2\pi i \sum_{n=1}^{\infty} \text{Res}[\frac{z}{z^2 - 1}, a_n]$

$$= 2\pi i \sum_{n=1}^{\infty} \frac{z}{z^2 - 1} \Big|_{z=a_n} = \frac{2\pi i}{n} \sum_{n=1}^{\infty} \frac{a_n}{a_n^2 - 1} = 2\pi i \sum_{n=1}^{\infty} \frac{a_n}{a_n^2 - 1} = \begin{cases} 2\pi i, n=2 \\ 0, n \neq 2 \end{cases} \text{ (重数)}$$

(2) $-2\pi i \text{Res}[\frac{z}{z^2 - 1}, \infty] = 2\pi i \text{Res}[\frac{1}{z}f(\frac{1}{z}), 0]$

$= 2\pi i \text{Res}[\frac{1}{z}f(\frac{1}{z}), 0] = 2\pi i \text{Res}[\frac{1}{z}f(\frac{1}{z}), 0]$

(2) $f(z) = \frac{z}{z^2 - 1}, I = \oint_{|z|=1} f(z) dz$

$f(z) = \sum_{n=0}^{\infty} a_n z^n, 0 < |z| < \infty \Rightarrow \frac{1}{z}f(\frac{1}{z}) = \sum_{n=0}^{\infty} c_n z^n, 0 < |z| < 1$

$\Rightarrow -2\pi i \text{Res}[f(z), \infty] = 2\pi i \text{Res}[\frac{1}{z}f(\frac{1}{z}), 0] = 2\pi i \text{Res}[\frac{1}{z}f(\frac{1}{z}), 0]$

$= 2\pi i \text{Res}[\frac{1}{z}f(\frac{1}{z}), 0] = 2\pi i \text{Res}[\frac{1}{z}f(\frac{1}{z}), 0]$

(3) $f(z) = \frac{1}{(z-1)^2(z-2)^2}$

$I = \oint_{|z|=2} f(z) dz$

$\oint_{|z|=2} 2\pi i [\text{Res}[f(z), -1] + \text{Res}[f(z), 1]]$

$\oint_{|z|=2} 2\pi i [\text{Res}[f(z), 2] + \text{Res}[f(z), \infty]]$

$\text{Res}[f(z), \infty] = -\text{Res}[\frac{1}{z}f(\frac{1}{z}), 0]$

$= -\text{Res}[\frac{1}{z}f(\frac{1}{z}), 0] = 0$

$= -\text{Res}[\frac{1}{z}f(\frac{1}{z}), 0] = 0$

Thm $f(z) = \frac{P(z)}{Q(z)}$, 有理函数, $m < n \Rightarrow \text{Res}[f(z), \infty] = 0$

$f(z)$ 以 ∞ 为可去奇点 $\Rightarrow \text{Res}[f(z), \infty] = 0, f(z) = \frac{1}{z}$

$$= \int_0^{\frac{\pi}{2}} \frac{d\theta}{b^2 \sin^2 \theta} + \int_{\frac{\pi}{2}}^{\pi} \frac{d\theta}{b^2 \sin^2 \theta}$$

$$\int_{-\pi}^{\pi} \frac{d\theta}{b^2 \sin^2 \theta} = \int_0^{\pi} \frac{d\theta}{b^2 \sin^2 \theta} + \int_{\pi}^0 \frac{d\theta}{b^2 \sin^2 \theta}$$

$$= \int_0^{\pi} \frac{2b d\theta}{b^2 \sin^2 \theta} = \frac{1}{b} \int_0^{\pi} \frac{2b d\theta}{\sin^2 \theta} = \frac{1}{b} \int_0^{\pi} \frac{2b d\theta}{\sin^2 \theta}$$

所有函数都有无穷奇点, 只是未必连续

§3. 留数在定积分中的应用

1^0 形如 $I = \int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta$

$R = R(x, y)$ 是二元有理函数 $= \frac{P(x, y)}{Q(x, y)}$, P, Q 是二元多项式

$z = |z|=1 \Rightarrow z = \cos \theta + i \sin \theta = x + iy$

$\begin{cases} x = \cos \theta \\ y = \sin \theta \end{cases} \Rightarrow \begin{cases} \bar{z} = \cos \theta - i \sin \theta \end{cases}$

$$\Rightarrow \cos \theta = \frac{z + \bar{z}}{2} = \frac{z + \frac{1}{z}}{2}, \sin \theta = \frac{z - \bar{z}}{2i} = \frac{z - \frac{1}{z}}{2i}$$

$$d\theta = d e^{i\theta} = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$$I = \oint_{|z|=1} R(\frac{z + \frac{1}{z}}{2}, \frac{z - \frac{1}{z}}{2i}) \frac{dz}{iz}, f(z) = R(\frac{z + \frac{1}{z}}{2}, \frac{z - \frac{1}{z}}{2i}) \frac{1}{iz}$$

要求 $f(z)$ 在 $|z|=1$ 无极点 $\Rightarrow I = 2\pi i \sum \text{Res}[f(z), a_n]$, a_n 表示 $f(z)$ 在 $|z| < 1$ 内所有极点

Ex: (1) $I(A) = \int_0^{2\pi} \frac{d\theta}{A - \cos \theta}, A > 1 = \oint_{|z|=1} \frac{dz}{A - \frac{z + \frac{1}{z}}{2}} = -\frac{2}{i} \oint_{|z|=1} \frac{dz}{2A - z - \frac{1}{z}} = -\frac{2}{i} \cdot 2\pi i \text{Res}[\frac{1}{2A - z - \frac{1}{z}}, 0]$

$$= -4\pi \frac{1}{(2A - 1)^2} = -4\pi \frac{1}{2^2 (A - \frac{1}{2})^2} = \frac{2\pi}{(A - \frac{1}{2})^2} \cos 2\theta = (e^{2i\theta} + e^{-2i\theta}) \frac{1}{2}$$

$$(2) I(P) = \int_0^{2\pi} \frac{\cos(2\theta) d\theta}{1 + P^2 - 2P \cos \theta} d\theta (0 < P < 1)$$

$$\text{令 } \tilde{I}(P) = \int_0^{2\pi} \frac{\sin(2\theta) d\theta}{1 + P^2 - 2P \cos \theta} d\theta$$

$$I = I(P) + i\tilde{I}(P) = \oint_{|z|=1} \frac{z^2 + \frac{1}{z^2}}{1 + P^2 - 2P \cos \theta} dz$$

$$= \oint_{|z|=1} \frac{z^2 + \frac{1}{z^2}}{1 + P^2 - P(z + \frac{1}{z})} dz = \frac{1}{i} \oint_{|z|=1} \frac{z^2 + \frac{1}{z^2}}{(1 + P^2 - Pz - \frac{P}{z})} dz = -\frac{1}{i} \oint_{|z|=1} \frac{z^2 + \frac{1}{z^2}}{(1 + P^2 - Pz - \frac{P}{z})} dz$$

$$= \frac{2\pi P^2}{1 - P^2} \Rightarrow \begin{cases} I(P) = \frac{2\pi P^2}{1 - P^2} \\ \tilde{I}(P) = 0 \end{cases}$$

2^0 形如 $I = \int_{-\infty}^{\infty} R(x) dx, R(x) = \frac{P(x)}{Q(x)}$ 有理函数

要求: $R(x) = \frac{P(x)}{Q(x)}$ 在 $\mathbb{R}$ 上无极点, a_n 为 $R(x)$ 在 $\mathbb{C}$ 上所有极点

$I = 2\pi i \sum_{n=1}^{\infty} \text{Res}[R(x), a_n]$

$= -2\pi i \sum_{n=1}^{\infty} \text{Res}[R(x), \bar{a}_n]$

$(\Leftrightarrow \sum_{n=1}^{\infty} \text{Res}[R(x), a_n] = 0)$

$\oint_{|z|=2} R(z) dz = \int_{-\infty}^{\infty} R(x) dx + \int_{\gamma} R(z) dz$

$(\gamma \rightarrow \infty) = 2\pi i \sum_{n=1}^{\infty} \text{Res}[R(z), a_n]$

$|R(z)| = \left| \frac{a_n z^n + \dots + a_0}{b_m z^m + \dots + b_0} \right| = \frac{1}{|z|^m} \left| \frac{a_n z^n + \dots + a_0}{b_m + \dots + \frac{b_0}{z^m}} \right|$

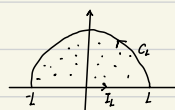
$\Rightarrow |R(z)| \leq \frac{1}{|z|^m}$

$\Rightarrow \int_{\gamma} R(z) dz \leq \int_{\gamma} |R(z)| |dz| \leq \int_{\gamma} \frac{1}{|z|^m} |dz| \leq \frac{2\pi}{1-m} \int_{\gamma} \frac{1}{|z|^m} |dz| \xrightarrow{m \rightarrow \infty} 0$

Ex: $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}, I_1 = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}, I_2 = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}, n=1$

$= 2\pi i (\text{Res}[\frac{1}{1+z^2}, i] + \text{Res}[\frac{1}{1+z^2}, -i])$

$= 2\pi i (\frac{1}{2i} - \frac{1}{-2i}) = \frac{2\pi}{2} = \pi$



$$\gamma = I_1 + C_1$$

3° 计算 $I = \int_{-\infty}^{\infty} R(x) e^{iax} dx \quad (a \neq 0, a \in \mathbb{R})$

$R(x) = \frac{1}{x^2 + 1}$

$R(x)$ 在 $\mathbb{R}$ 上无极点 且 $n=2$

$I = \begin{cases} 2\pi i \sum_{\substack{z_k \in \mathbb{C} \\ \text{Im } z_k > 0}} \text{Res}[R(z)e^{iaz}, z_k], & a > 0 \\ -2\pi i \sum_{\substack{z_k \in \mathbb{C} \\ \text{Im } z_k < 0}} \text{Res}[R(z)e^{iaz}, z_k], & a < 0 \end{cases}$ a > 0 上面
a < 0 下面

仍有 $\oint_{\Gamma} R(z) e^{iaz} dz = \int_{-1}^1 R(x) e^{iax} dx + \int_{\Gamma} R(z) e^{iaz} dz$

$|\oint_{\Gamma} R(z) e^{iaz} dz| \leq \int_{\Gamma} |R(z)| |e^{iaz}| |dz|$
 $\leq \frac{2A}{L} \int_{\Gamma} e^{-ay} ds = \frac{2A}{L} \int_0^{2\pi} e^{-aL \sin \theta} L d\theta = \frac{2A}{L^{a-1}} \int_0^{2\pi} e^{-aL \sin \theta} d\theta$

熟知不等式 $\sin \theta \leq \sin \theta, \theta \in [0, \pi]$

a > 0 很重要

$= \frac{2A}{L^{a-1}} \int_0^{2\pi} e^{-aL \sin \theta} d\theta \leq \frac{2A}{L^{a-1}} \int_0^{2\pi} e^{-\frac{2a}{\pi} \theta} d\theta = \frac{2A\pi}{aL^{a-1}} e^{-\frac{2a}{\pi} \theta} \Big|_0^{2\pi} = \frac{2A\pi}{aL^{a-1}} (1 - e^{-aL}) \xrightarrow{L \rightarrow \infty} 0$

$I = \int_{-\infty}^{\infty} R(x) e^{iax} dx = \int_{-\infty}^{\infty} R(x) \cos ax dx + \int_{-\infty}^{\infty} R(x) \sin ax dx \quad (b > 0)$

$\mathcal{E}_x I(b) = \int_{-\infty}^{\infty} \frac{x \sin bx}{x^2 + 1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} dx = \frac{1}{2} \mathcal{I}_m J(b), J(b) = \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + 1} dx$

$J(b) = 2\pi i \text{Res} \left[\frac{x e^{ix}}{x^2 + 1}, bi \right] = 2\pi i \cdot \frac{b e^{-b}}{2bi} = i\pi e^{-b}$

$\Rightarrow I = \frac{1}{2} \pi e^{-b}$

$\frac{1}{2} = \lim_{b \rightarrow 0} I(b) = \frac{1}{2} \lim_{b \rightarrow 0} \int_{-\infty}^{\infty} \frac{x \sin bx}{x^2 + 1} dx$ PDS

$= \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} dx = \int_{-\infty}^{\infty} \frac{x \sin x}{x} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \frac{1}{2} \pi$

$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \mathcal{I}_m \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx$

$f(z) = i \cos \frac{z}{2-1} + \frac{1}{z} - \frac{1}{\sin z}$

留数

无穷远点留数: 若 $\exists R, f(z)$ 在 $|z| > R$ 是主解析.

那么无穷远点对应一个孤立奇点, 存在对应留数

Que $\int_0^{\infty} \frac{x^2 dx}{e^x - 1} = \frac{\pi^2}{15}$

复变函数求实值函数积分 $\Rightarrow$ 在某个复轴上积分, 最后消复数积分部分即可

正向曲线是逆时针