

$$f(x) = \begin{cases} e^{-x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f'(x) = \begin{cases} -\frac{2}{x^3} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f''(x) = \begin{cases} (\frac{2}{x^5} + \frac{6}{x^4}) e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

归纳法, $f^{(n)}(x) = \begin{cases} e^{-\frac{1}{x^2}} (\frac{A_1}{x^{m_1}} + \dots + \frac{A_n}{x^{m_n}}), & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$\lim_{x \rightarrow 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = f^{(n+1)}(0) = 0.$$

$$\lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}} (\frac{A_1}{x^{m_1}} + \dots + \frac{A_n}{x^{m_n}})}{x} \quad \text{令 } t = \frac{1}{x}$$

$$= \lim_{t \rightarrow \infty} \frac{A_1 t^{m_1+1} + \dots + A_n t^{m_n+1}}{e^{t^2}} = 0.$$

$$f^{(n)}(0) = 0, \quad \forall n \in \mathbb{N}^+.$$

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0) x^n}{n!} = 0$$

$$f(z) = \begin{cases} e^{-\frac{1}{z^2}}, & z \neq 0 \\ 0, & z = 0 \end{cases} \quad \text{不解析}$$

$$\text{令 } z = yi, y \in \mathbb{R}. \quad f(yi) = \begin{cases} e^{\frac{1}{y^2}}, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

$f(z)$ 在 $z=0$ 点解析.

例1. $f(z) = |x|^\alpha x, \alpha > 0$ - 奇次多项式的推广 (当 $\alpha \in \mathbb{N}$ 时, $f(x)$ 是 $\alpha+1$ 次奇次多项式)

$$\text{例 } \alpha = \frac{1}{2}, f_\alpha(x) = |x|^{\frac{1}{2}} x = \begin{cases} x^{\frac{3}{2}}, & x \geq 0 \\ -|x|^{\frac{3}{2}}, & x < 0 \end{cases} \quad \text{在 } z=0 \text{ 处不解析}$$

同理 $\alpha = \frac{3}{2}$ 时 $f_\alpha(x) = |x|^{\frac{3}{2}} x$
 $\vdots$
 $\alpha = 3$ 时 $f_\alpha(x) = |x|^3 x$ 均在原点 $z=0$ 不解析

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad f \in C^\infty(\mathbb{R})$$

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \quad |x| < 1.$$

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{Euler公式}$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\begin{cases} \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \end{cases}$$

$$\text{令 } z = \theta, \quad \begin{cases} \cos z = \frac{e^{iz} + e^{-iz}}{2} \\ \sin z = \frac{e^{iz} - e^{-iz}}{2i} \end{cases}$$

$$\begin{aligned} \cos(x+iy) &= \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \frac{e^{ix} e^{-y} + e^{-ix} e^y}{2} \\ &= \frac{(\cos x + i\sin x) e^{-y} + (\cos x - i\sin x) e^y}{2} \\ &= \frac{e^{-y} + e^y}{2} \cos x + i \frac{e^{-y} - e^y}{2} \sin x \end{aligned}$$

$$\begin{aligned} 2i \sin(x+iy) &= e^{i(x+iy)} - e^{-i(x+iy)} \\ &= (\cos x + i\sin x) e^{-y} - (\cos x - i\sin x) e^y \\ &= \cos x (e^{-y} - e^y) + i(\sin x (e^{-y} + e^y)) \\ \sin(x+iy) &= \frac{e^{-y} + e^y}{2} \sin x + i \frac{e^{-y} - e^y}{2} \cos x. \end{aligned}$$

$$\begin{aligned} |\cos(x+iy)| &= \sqrt{\left(\frac{e^{-y} + e^y}{2} \cos x\right)^2 + \left(\frac{e^{-y} - e^y}{2} \sin x\right)^2} \\ &= \sqrt{\frac{e^{-2y} + 2 + e^{2y}}{4} \cos^2 x + \frac{e^{-2y} - 2 + e^{2y}}{4} \sin^2 x} \\ &= \sqrt{\frac{e^{-2y} + e^{2y}}{4} \cos^2 x - \sin^2 x} \end{aligned}$$

$$\cos z \in \mathbb{C} \quad \cos z = A + Bi, \quad A, B \in \mathbb{R}$$

有无穷多解

$$\begin{aligned} \sin(x+iy) &= A + Bi \\ \begin{cases} \frac{\sin x}{2} (e^{-y} + e^y) = A \\ \frac{e^y - e^{-y}}{2} \cos x = B \end{cases} &\Rightarrow \begin{cases} \frac{\sin x}{2} = \frac{A}{e^{-y} + e^y} \\ \frac{\cos x}{2} = \frac{B}{e^y - e^{-y}} \end{cases} \end{aligned}$$

$$\frac{1}{4} = \frac{A^2}{(e^{-y} + e^y)^2} + \frac{B^2}{(e^y - e^{-y})^2}$$

在 $z=0$ 处 $h(y) \Rightarrow h(y) \rightarrow +\infty, y \rightarrow 0 (B \neq 0)$
 $y \rightarrow +\infty, h(y) \rightarrow 0$
 $y \rightarrow -\infty, h(y) \rightarrow 0$

例 $f(z) = e^{ax}(\cos by + i \sin by) = e^{ax+iby}$

$f^{(n)}(z) = a^n(u+v) = a^n f(z)$

$u = e^{ax} \cos by, v = e^{ax} \sin by$

$\frac{\partial^n u}{\partial x^n} = a^n e^{ax} \cos by = a^n u$

$\frac{\partial^n v}{\partial x^n} = a^n e^{ax} \sin by = a^n v$

$f(z) = x^2 + y^2 = |z|^2 = z\bar{z} = u, v=0$

$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \Rightarrow x=0 \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0 \Rightarrow y=0 \end{cases}$

只在 $z=0$ 点可导。实部和虚部所具阶可导，复函数在点可导 $\max|z^n+\alpha| = r^n + |\alpha|$ 且

$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

可证 $f \in C^\infty(\mathbb{R}) \Leftrightarrow f^{(n)}(x)$ 连续, $\forall n \in \mathbb{N}$

$n=0, f^{(0)}(0)$ 存在 $f^{(n)}(0) = 0$
 $f'(x) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x}$

令 $t = \frac{1}{x}, \Rightarrow x = \frac{1}{t}, \Rightarrow t \rightarrow \infty$

$= \lim_{x \rightarrow 0} t e^{-t^2} = \lim_{x \rightarrow 0} \frac{t}{e^{t^2}} = 0$

$n=1, f^{(1)}(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f'(x)}{x}$
 $= \lim_{x \rightarrow 0} \frac{2e^{-\frac{1}{x^2}}}{x^4}$ 令 $t = \frac{1}{x^2} \Rightarrow t \rightarrow +\infty$

Taylor 级数展开: $e^{-\frac{1}{x^2}} = f(x) / \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n \equiv 0$

最大模原理

$f(z)$ 在 D 处处可导, 则 $|f(z)|$ 必在 D 的边界上取得

其最大模, 即 $\max_{z \in D} |f(z)| = \max_{z \in \partial D} |f(z)|$

求 $\max_{|z|=r} |z^n + \alpha|$

由 $|z^n + \alpha| \leq |z^n| + |\alpha| \leq r^n + |\alpha|$
 $\Leftrightarrow \max |z^n + \alpha| \leq r^n + |\alpha|$

等式(1)成立 $\Leftrightarrow |z^n + \alpha| = |z^n| + |\alpha|$

$\Leftrightarrow z^n$ 与 α 同向, 即存在 $k \geq 0$, 使 $z^n = k\alpha$

Euler $e^{i\theta} = \cos \theta + i \sin \theta \quad \theta = by$

等式(2)成立 $\Leftrightarrow |z| = r$

① $\alpha = 0$ 时, $|z^n + \alpha| = |z^n| \leq r^n$ 且等号成立 $\Leftrightarrow |z| = r$

$\Leftrightarrow z = r e^{i\theta}, \theta \in [0, 2\pi)$ 或 $\theta \in \partial D$

$\max |z^n + \alpha| = r^n$

② $\alpha \neq 0$ 时, $z^n = k\alpha, |z| = r$, 两边取模 $|z^n| = |k\alpha| = |k||\alpha|$

$k = \frac{r^n}{|\alpha|} \Rightarrow z^n = \frac{r^n}{|\alpha|} \alpha = \frac{r^n}{|\alpha|} |\alpha| e^{i \arg \alpha}$

$\Rightarrow z = r e^{i \frac{\arg \alpha + 2(k-1)\pi}{n}} = r e^{i \arg \alpha}$
 $k = 1, 2, \dots, n$

$z_1 + z_2 + z_3 = 0, |z_i| = r > 0, i = 1, 2, 3$

M2: $(z-z_1)(z-z_2)(z-z_3) = (z^3 - (z_1+z_2+z_3)z^2 + (z_1z_2+z_2z_3+z_3z_1)z - z_1z_2z_3)$

$z_1z_2+z_2z_3+z_3z_1 = z_1z_2z_3(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3})$
 $z_i \bar{z}_i = |z_i|^2 = r^2$
 $\frac{1}{z_i} = \frac{\bar{z}_i}{r^2}$
 $= z_1z_2z_3 \cdot \frac{1}{r^2} (\bar{z}_1 + \bar{z}_2 + \bar{z}_3)$
 $= z_1z_2z_3 \frac{1}{r^2} \overline{z_1+z_2+z_3} = 0$

$z^3 - z_1z_2z_3 = 0$
 $z^3 = z_1z_2z_3 = r^3 e^{i(\theta_1+\theta_2+\theta_3)}$
 $\theta_1+\theta_2+\theta_3 = \alpha$
 $z^3 = r^3 e^{i\alpha} \quad z = z_k = r e^{i \frac{\alpha + 2(k-1)\pi}{3}}, k=1, 2, 3$

推广: $|z_k| = r, k=1, 2, 3, 4$

$z_1 + z_2 + z_3 + z_4 = 0$

结论: 四点 z_1, z_2, z_3, z_4 是圆内接正方形

$\prod_{k=1}^4 (z - z_k) = z^4 - (\sum_{k=1}^4 z_k) z^3 + (\sum_{1 \leq m < n \leq 4} z_m z_n) z^2 - (\sum_{1 \leq k < l < m < n \leq 4} z_k z_l z_m z_n) z$
 $+ \prod_{k=1}^4 z_k$

$= z^4 + az^2 + b = 0 \Leftrightarrow \begin{cases} z_3 = -z_1 \\ z_4 = -z_2 \end{cases}$

推广: $|z_k| = r, k=1, 2, \dots, n$

$\sum_{k=1}^n z_k = 0$

从单位圆出发

$$\Rightarrow \begin{cases} A = e^x \cos y \\ B = e^x \sin y \end{cases} \Rightarrow A^2 + B^2 = e^{2x} > 0 \quad x = \frac{1}{2} \ln(A^2 + B^2)$$

多项式 $P_n(z) = \sum_{k=0}^n a_k z^k \quad (a_n \neq 0)$

$$= a_n \prod_{k=1}^n (z - z_k) \quad [\text{代数基本定理}]$$

$$\sin z = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!} \quad z \in \mathbb{C} \quad \left. \begin{array}{l} \text{周期性} \\ \text{零点} \end{array} \right\}$$

$$= z \prod_{n=1}^{+\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right) \quad \text{无穷积形式}$$

$$= z \prod_{n=1}^{+\infty} \frac{(n\pi - z)(n\pi + z)}{n^2 \pi^2}$$

$$\cos z = \sum_{n=0}^{+\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \prod_{n=1}^{+\infty} \left(1 - \frac{z^2}{(n\pi)^2}\right)$$

$$\frac{\sin x}{x} = \prod_{n=1}^{+\infty} \left(1 - \frac{x^2}{n^2 \pi^2}\right) \leq 1$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \left| \frac{\sin x}{x} \right| \leq 1 \text{ 当且仅当 } x=0$$

$$\sin(z+\pi) = -\sin z \Leftrightarrow \sin(z+2\pi) = \sin z \quad (\text{奇函数})$$

$$\sin z = \frac{z(z-\pi)(z+\pi)(z-2\pi)(z+2\pi)\cdots}{\prod_{n=1}^{+\infty} n^2 \pi^2}$$

对数函数

$$\ln z = \ln z + 2k\pi i, \quad k \in \mathbb{Z}$$

$$\ln z = \ln|z| + i \arg z, \quad z \neq 0, \quad \arg z \in [0, 2\pi) / (-\pi, \pi]$$

指数函数 $a^b = e^{b \cdot \ln a}$

C-R 条件 $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$I_n = \oint_{C_r} \frac{dz}{(z-z_0)^{n+1}} = \begin{cases} 2\pi i, & n=0 \\ 0, & n \neq 0 \end{cases} \quad n \in \mathbb{Z}$$

① $n \leq -1$ 时, $n+1 \leq 0 \Rightarrow \frac{1}{(z-z_0)^{n+1}} = (z-z_0)^{-(n+1)}$ 解析

② 由 C-G 定理, 这时有 $I_n = 0$.

③ $n \geq 0$, $C_r: |z-z_0| \leq r \Leftrightarrow z = z_0 + re^{i\theta} \quad 0 \leq \theta \leq 2\pi$

$$dz = ri e^{i\theta} d\theta \quad (z-z_0)^n = r^n e^{in\theta}$$

$$\Rightarrow I_n = \int_0^{2\pi} \frac{ri e^{i\theta} d\theta}{r^{n+1} e^{i(n+1)\theta}} = \frac{i}{r^n} \int_0^{2\pi} e^{-in\theta} d\theta = \frac{i}{r^n} \left(\int_0^{2\pi} \cos n\theta d\theta + i \int_0^{2\pi} \sin n\theta d\theta \right)$$

复变 Cauchy 积分公式

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{C_r} \frac{f(z)}{(z-z_0)^{n+1}} dz \quad n \geq 0$$

证明: $f(z) = \sum_{m=0}^{+\infty} \frac{f^{(m)}(z_0)}{m!} (z-z_0)^m$

两边同乘以 $(z-z_0)^{n+1}$, 作 C_r 的积分

$$\begin{aligned} \text{左} &= \oint_{C_r} \frac{f(z)}{(z-z_0)^{n+1}} dz = \sum_{m=0}^{+\infty} \frac{f^{(m)}(z_0)}{m!} \oint_{C_r} (z-z_0)^{m-n-1} dz \\ &= \frac{f^{(n)}(z_0)}{n!} \cdot 2\pi i \quad \text{|| } I_{n-n} \end{aligned}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{C_r} \frac{f(z)}{(z-z_0)^{n+1}} dz \quad (n=1, 2, \dots)$$

$$0 \leq |f^{(n+j)}(0)| < \frac{M(r)}{r^{n+j}} \leq \frac{M_0 \sum_{k=0}^{\infty} r^k}{r^{n+j}} \rightarrow 0$$

$$\Rightarrow f^{(n+j)}(0) = 0, \quad j=1, 2, \dots$$

$$f^{(n+1)}(0) = f^{(n+2)}(0) = \dots = 0$$

$$f(z) = \sum_{m=0}^{+\infty} \frac{f^{(m)}(0)}{m!} z^m (= f(0))$$

$$= f(0) + z f'(0) + \dots + \frac{z^n}{n!} f^{(n)}(0)$$

代数基本定理

$$P_n(z) = \sum_{k=0}^n a_k z^k, \quad n \geq 1, \quad a_n \neq 0$$

复系数多项式 $\Rightarrow P_n(z) = 0$ 至少有一根.

证明取意: 反证法. 设 $P_n(z) \neq 0, \forall z \in \mathbb{C}$

$$\text{则 } f(z) = \frac{1}{P_n(z)} \text{ 处处可导, } f'(z) = \frac{-P_n'(z)}{P_n^2(z)}$$

引理: $|P_n(z)| \rightarrow +\infty$ (当 $|z| \rightarrow +\infty$ 时)

事实上, $P_n(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$

$$\Rightarrow |P_n(z)| = |z|^n \left| a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right|$$

$$\Rightarrow |P_n(z)| \geq \frac{|a_n|}{2} |z|^n \rightarrow \infty \quad (|z| \rightarrow \infty, |a_n| \neq 0)$$

$$\Rightarrow f(z) \rightarrow 0, \quad z \rightarrow \infty$$

又 f 处处可导, $\Rightarrow f$ 连续 $\Rightarrow f$ 解析

$\therefore \exists R > 0$, 当 $|z| > R$ 时, $|f(z)| \leq 1$. 而当 $|z| \leq R$ 时, $f(z)$ 连续, $\Rightarrow |f(z)| \leq M, |z| \leq R$

$$\text{这里 } M_0 = \max_{|z| \leq R_0} |f(z)|$$

$$\text{令 } M = M_0 + 1, \text{ 则 } |f(z)| < M, \forall z \in \mathbb{C}$$

由 Liouville 定理 $f(z) = \text{常数} \Rightarrow R P_1(z) = \frac{1}{f(z)} = \text{常数}$.

这与 $n \geq 1$ 矛盾?

调和函数 harmonic function

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \left(+ \frac{\partial^2}{\partial z^2} \right) \quad \Delta: \text{Laplace 算子.}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \left(+ \frac{\partial^2 u}{\partial z^2} \right) \quad C^2 \rightarrow C^0$$

$\Delta u \equiv 0$ 称 u 是调和函数. Laplace 方程.

结论: 解析函数的实虚部函数均是调和函数.

$$f(z) = u + i v, u, v \in \mathbb{R}$$

若 f 处处可导 $\Rightarrow \Delta u = \Delta v \equiv 0$.

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \end{cases} \xrightarrow{u, v \in C^2} \underline{\Delta u \equiv 0}.$$

定积分 } 复积分
实积分

$$dz = ie^{i\theta} d\theta = iz d\theta$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

复合闭路定理: $f(z)$ 在 D 内只有有限个奇点, $z_1, z_2, \dots, z_n$.

C 是 D 内任一简单闭曲线, 其内有 f 的 n 个奇点 $z_1, z_2, \dots, z_n$.

$f(z)$ 在 z_k 点的 Laurent 级数为

$$f(z) = \sum_{n=-\infty}^{+\infty} C_n^{(k)} (z - z_k)^n$$

$$\oint_C f(z) dz = 2\pi i \cdot C_n^{(k)}$$

$$\Rightarrow I = \frac{1}{a} \int_0^{2\pi} \frac{d\theta}{1+k \cos \theta} = \left(\frac{1}{a} \int_0^{2\pi} \frac{d\theta}{1+k \sin \theta} \right) \stackrel{z=e^{i\theta}}{\Rightarrow} \frac{1}{a} \int_{|z|=1} \frac{dz}{1+k \frac{z-1}{z}}$$

$$= \frac{1}{a} \int_{|z|=1} \frac{dz}{1+k \frac{z-1}{z}} = \int_{|z|=1} \frac{dz}{1+k \frac{z-1}{z}}$$

$$= \frac{2}{i} \oint_{|z|=1} \frac{dz}{2z + k(z^2 + 1)}$$

$$\Rightarrow \oint_C f(z) dz = \sum_{k=1}^n \oint_{C_k} f(z) dz = 2\pi i \left(\sum_{k=1}^n C_{-1}^{(k)} \right)$$

记 $\text{Res}[f, z_k] = C_{-1}^{(k)}$ Def: f 在 z_k 的留数.

$$\Rightarrow \oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}[f, z_k]$$

$$f(z) = \frac{p(z)}{Q(z)} \quad \left. \begin{array}{l} p, Q \text{ 互质} \\ p(z_k) \neq 0 \\ Q(z_k) = 0 \\ Q'(z_k) \neq 0 \end{array} \right\}$$

$$\Rightarrow \text{Res}[f, z_k] = \frac{p(z_k)}{Q'(z_k)}$$

例: $I_n = \oint_{|z|=r>1} \frac{1}{1+z^n} dz, n \in \mathbb{N}$

$$z_k^n = -1 = e^{z_i} = e^{zi+2(k-1)\pi i} = e^{(2k-1)\pi i}$$

$$= 2\pi i \sum_{k=1}^n \text{Res}\left[\frac{1}{1+z^n}, z_k\right] = 2\pi i \sum_{k=1}^n \frac{1}{n z_k^{n-1}} = \frac{2\pi i}{n} \sum_{k=1}^n z_k^{-n+1}$$

$$= -\frac{2\pi i}{n} \sum_{k=1}^n z_k^n$$

[韦达定理] $z^{n+1} = (z-z_1)(z-z_2)\dots(z-z_n)$

$$= z^n - \left(\sum_{k=1}^n z_k\right) z^{n-1} + \dots + (-1)^n \prod_{k=1}^n z_k$$

$n=1 \Rightarrow z+1 = z-z_1 \quad z_1 = -1 \quad \sum_{k=1}^1 z_k = -1$

$n \geq 2$ 时, $\sum_{k=1}^n z_k = 0, \quad I_n = \frac{-2\pi i}{n} \left(\sum_{k=1}^n z_k \right) = 0$

(综上, $I_n = \begin{cases} 2\pi i, & n=0 \\ 0, & n \geq 1 \end{cases}$

$$I = \int_0^{2\pi} \frac{d\theta}{a+b \cos \theta} = \int_0^{2\pi} \frac{d\theta}{a+b \sin \theta}, \quad a>b>0, \quad \left(I = \frac{2\pi}{\sqrt{a^2-b^2}} \right)$$

思路: 令 $z = e^{i\theta}$, 将积分转化为复积分

$$\Rightarrow \begin{cases} \cos \theta = \frac{z+z^{-1}}{2} = \frac{z^2+1}{2z} \\ \sin \theta = \frac{z-z^{-1}}{2i} = \frac{z^2-1}{2iz} \end{cases}$$

$$= \frac{2}{i} 2\pi i \text{Res}\left[\frac{1}{2z+k(z^2+1)}, z_1\right]$$

$$= 4\pi \cdot \frac{1}{2+2kz_1}$$

$$= \frac{2\pi}{1+kz_1} = \frac{2\pi}{\sqrt{1+k^2}}$$

$$I = \frac{1}{a} \int = \frac{1}{a} \frac{2\pi}{\sqrt{1-\frac{b^2}{a^2}}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$z_1^2 + \frac{2}{k} z_1 + 1 = 0$$

$$z_2 = \frac{-\frac{2}{k} \pm \sqrt{\frac{4}{k^2} - 4}}{2}$$

$$= \frac{-1 \pm \sqrt{1-k^2}}{k}$$

$$z_2 = -\frac{1}{k} - \sqrt{1-k^2}, \quad -\frac{1}{k} < 1$$

$$z_1 = \frac{1}{z_2}, \quad z_1 = -\frac{1}{k} + \sqrt{1-k^2}$$

$$-1 < z_1 < 0$$

$$kz_1 = -1 + \sqrt{1-k^2}$$

$$kz_1 + 1 = \sqrt{1-k^2}$$

$$I = \int_0^{2\pi} \frac{d\theta}{1-2p \cos \theta + p^2} = \frac{2\pi}{1-p^2}, \quad |p| < 1$$

$$a=1+p^2, \quad b=2p, \quad a>b>0$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{(1+p^2)^2 - (2p)^2}} = \frac{2\pi}{1-p^2}$$

$$I = \int_0^{2\pi} \frac{d\theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta} = \frac{2\pi}{ab}, \quad \begin{cases} a>0 \\ b>0 \end{cases}$$

$$I_n = \int_0^{+\infty} \frac{dx}{1+x^{2n}}, \quad n \in \mathbb{N} \Rightarrow \left(I_n = \frac{\pi}{n \sin \frac{\pi}{2n}} \right)$$

$$I_1 = \frac{\pi}{2}$$

$$I_2 = \frac{\pi}{\sqrt{3}}$$

$$I_n = \int_0^{+\infty} \frac{dx}{1+x^{2n}} \quad n \in \mathbb{N} \quad \left(I_n = \frac{\pi}{2n \sin \frac{\pi}{2n}} \right)$$

$$\sum_{k=1}^n z_k = \frac{z_1(1-z_1^n)}{1-z_1} = \frac{z_1}{1-z_1} = \frac{2e^{\frac{\pi i}{2n}}}{1-e^{\frac{\pi i}{n}}}$$

$$= \frac{2}{e^{\frac{\pi i}{2n}} - e^{-\frac{\pi i}{2n}}} = \frac{2}{-2i \sin \frac{\pi}{2n}}$$

$$= \frac{-1}{i \sin \frac{\pi}{2n}} \Rightarrow I_n = -\frac{z_1}{2n} \cdot \frac{-1}{i \sin \frac{\pi}{2n}} = \frac{\pi}{2n \sin \frac{\pi}{2n}}$$

$$I = \int_{-R}^R \frac{dx}{1+x^{2n}} + \int_{\text{arc}} \frac{dx}{1+x^{2n}}$$

$$= \int_{-R}^R \frac{dx}{1+x^{2n}} + \int_0^R \frac{dx}{1+x^{2n}} + \int_R^0 \frac{dx}{1+x^{2n}}$$

$$2I_n = \int_{-\infty}^{+\infty} \frac{dx}{1+x^{2n}} = \lim_{R \rightarrow \infty} \left(\int_{-R}^R \frac{dx}{1+x^{2n}} \right)$$

$$\sum_{k=1}^n \sin \frac{2k-1}{2n} \pi = \frac{\sum_{k=1}^n \sin \left(\frac{k-1}{2n} \pi \right) \cos \frac{\pi}{2n}}{2 \sin \frac{\pi}{2n}}$$

$$= 2\pi i \sum_{k=1}^n \operatorname{Res} \left[\frac{1}{1+z^{2n}}, z_k \right]$$

$$\lim_{R \rightarrow \infty} \int_{\text{arc}} \frac{p(z)}{Q(z)} dz = 0$$

条件: $\deg Q \geq \deg P + 1$

$$I = \oint_{|z|=2} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz$$

$z=0$ 本性奇点.
 $z=-1$ 一阶极点

$$= 2I + 0 = 2I$$

$$\Rightarrow I = \pi i \sum_{k=1}^n \frac{1}{2n z_k^{2n-1}}$$

$$= \pi i \sum_{k=1}^n \frac{1}{z_k^{2n-1}}$$

$$= -\frac{\pi i}{2n} \sum_{k=1}^n z_k$$

$$z_k^{2n} = -1 = e^{z_i + 2i(k-1)\pi} = e^{(2k-1)\pi i} \quad k=1, \dots, n$$

$$z_k = e^{\frac{(2k-1)\pi i}{2n}} \quad k=1, 2, \dots, n$$

$$= \cos \frac{(2k-1)\pi}{2n} + i \sin \frac{(2k-1)\pi}{2n}$$

$$I = \frac{-\pi i}{2n} \left[\sum_{k=1}^n \cos \frac{(2k-1)\pi}{2n} + i \sin \frac{(2k-1)\pi}{2n} \right] > 0$$

$$= \frac{\pi \sin \frac{\pi}{2n}}{2n} \sum_{k=1}^n \sin \frac{(2k-1)\pi}{2n}$$

$$\text{求 } I = \sum_{k=1}^n \sin \frac{(2k-1)\pi}{2n}$$

$$= \frac{1}{2 \cos \frac{\alpha}{2}}$$

$$\sum_{k=1}^n \sin(k\alpha + \beta) \cos \alpha = \sum_{k=1}^n \frac{\sin((k+1)\alpha + \beta) - \sin((k-1)\alpha + \beta)}{2} = (-1 + 1 - \frac{1}{2} + \frac{1}{2}) \sin \beta$$

$$= \sin(k\alpha + \beta) \cos \alpha = \left(\sin \frac{(2k+1)\alpha}{2} \cos \frac{\alpha}{2} \right) + \left(\sin \frac{(2k-1)\alpha}{2} \cos \frac{\alpha}{2} \right)$$

$$= \sin \frac{(2k+1)\alpha}{2} \cos \frac{\alpha}{2} + \sin \frac{(2k-1)\alpha}{2} \cos \frac{\alpha}{2}$$

$$= \sin \left(\frac{2k+1}{2} \alpha + \beta \right) - \sin \left(\frac{2k-1}{2} \alpha + \beta \right) = 2 \cos \frac{\alpha}{2} \sin \beta$$

$$\text{令 } t = \frac{1}{z} \quad dz = -\frac{dt}{t^2}$$

$$z = 2e^{i\theta} \quad \theta: 0 \rightarrow 2\pi$$

$$t = \frac{1}{2} e^{-i\theta} \quad -\theta: 0 \rightarrow -2\pi \quad \theta: 0 \rightarrow -2\pi$$

$$\Rightarrow I = -\oint_{|t|=\frac{1}{2}} \frac{e^t}{t^3(1+t)} \left(-\frac{dt}{t^2} \right)$$

$$= + \oint_{|t|=\frac{1}{2}} \frac{e^t}{t^5(1+t)} dt = \oint_{|t|=\frac{1}{2}} \frac{e^t dt}{t^5(1+t)} = 2\pi i \left(-\frac{1}{3} \right) = -\frac{2\pi i}{3}$$

$$\int_{\text{arc}} \frac{e^t}{t^5(1+t)} dt \quad \operatorname{Res} [f(t), 0] = C_1$$

$$f(t) = \frac{1}{t} g(t) \quad g(t) = \frac{e^t}{t^5(1+t)} = g^{(0)} + g^{(1)}t + \dots + \frac{g^{(4)}(0)}{4!} t^4 + \dots$$

$$\frac{1}{t} \left(\frac{e^t}{t^5} \right) = \frac{1}{t^6} (1+t+t^2+\dots) = (1+t+t^2+\dots) (1-t+t^2-t^3+\dots)$$

$$\text{Res}\left[\frac{z-\sin z}{z^8}, 0\right]$$

$$= \frac{1}{7!}$$

$$\frac{z-\sin z}{z^8} = \frac{1}{z^8} \left(z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right) \right)$$

$$= \frac{1}{z^8} \left(\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \dots \right)$$

$$\left| \oint_{|z|=1} \frac{dz}{(z-a)^n(z-b)^n} \right|$$

$$0 < |a|, |b| < 1$$

$$a \neq b$$

$$\frac{1}{z-b} \text{ in } z=a \text{ is Taylor (323)}$$

$$\frac{1}{z-b} = \frac{1}{z-a+a-b} = \frac{1}{(a-b)\left(1-\frac{z-a}{b-a}\right)}$$

$$= \frac{1}{a-b} \sum_{n=0}^{\infty} \left(\frac{z-a}{b-a} \right)^n = - \sum_{n=0}^{\infty} \frac{(z-a)^n}{(b-a)^{n+1}}$$

$$J_n = \int_0^{\infty} \frac{dx}{(1+x^2)^n}, \quad n \leq N.$$

$$2J_n = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^n}$$

$$\Rightarrow \int_{-R}^R \frac{dx}{(1+x^2)^n} + \int_{C_R} \frac{dz}{(1+z^2)^n} = 2\pi i \text{Res}\left[\frac{1}{(1+z^2)^n}, i\right] \quad q: 30-1430.$$

$$q: \infty - 12^{\infty}$$

$$\text{Let } R \rightarrow \infty \quad 2J_n = 2\pi i \text{Res}\left[\frac{1}{(1+z^2)^n}, i\right]$$

$$\Rightarrow J_n = \pi i \text{Res}\left[\frac{1}{(1+z^2)^n}, i\right] = \pi i \cdot \frac{1}{(2i)^n} \frac{(-1)^{n-1}}{(2i)^{n-1}} \cdot 2n(2n-1) \frac{1}{(2i)^{n-1}} = \frac{\pi (-1)^{n-1} \cdot 2n(2n-1) \cdot n}{(2i)^n \cdot 2^{n-1} \cdot i^{2n-1}}$$

$$\frac{1}{(1+z^2)^n} = \frac{1}{(z+i)^n(z-i)^n} = \frac{1}{(z+i)^n} \cdot \frac{1}{(z-i)^n} = \frac{1}{(2i)^n \left(1 + \frac{z+i}{-i}\right)^n} \cdot \frac{1}{(2i)^n}$$

$$= \frac{1}{2^n i^n} \cdot \frac{1}{(2i)^n} (1+x)^{-n}$$

$$(1+x)^{-n} = 1 - nx + \frac{(-n)(-n-1)}{2} x^2$$

$$J_n = \int_0^{\infty} \frac{dx}{(1+x^2)^n} = \frac{(2n-2)! \cdot \pi}{((n-1)!)^2 \cdot 2^{2n-1}}$$

$$I_n = \int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx \quad a > 0$$

$$(J_1 = \frac{\pi}{2}) \quad J_2 = \frac{\pi}{4} \quad J_3 = \frac{3\pi}{16} \quad J_4 = \frac{5\pi}{32} \quad J_5 = \frac{35\pi}{256}$$

$z_1, z_2, z_3 \rightarrow w_1, w_2, w_3$ 的分式线性映射为

$$\frac{w-w_1}{w-w_2} / \frac{w_3-w_1}{w_3-w_2} = \frac{wz-z_1}{z-z_1} \cdot \frac{z_3-z_1}{z_3-z_2} \quad \text{— 硬式}$$

若 $z_3 = \infty$ 或 $w_3 = \infty$ 时 上式变为

$$\left\{ \begin{array}{l} \frac{w-w_1}{w-w_2} / \frac{w_3-w_1}{w_3-w_2} = \frac{z-z_1}{z-z_2} \\ \text{或} \frac{w-w_1}{w-w_2} = \frac{z-z_1}{z-z_2} \cdot \frac{z_3-z_1}{z_3-z_2} \end{array} \right. \quad \left| \begin{array}{l} z_3 = w_3 = \infty \\ \frac{w-w_1}{w-w_2} = \frac{z-z_1}{z-z_2} \end{array} \right.$$

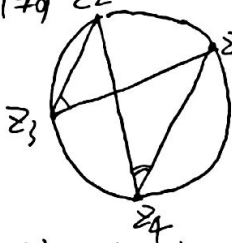
记 $\langle z_1, z_2, z_3, z_4 \rangle = \frac{z_4-z_1}{z_4-z_2} / \frac{z_3-z_1}{z_3-z_2} \quad (z_4 \neq z_2)$
为 z_1, z_2, z_3, z_4 的交比.

定理: z_1, z_2, z_3, z_4 共圆 (或直线) 的充要条件是
其交比为实数.

证: $\langle z_1, z_2, z_3, z_4 \rangle = \text{实数} \begin{cases} > 0 \\ < 0 \end{cases} \quad (z_4 \neq z_2)$

$$\Leftrightarrow \arg \langle z_1, z_2, z_3, z_4 \rangle = \begin{cases} 0 \\ \pi \end{cases}$$

$$\Leftrightarrow \arg \frac{z_4-z_1}{z_4-z_2} = \arg \frac{z_3-z_1}{z_3-z_2}$$



$$\Leftrightarrow \arg(z_4-z_1) - \arg(z_4-z_2) = \arg(z_3-z_1) - \arg(z_3-z_2)$$

1) $z_1 \rightarrow w_1: w = (z-z_1) + w_1 = z + w_1 - z_1$

2) $z_1, z_2 \rightarrow w_1, w_2: \begin{cases} z_1 \rightarrow w_1 \\ z_2 \rightarrow w_2 \end{cases} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$w = \frac{az+b}{cz+d} \Rightarrow z = \frac{A w + B}{C w + D}$$

$$w = \alpha z + \beta, \quad A = \begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} \alpha = \frac{w_2 w_1}{z_2 z_1} \\ \begin{cases} w_1 = \alpha z_1 + \beta \\ w_2 = \alpha z_2 + \beta \end{cases} \Rightarrow \beta = \end{cases}$$

3) $z_1, z_2, z_3 \rightarrow w_1, w_2, w_3$

4) $z_1, z_2, \dots, z_n \rightarrow w_1, w_2, \dots, w_n$ (Lagrange 插值多项式)

$$l_j(z) = \frac{(z-z_1)(z-z_2)\dots(z-z_{j-1})(z-z_{j+1})\dots(z-z_n)}{(z_j-z_1)(z_j-z_2)\dots(z_j-z_{j-1})(z_j-z_{j+1})\dots(z_j-z_n)} \quad j=1,2,\dots,n$$

$$= \frac{\prod_{k \neq j} (z-z_k)}{\prod_{k \neq j} (z_j-z_k)} \quad \deg l_j(z) = n-1$$

$$L(z) = \sum_{k=1}^n w_k \cdot l_k(z) \quad \deg L(z) \leq n-1 \quad \text{而} \quad \deg L(z) = n-1$$

$$L(z_j) = w_j, \quad j=1,2,\dots,n.$$

$$l_j(z_k) = \delta_{jk} = \begin{cases} 1, & j=k \\ 0, & j \neq k \end{cases} \quad (\text{自验证})$$

$$P_n(z) \sim n \text{ 次多项式} = \sum_{k=0}^n a_k \cdot z^k$$

$$z_1, z_2, \dots, z_{n+1} \rightarrow w_1, w_2, \dots, w_{n+1}$$

$$A \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$A^{-1} \text{ 存在, } \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = A^{-1} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}$$

1) 平移

$$w = z + \alpha \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + \alpha \\ y \end{pmatrix}$$

2) 旋转

$$w = \alpha \cdot z \quad \alpha = r_0 e^{i\theta_0}, r_0 > 0$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = w = \begin{pmatrix} r_0 e^{i\theta_0} \cdot x \\ r_0 e^{i\theta_0} \cdot y \end{pmatrix} = r_0 e^{i\theta_0} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = r_0 \cdot e^{i(\theta_0)}$$

$$x = r_0 \cdot x, y = r_0 \cdot y \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = r_0 e^{i\theta_0} \begin{pmatrix} x \\ y \end{pmatrix} = r_0 e^{i\theta_0} (x + iy)$$

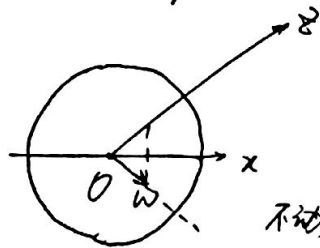
$$= r_0 (\cos \theta_0 + i \sin \theta_0) (x + iy)$$

$$= r_0 (x \cos \theta_0 - y \sin \theta_0 + i(x \sin \theta_0 + y \cos \theta_0))$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta_0 & -\sin \theta_0 \\ \sin \theta_0 & \cos \theta_0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = r_0 \begin{pmatrix} x \cos \theta_0 - y \sin \theta_0 \\ x \sin \theta_0 + y \cos \theta_0 \end{pmatrix}$$

3) 倒数. 莫比乌斯变换

$$z = z r e^{i\theta} \quad w = \frac{1}{r} e^{-i\theta} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



不动点.

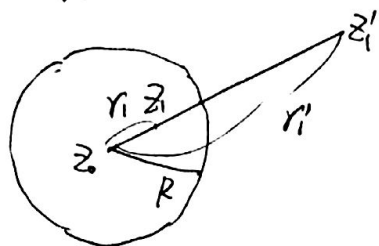
$$z = \frac{1}{z} \Rightarrow z = \pm 1$$

不动点集合: 单位圆 $|z|=1$
实轴 $-x$

$$w = \frac{az+b}{cz+d} = \begin{cases} \frac{a}{d} z + \frac{b}{d}, & c=0 \\ \frac{a}{c} + \frac{ad-bc}{c(z+d)}, & c \neq 0 \end{cases}$$

$$= \frac{a(z+\frac{d}{c}) + \frac{1}{c}(ad-bc)}{c(z+\frac{d}{c})} = \frac{a}{c} + \frac{ad-bc}{c^2(z+\frac{d}{c})}$$

圆的对称点



z_1' 与 z_1 点关于圆对称

$$\Leftrightarrow |z_1 - z| |z_1' - z| = R^2$$

且 z_1', z_1, z 共线

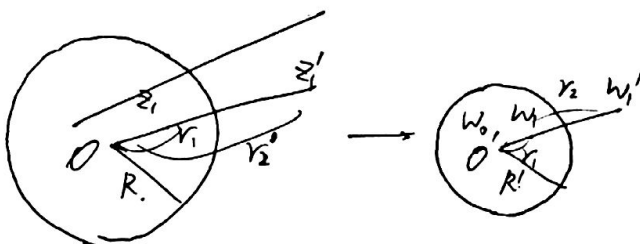
$$\frac{R}{r_1} = \frac{r_1'}{R}$$

$$\left\{ \begin{aligned} \frac{|z_1 - z| \cdot |z_1' - z|}{R^2} &= 1 \\ (z_1' - z) \bar{z} &= k(z_1 - z) \end{aligned} \right.$$

($k > 0$, 可以用 R)

$$R - z_1' = R \rightarrow L$$

5. 圆的对称点



$$\begin{cases} |w_1' - w_0| = k |w_1 - w_0| \\ |w_1' - w_0| / |w_1 - w_0| = R'^2 \end{cases}$$

分式线性映射对圆映射
构成单叶群

2. 保圆性

定理: 分式线性映射是保圆的 (广义圆 = 圆 ∪ 直线)

$$w = \frac{az+b}{cz+d}, \quad z - z_0 = r e^{i\theta}$$

(w_0, R)

$$w = w_0 + R e^{i\phi}$$

保圆: $\checkmark$ 平移 $\checkmark$ z 分射 $w = \alpha \cdot z$

引例: 直线: $\alpha \bar{z} + \bar{\alpha} z + \beta = 0$

圆: $\alpha \bar{z} + \bar{\alpha} z + \beta = 0$

$$z = \frac{1}{w}, \quad \bar{z} = \frac{\alpha}{w} + \frac{\bar{\alpha}}{\bar{w}} + \beta = 0$$

$$\beta w \bar{w} + \alpha w + \bar{\alpha} \bar{w} = 0$$

$$\beta = 0, \text{ 直线过原点. } w \bar{w} + \frac{\alpha}{\beta} \cdot \bar{w} + \frac{\bar{\alpha}}{\beta} \cdot w = 0 \text{ (圆)}$$

$$\text{圆. } z = \frac{1}{w}, \quad \frac{1}{w} \cdot \frac{1}{\bar{w}} + \frac{\alpha}{w} + \frac{\bar{\alpha}}{\bar{w}} + \beta = 0$$

$$\beta w \bar{w} + \alpha w + \bar{\alpha} \bar{w} + 1 = 0$$

$$\beta = 0 \text{ 直线 } \alpha w + \bar{\alpha} \bar{w} + 1 = 0$$

$$\beta \neq 0, \text{ 圆. } w \bar{w} + \frac{\alpha}{\beta} w + \frac{\bar{\alpha}}{\beta} \bar{w} + \frac{1}{\beta} = 0$$

求单位圆内 \$|z| < 1\$ 映成单位圆内 \$|w| < 1\$ 的分式线性映射
 \$w(z)\$ 并使 \$a \mapsto 0\$ 即 \$(w(a)=0)\$

单位圆的对称点, \$a' = ka, k > 1, |a| < 1\$ ①

同时有 \$|a'| \cdot |a| = 1 \Rightarrow |a'| = \frac{1}{|a|}\$ ②

$$\Rightarrow k|a|^2 = 1 \Rightarrow k = \frac{1}{|a|^2} = \frac{1}{a\bar{a}}$$

$$\Rightarrow a' = ka = \frac{a}{a\bar{a}} = \frac{1}{\bar{a}} \quad (0 \mapsto \infty)$$

$$w = \frac{az+b}{cz+d} = \frac{\lambda(z-a)}{z-a'} \quad (\text{对称点})$$

\$\lambda = ?\$ 边界 \$\mapsto\$ 边界, \$|z|=1 \Rightarrow |w|=1\$.

$$\begin{aligned} 1 = |w| &= |\lambda| \cdot \left| \frac{z-a}{z-a'} \right| = |\lambda| \cdot \left| \frac{z-a}{z-\frac{1}{\bar{a}}} \right| \cdot a' = \frac{1}{a} \\ &= |\lambda| \cdot \left| \frac{z-a}{z-\frac{1}{\bar{a}}} \right| = |\lambda \bar{a}| \cdot \left| \frac{z-a}{z-\frac{1}{\bar{a}}} \right| \cdot a' = \frac{1}{a} \\ &= |\lambda \bar{a}| \cdot \left| \frac{z-a}{z-\frac{1}{\bar{a}}} \right| = \left| \frac{\lambda \bar{a}}{z} \right| \cdot \left| \frac{z-a}{z-\frac{1}{\bar{a}}} \right| \\ &= |\lambda \bar{a}| \cdot \left| \frac{z-a}{z-\frac{1}{\bar{a}}} \right| = |\lambda \bar{a}| = 1 \end{aligned}$$

$$\Rightarrow \lambda \bar{a} = e^{i\theta} \Rightarrow w(z) = e^{i\theta} \frac{z-a}{z-\frac{1}{\bar{a}}}, \theta \in [0, 2\pi)$$

$$w'(a) = e^{i\theta} \quad a=0 \text{ 时 } \Rightarrow w(z) = e^{i\theta} z$$

$$\text{可证: } \frac{|dw|}{1-|w|^2} = \frac{|dz|}{1-|z|^2} \quad (\text{不变式: } 16.2)$$

$$\Leftrightarrow 0 < \left| \frac{dw}{dz} \right| = \frac{|w'|}{1-|z|^2}$$

$$w(z) = \frac{az+b}{cz+d}, \quad w'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0.$$

$$\begin{aligned} w(z) = e^{i\theta} \frac{z-a}{z-\frac{1}{\bar{a}}} &\Rightarrow w'(z) = \frac{e^{i\theta} \cdot 1 - (-ae^{i\theta} \cdot \bar{a})}{(1-\bar{a}z)^2} \\ &= e^{i\theta} \cdot \frac{1+a\bar{a}}{(1-\bar{a}z)^2} \end{aligned}$$

$$\Rightarrow \left| \frac{dw}{dz} \right| = |w'(z)| = \frac{1+a\bar{a}}{(1-\bar{a}z)^2} > 0, \quad |a|^2 = a\bar{a} < 1$$

$$\text{右端} = \frac{1-|w|^2}{1-|z|^2} > 0.$$

\$|1-|w|^2| > 0 \Leftrightarrow |1-|z|^2| > 0\$ 国内 \$\Rightarrow\$ 国内
 \$(1-|w|^2) < 0 \Leftrightarrow (1-|z|^2) < 0\$ 国外 \$\Rightarrow\$ 国外

$$\text{反之} \quad \frac{1}{1-|z|^2} = \frac{1}{(1-\bar{a}z)^2} \quad \text{证}.$$

$$\Leftrightarrow 1-w\bar{w} = (1-|z|^2) \cdot \frac{1-a\bar{a}}{(1-\bar{a}z)^2}$$

$$1-w\bar{w} = 1-e^{i\theta} \frac{z-a}{z-\frac{1}{\bar{a}}} \cdot \frac{1}{1-\bar{a}\bar{z}}$$

$$= 1-e^{i\theta} \frac{z-a}{z-\frac{1}{\bar{a}}} \cdot \frac{1}{1-\bar{a}\bar{z}}$$

$$= 1 - \frac{z-a}{z-\frac{1}{\bar{a}}} \cdot \frac{1}{1-\bar{a}\bar{z}} = \frac{(1-\bar{a}\bar{z})(z-\frac{1}{\bar{a}}) - (z-a)(\bar{z}-\bar{a})}{(1-\bar{a}\bar{z})(z-\frac{1}{\bar{a}})(1-\bar{a}\bar{z})}$$

$$= \frac{1-a\bar{z}-\bar{a}z+\bar{a}a\bar{z}\bar{z} - (z\bar{z}-z\bar{a}-\bar{z}a+\bar{z}\bar{a})}{(1-\bar{a}\bar{z})^2}$$

$$= \frac{1-a\bar{z}-\bar{a}z+\bar{a}a\bar{z}\bar{z} - (z\bar{z}-z\bar{a}-\bar{z}a+\bar{z}\bar{a})}{(1-\bar{a}\bar{z})^2} = \frac{1-a\bar{z}-\bar{a}z+\bar{a}a\bar{z}\bar{z} - z\bar{z} + z\bar{a} + \bar{z}a - \bar{z}\bar{a}}{(1-\bar{a}\bar{z})^2}$$

一般地, \$|z-z_0| < r \mapsto |w-w_0| < R\$ 并设 \$z_0 \mapsto 0\$

$$\begin{cases} z' = \frac{z-z_0}{r} \\ w' = \frac{w-w_0}{R} \end{cases}$$

$$w' = e^{i\theta} \frac{z'-z'_0}{1-\bar{z}'_0 z'}$$

$$\begin{aligned} \frac{w-w_0}{R} &= e^{i\theta} \cdot \frac{\frac{z-z_0}{r} - \frac{z_0-z_0}{r}}{1-\bar{z}'_0 \frac{z-z_0}{r}} \\ &= e^{i\theta} \cdot \frac{z-z_0}{r} \cdot \frac{1}{1-\frac{\bar{z}_0-z_0}{r} \frac{z-z_0}{r}} \end{aligned}$$

$$w = w_0 + R \cdot e^{i\theta} \frac{z-z_0}{r^2 - (\bar{z}_0-z_0)(z-z_0)} \quad \theta \in [0, 2\pi)$$

例2. 求将上半平面 $\text{Im} z > 0$ 映成单位圆 $|w| < 1$ 的线性映射.

作: 设 $z_0 (\text{Im} z_0 > 0) \rightarrow 0$. z 的对应点 $(z \rightarrow w)$
故有 $w(z) = \lambda \frac{z - z_0}{z - \bar{z}_0}$, 下面求 $\lambda = ?$

$$\begin{aligned} \text{令 } z = x \in \mathbb{R}, & \Rightarrow |w(z)| = 1 \Rightarrow \\ \Rightarrow |w(z)| = 1 & \Rightarrow |\lambda| \cdot \left| \frac{x - z_0}{x - \bar{z}_0} \right| = 1 \quad x \in \mathbb{R} \\ |\lambda| \cdot \left| \frac{x - z_0}{x - \bar{z}_0} \right| &= 1 \Rightarrow |\lambda| \cdot \left| \frac{x - z_0}{x - \bar{z}_0} \right| = 1 \\ \Rightarrow |\lambda| = 1, & \lambda = e^{i\alpha}, \alpha \in [0, 2\pi] \end{aligned}$$

$$\Rightarrow w(z) = e^{i\alpha} \frac{z - z_0}{z - \bar{z}_0}$$

半平面 $\rightarrow$ 圆盘



1. 平移: $z' = z - z_0$

$$\begin{aligned} 2. z'' &= z' \cdot e^{-i\alpha} \\ &= (z - z_0) \cdot e^{-i\alpha} \end{aligned}$$

$$w' = e^{i\alpha} \frac{z' - z_0}{z' - \bar{z}_0}$$

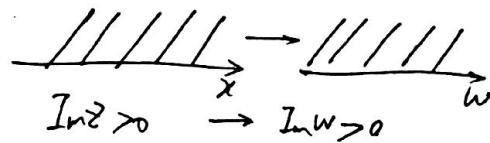
$$\begin{aligned} \Leftrightarrow \frac{w' - w_0}{R} &= e^{i\alpha} \frac{(z - z_0) - z_0 e^{-i\alpha}}{(z - z_0) e^{-i\alpha} - (\bar{z} - \bar{z}_0) e^{-i\alpha}} \\ &= e^{i\alpha} \frac{z - z_0}{z - z_0 - (\bar{z} - \bar{z}_0) e^{2i\alpha}} \end{aligned}$$

$$\text{得: } w = w_0 + R \cdot e^{i\alpha} \frac{z - z_0}{z - z_0 - (\bar{z} - \bar{z}_0) e^{2i\alpha}}$$

圆盘 $\rightarrow$ 半平面, $z = z(w)$

$$e^z, z^\alpha, \alpha > 1$$

例3. 半 $\rightarrow$ 半平面.



$$w(z) = \frac{az + b}{cz + d} \quad \text{结论: 这时可取 } \boxed{a, b, c, d \in \mathbb{R}}$$

$$\text{且有 } \boxed{ad - bc > 0}$$

$$\begin{aligned} w(z) &= z = \frac{iz}{i} \\ a &= i, b = 0, c = 0, d = i \\ ad - bc &= i^2 = -1 < 0 \end{aligned}$$

实轴 x : $x_1 < x_2 < x_3 \rightarrow u_1 \leq u_2 \leq u_3$: 实轴 u .

$$u_k = \frac{ax_k + b}{cx_k + d} \Leftrightarrow ax_k + b = u_k (cx_k + d) = cu_k x_k + bu_k$$

$$x_k \cdot a + 1 \cdot b - cu_k x_k + bu_k = 0$$

$$A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \vec{0}, A \text{ 为实矩阵.}$$

故 $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ 可为实数.

$$w(x) = \frac{ax + b}{cx + d}, a, b, c, d \in \mathbb{R}. \quad \text{单调性?}$$

$$w'(x) = \frac{ad - bc}{(cx + d)^2} > 0. \quad \underline{ad - bc > 0} \quad w(x) \text{ 单调增}$$

$$\text{Im} z > 0 \Rightarrow \text{Im} w > 0$$

法二: 令 $z = x + iy, y > 0 \rightarrow w(z) = w(x + iy)$

$$= \frac{a(x + iy) + b}{c(x + iy) + d}$$

$$= \frac{ax + b + aiy}{cx + d + cyi} = \frac{(ax + b + aiy)(cx + d - cyi)}{(cx + d + cyi)(cx + d - cyi)}$$

$$= \frac{(ax + b)(cx + d) + acy^2}{(cx + d)^2 + (cy)^2} + i \frac{ay(cx + d) - cy(ax + b)}{(cx + d)^2 + (cy)^2}$$

$$\text{Im} w = \frac{y(abd - bc)}{(cx + d)^2 + (cy)^2} > 0$$

$$\Leftrightarrow u > 0 \rightarrow u = 0 \quad \text{上} \leftrightarrow \text{下}$$

1/ 单位圆盘 $|z| < 1 \rightarrow$ 单位圆盘 $|w| < 1$

$$w = e^{i\theta} \frac{z-a}{1-\bar{a}z}, |a| < 1, \theta \in [0, 2\pi)$$

2/ 上半平面 $\text{Im} z > 0 \rightarrow$ 单位圆盘 $|w| < 1$

$$w = e^{i\theta} \frac{z-z_0}{z-\bar{z}_0}, \text{Im} z_0 > 0, \theta \in [0, 2\pi)$$

若令 $\theta = 0, z_0 = i$, 则有 $w = \frac{z-i}{z+i}$

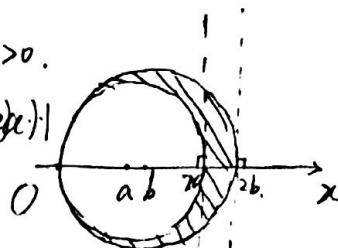
3/ 上半平面 $\text{Im} z > 0 \rightarrow$ 上半平面 $\text{Im} w > 0$

$$w = \frac{az+b}{cz+d}, \text{若 } a, b, c, d \in \mathbb{R}, \text{ 有 } ad-bc > 0$$

例2: 将 $|z-a| > r$ 与 $|z-b| < b$ 之间的部分映成单位圆盘 $|w| < 1$. 这里 a, c, b

$$z_1 = \frac{2b}{z}, 2b \rightarrow \frac{b-a}{b} > 0$$

$$a_1 - b_1 = 1 \cdot 0 - (-2a) = 2a > 0$$



分式线性映射 $\rightarrow$ 保圆

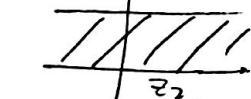
大圆 (同射线)

有极点 $\rightarrow$ 直线

$$z_2 = \frac{z-z_1}{b-a} = \frac{z-2a}{b-a}$$

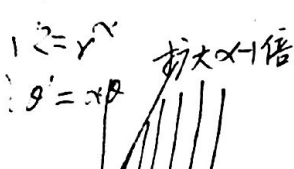
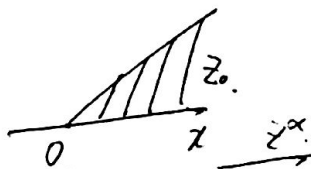
$$z_3 = e^{z_2}$$

$$z_4 = \frac{z_3-i}{z_3+i} = \frac{e^{z_2}-i}{e^{z_2}+i} = \frac{e^{\frac{z-2a}{b-a}} - i}{e^{\frac{z-2a}{b-a}} + i}$$



保角 $(e^z)' = e^z \neq 0$

$$W = z^\alpha, \alpha > 0, W' = \alpha z^{\alpha-1} \neq 0, z \neq 0, z = re^{i\theta}, \frac{0 < \alpha < 1}{|w| = r^\alpha e^{i\alpha\theta} = R e^{i\theta'}}$$

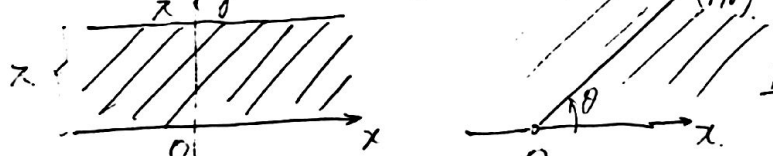


$$W(0) = 0, (\alpha > 0)$$

$$0 < \theta_1 < \theta_2$$

$$z_1 = z^{\frac{1}{\alpha}}, w = \frac{z-i}{z+i} = \frac{z^{\frac{1}{\alpha}}-i}{z^{\frac{1}{\alpha}}+i}$$

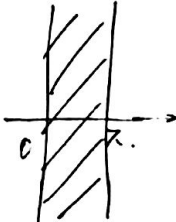
$$1 > r = e^x > 0, (\neq 0) y = 0 \pmod{2\pi}$$



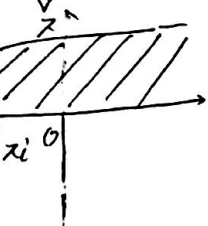
上半平面

$$0 < y < \infty, -\infty < x < \infty \Rightarrow \text{Im} w > 0$$

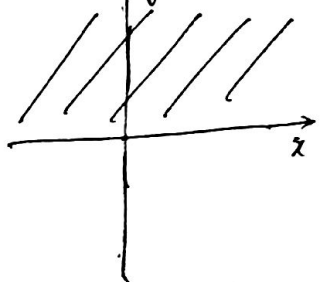
$$z_1 = \frac{z-a}{b-a}$$



$$z_2 = e^{\frac{z_1-i}{b-a}}$$

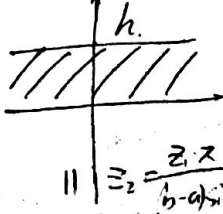


$$z_3 = e^{z_2} = e^{\frac{z-a}{b-a}}$$

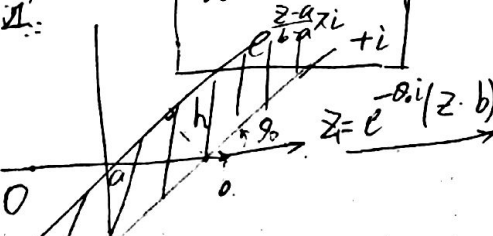


$$W = \frac{z-i}{z+i} = \frac{e^{\frac{z-a}{b-a}} - i}{e^{\frac{z-a}{b-a}} + i}$$

$$h = (b-a) \sin \theta_0$$



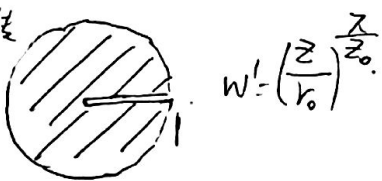
例2:



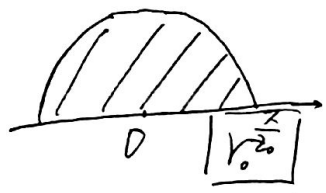


$$\begin{cases} 0 < \theta < \theta_0 \\ 0 < r < r_0 \end{cases}$$

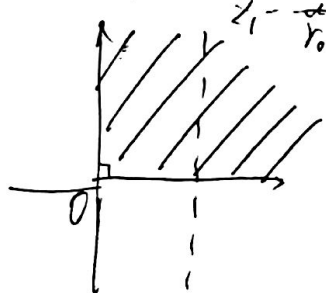
直接旋转



$$z_1 = z \frac{z}{z_0}$$



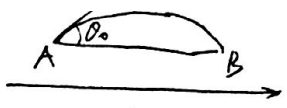
$$z_1 = \frac{z_1 + r_0 \frac{z}{z_0}}{z_1 - r_0 \frac{z}{z_0}}$$



$$z_1 = z^2$$



$$w = \frac{z-i}{z+i} = \frac{\left(\frac{z^2}{r_0^2} + \frac{z}{r_0}\right)^2 - i}{\left(\frac{z^2}{r_0^2} + \frac{z}{r_0}\right)^2 + i}$$



$$z_1 = \frac{z-A}{B-A}$$

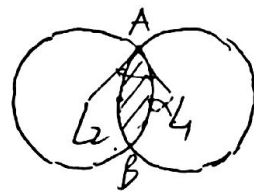
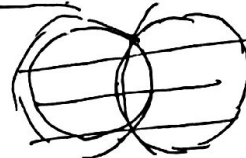


$$z_2 = \frac{z}{r_0} = \left(\frac{z-A}{B-A}\right)^{\frac{2}{\theta_0}}$$



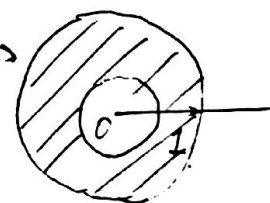
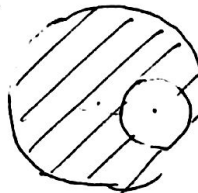
$$w = \frac{z-i}{z+i} = \frac{\left(\frac{z-A}{B-A}\right)^{\frac{2}{\theta_0}} - i}{\left(\frac{z-A}{B-A}\right)^{\frac{2}{\theta_0}} + i}$$

例4:

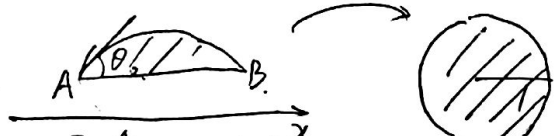


$$z_1 = \frac{z-A}{z-B}$$

思考:



例4:



$$z_1 = \lambda \frac{z-A}{z-B} \quad \frac{1+\lambda}{2} \quad z_1\left(\frac{A+B}{2}\right) = \lambda \frac{B-A}{A-B} = -\lambda > 0$$

$$\text{可取 } \lambda = -1, \quad z_1 = \frac{z-A}{B-z}$$