

复变函数期末复习

① 子时还材料

Chap 1. 复数

($\alpha, \beta \in \mathbb{C}, \alpha \neq 0, r > 0$)

$$1. \max_{|z| \leq r} |\alpha z^n + \beta| = \max_{|z| \leq r} |\alpha| |z^n + \frac{\beta}{\alpha}| \leq \max_{|z| \leq r} |\alpha| (|z^n| + \frac{|\beta|}{|\alpha|}) \stackrel{①}{\leq} |\alpha| (r^n + \frac{|\beta|}{|\alpha|}) = |\alpha| r^n + |\beta|$$

(1) $\beta \neq 0, |z| \leq r$

同时取等 $\Leftrightarrow |z| = r$ 且 $z^n = k \frac{\beta}{\alpha} (k \in \mathbb{R})$ ②

由①可设 $z = re^{i\theta}$, 将①代入②并取模有 $r^n = k \frac{|\beta|}{|\alpha|} \Rightarrow k = \frac{|\alpha|}{|\beta|} r^n$

$$\text{代入②有 } z^n = r^n \frac{|\alpha|}{|\beta|} \cdot \frac{\beta}{\alpha} = r^n \frac{|\alpha|}{|\beta|} \cdot \frac{|\beta|}{|\alpha|} e^{i \arg \frac{\beta}{\alpha}} = r^n e^{i \arg \frac{\beta}{\alpha}}$$

$$\Rightarrow z_k = r e^{i \frac{1}{n} [\arg \frac{\beta}{\alpha} + 2(k-1)\pi]}, k=1, 2, \dots, n$$

故: (1) $\beta \neq 0$ 时, $\max_{|z| \leq r} |\alpha z^n + \beta| = |\alpha| r^n + |\beta|$, 此时 $z = z_k = r e^{i \frac{1}{n} [\arg \frac{\beta}{\alpha} + 2(k-1)\pi]}$ $k=1, \dots, n$

(2) $\beta = 0$ 时, $\max_{|z| \leq r} |\alpha z^n + \beta| = |\alpha| r^n$, 此时 $z = re^{i\theta}, \theta \in [0, 2\pi)$

2. $f(z) = z^n + \sum_{k=1}^n C_k z^{n-k}$, 证明 $f(z)$ 为分圆多项式 $\Leftrightarrow f(z)$ 的 n 个零点为圆内接正 n 边形的顶点.

(1) $\Rightarrow$: 设 $f(z)$ 为分圆多项式, 即 $f(z) = z^n + C_n$

$$f(z) = 0 \Leftrightarrow z^n = -C_n, \text{ 解得 } z = z_k = |C_n|^{\frac{1}{n}} e^{i \frac{1}{n} [\arg(-C_n) + 2(k-1)\pi]}, k=1, \dots, n$$

$$\text{几何上有 } |z_k| = |C_n|^{\frac{1}{n}}, \arg z_{k+1} - \arg z_k = \arg \frac{z_{k+1}}{z_k} = \arg e^{\frac{i}{n} 2\pi} = \frac{2\pi}{n}, k=1, \dots, n$$

则 $z_1, \dots, z_n$ 构成圆内接正 n 边形的顶点.

(2) $\Leftarrow$: 设 $f(z)$ 的 n 个零点 $z_1, \dots, z_n$ 构成 $\dots$ 顶点, 由零点可设 $f(z) = \prod_{k=1}^n (z - z_k)$

则 $z_k = re^{i\theta_k}, \theta_k = \frac{1}{n} [\phi_0 + 2(k-1)\pi], k=1, \dots, n$ 且 ϕ_0 为实常数 (初相位)

令 $g(z) = z^n + (-1)^n \prod_{k=1}^n z_k$, 下证 $f(z) \equiv g(z)$, 即 $f(z)$ 为分圆多项式.

$$\text{由 } g(z_k) = z_k^n + (-1)^n \prod_{l=1}^n z_l = z_k^n + (-1)^n r^n e^{i \frac{1}{n} [\phi_0 + 2(n-1)\pi]} = z_k^n + (-1)^n r^n e^{i \phi_0 + i \frac{(n-1)2\pi}{n}}$$

$$= z_k^n + (-1)^n r^n e^{i \phi_0 + i(n-1)\pi} = z_k^n + (-1)^n r^n e^{i \phi_0} \cdot \underbrace{(-1)^{n-1}}_{[e^{i\pi}]}$$

$$= (re^{i \frac{1}{n} [\phi_0 + 2(k-1)\pi]})^n - r^n e^{i \phi_0} = r^n e^{i [\phi_0 + 2(k-1)\pi]} - r^n e^{i \phi_0}$$

$$= r^n e^{i \phi_0} \cdot \underbrace{1^{(k-1)}}_{[e^{i2\pi}]} - r^n e^{i \phi_0} = 0, k=1, \dots, n$$

而 $f(z), g(z)$ 均为次数为 n 的首一多项式, 则 $f(z) \equiv g(z)$. $\square$

$$\triangle z = |z| e^{i \arg z}, \text{ 解 } z^n = z_0 \text{ 得 } z = z_k = |z_0|^{\frac{1}{n}} e^{i \frac{\arg z_0 + 2(k-1)\pi}{n}} (k=1, \dots, n)$$

$$\arg \alpha - \arg \beta = \arg \frac{\alpha}{\beta}$$

对称性 $\begin{cases} e^{iz} + e^{-iz} = 2\cos z \\ e^{iz} - e^{-iz} = 2i\sin z \end{cases}$

Chap 2. 解析函数

1. (1) $\cos z = \cos(x+iy)$ 的实 & 虚部

由 $\begin{cases} e^{-iz} = e^{y+ic(-x)} = e^y(\cos x - i\sin x) \\ e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}(\cos x + i\sin x) \end{cases}$

* $\cos(x+iy) = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{e^y + e^{-y}}{2} \cos x + i \frac{e^y - e^{-y}}{2} \sin x$

* $\sin(x+iy) = \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{e^y - e^{-y}}{2i} \cos x + i \frac{e^y + e^{-y}}{2} \sin x = \frac{e^y + e^{-y}}{2} \sin x + i \frac{e^y - e^{-y}}{2} \cos x$

(2) 证明 $\cos(x+iy) = A + iB$ 有无穷多解:

即证 $\begin{cases} A = \frac{e^y + e^{-y}}{2} \cos x \\ B = \frac{e^y - e^{-y}}{2} \sin x \end{cases}$ 有无穷多解.

① $B=0$ 时 $\Rightarrow \sin x = 0$ 或 $y=0$

I) $|A| \leq 1$ 时, 取 $y=0$, 此时另一方程变为 $A = \cos x \in [-1, 1]$

解为 $\begin{cases} x = x_k = \arccos A + 2k\pi, k \in \mathbb{Z} \\ y = 0 \end{cases}$, 有无穷多个.

II) $|A| > 1$ 时, 取 $\sin x = 0$ 且 $\cos x = \begin{cases} 1, A > 0 \\ -1, A < 0 \end{cases}$, 另一方程为 $|A| = \frac{e^y + e^{-y}}{2}$

(可作出函数图象说明) y 有唯一解, 而 $x = \pi I_{A < -1} + 2k\pi$ 无穷多个.

② $B \neq 0$ 时 $\Rightarrow$ 用 y 表示 $\sin x, \cos x$ 并代入 $\cos^2 x + \sin^2 x = 1$ 得

$\left(\frac{2A}{e^y + e^{-y}}\right)^2 + \left(\frac{2B}{e^y - e^{-y}}\right)^2 = 1$, 令 $g(y) = \text{LHS}$ 为偶函数

由 $\lim_{y \rightarrow 0^+} g(y) = +\infty, \lim_{y \rightarrow +\infty} g(y) = 0$, $g(y)$ 连续,

故必 $\exists y_{AB} > 0$, s.t. $g(\pm y_{AB}) = 1$, 即 $g(y) = 1$ 必有解.

代回可知 $\cos x = \frac{2A}{e^{y_{AB}} + e^{-y_{AB}}} \in (-1, 1)$ 有无穷多解. $\square$

(3) 证明 $\sin(x+iy) = A + iB$ 有无穷多解:

即 $\begin{cases} A = \frac{e^y + e^{-y}}{2} \sin x \\ B = \frac{e^y - e^{-y}}{2} \cos x \end{cases}$

① $B=0$ 时 $\Rightarrow \cos x = 0$ 或 $y=0$

I) $|A| \leq 1$ 时, 取 $y=0$, 另一为 $A = \sin x$ 无穷多个

II) $|A| > 1$ 时, 取 $\cos x = 0$ 且 $\sin x = \begin{cases} 1, A > 0 \\ -1, A < 0 \end{cases}$, 另一为 $|A| = \frac{e^y + e^{-y}}{2}$ 有解 (无穷多个)

② $B \neq 0$ 时 $\Rightarrow$ 令 $\left(\frac{2B}{e^y - e^{-y}}\right)^2 + \left(\frac{2A}{e^y + e^{-y}}\right)^2 = 1$, $g(0^+) = +\infty, g(+\infty) = 0, g(y) = 1$ 有解且不恒为 0.

则 $\sin x = \frac{2A}{e^{y_{AB}} + e^{-y_{AB}}}$ 有无穷多解.

或 $f = u + iv$ 中 u/v 为 const (由 C-R 条件, 处处可导 $\Leftrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$)

2. $f(z)$ 的模为 const, 或幅角为 const 且 $f(z)$ 处处可导, 则 $f(z) \equiv \text{const}$.

(分 $f(z) \equiv 0$ 与 $f(z) \neq 0$, 后一类转化为 $g(z) = \ln f(z) = \ln|f(z)| + i \arg f(z) \triangleq u + iv$)

Chap 3. 复积分

1. Cauchy 高阶导数公式:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad \forall n \in \mathbb{N}, \quad f(z) \text{ 在圆盘内可导, 圆周上连续}$$

[例] $C_r: |z|=r>0$ 在复平面处处可导, 则 $f^{(n)}(0) = \frac{n!}{2\pi i} \oint_{C_r} \frac{f(z)}{z^{n+1}} dz$.

① 注意是 $n+1$ 次方

(由 Cauchy ...)

2. 证明: ① $f(z)$ 可导, 若令 $M(r) = \max_{|z|=r} |f(z)|$, 则 $|f^{(n)}(0)| \leq \frac{n! M(r)}{r^n}$

$$f^{(n)}(0) = \frac{n!}{2\pi i} \oint_{|z|=r} \frac{f(z)}{z^{n+1}} dz$$

$$\text{放缩有 } |f^{(n)}(0)| = \frac{n!}{2\pi} \left| \oint_{C_r} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{n!}{2\pi} \oint_{C_r} \frac{|f(z)|}{|z|^{n+1}} |dz| \leq \frac{n!}{2\pi} \oint_{C_r} \frac{\max_{|z|=r} |f(z)|}{|z|^{n+1}} |dz|$$

$$\text{而 RHS} = \frac{n!}{2\pi} \int_0^{2\pi} \frac{M(r)}{r^{n+1}} |d(re^{i\theta})| = \frac{n!}{2\pi} \int_0^{2\pi} \frac{M(r)}{r^n} d\theta = \frac{n! M(r)}{r^n}. \quad \square$$

② 若 $f(z)$ 有界, 则 $f(z) \equiv \text{const}$. (Liouville 定理)

由①知 $\forall n \in \mathbb{N}^*$, $|f^{(n)}(0)| \leq \frac{n! M(r)}{r^n}$, 其中 $M(r) = \max_{|z|=r} |f(z)|$.

由题 $M(r) < C$, 则 $\forall n \geq 1$ 且 $n \in \mathbb{N}$, 令 $r \rightarrow +\infty$ 则 $\text{RHS} \rightarrow 0$, 故 $f^{(n)}(0) = 0$.

将 $f(z)$ 在 $z=0$ Taylor 展开得:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = f(0) \equiv \text{const}. \quad \square$$

③ 若 $\exists$ 常数 $M_0 > 0$, $m \in \mathbb{N}$, $R_0 > 0$ s.t. $|z| \geq R_0$ 时有 $|f(z)| \leq M_0 \sum_{k=0}^m |z|^k$.

则 $f(z)$ 为次数不超过 m 的多次式.

(只须仿照②证 $f^{(m+k)}(0) = 0$, $k \geq 1$ 且 $k \in \mathbb{N}$)

$$\text{有 } |f^{(n)}(0)| \leq \frac{n! M(r)}{r^n} \leq \frac{n! M_0 \sum_{k=0}^m r^k}{r^n} = \frac{n! M_0 \sum_{k=0}^m r^{k-m}}{r^{n-m}} = \frac{n! M_0 \sum_{j=-m}^0 r^j}{r^{n-m}}$$

$$\forall n \geq m+1, |f^{(n)}(0)| \rightarrow 0 \quad (r \rightarrow +\infty), \text{ 即 } f^{(n)}(0) = 0, \text{ 也即 } f^{(m+k)}(0) = 0, \forall k \geq 1. \quad \square$$

3. 由平均性公式 (令 $n=0$) 证最大模原理 (z_0 为圆心的圆周上均为 const).

4. 由 Liouville 证代数基本定理

Chap 4. 级数

1. Abel 定理 (注意均不取等)

若 $f(z) = \sum_{n=0}^{+\infty} C_n z^n$ 在 z_0 收敛, 则 $\forall z: |z| < |z_0|$ 时, $f(z)$ 绝对收敛.

反之若 $f(z)$ 在 z_1 发散, 则 $\forall z: |z| > |z_1|$, $f(z)$ 发散.

2. 收敛半径 $R (> 0)$ 的定义 (注意不取等, 即不含收敛圆周)

$f(z) = \sum_{n=0}^{+\infty} C_n z^n$ 的收敛半径为 $R (> 0)$

$\Leftrightarrow \forall z, |z| < R$ 时 $f(z)$ 绝对收敛, 而 $|z| > R$ 时 $f(z)$ 发散.

3. 举出 3 个级数的例子, 在收敛圆周上分别

(1) 发散: $\sum_{n=1}^{+\infty} z^n$ 在收敛圆周 $|z|=1$ 上发散.
(注意从 1 开始)

* 通项 $\not\rightarrow 0$, 必发散
 $(z^n = e^{in\theta} = \cos n\theta + i \sin n\theta \not\rightarrow 0, n \rightarrow +\infty)$

(2) 有的收敛有的发散: $\sum_{n=1}^{+\infty} \frac{z^n}{n}$, $f(-1) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$ 交错级数, 收敛

(实际上除 $z=-1$ 都发散) $f(1) = \sum_{n=1}^{+\infty} \frac{1}{n}$ 调和级数, 发散

(3) 收敛: $\sum_{n=1}^{+\infty} \frac{z^n}{n^2}$ (可证 $\sum_{n=1}^{+\infty} |\frac{z^n}{n^2}| = \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, 绝对收敛)

Chap 5. 留数

$$1. I = \oint_{|z|=r>1} \frac{z^3 e^{\frac{1}{z}}}{1+z} dz = -\frac{2\pi i}{3}$$

[倒取法] 令 $t = \frac{1}{z}$, 则 $I = \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{1+t} \cdot \frac{1}{t^2} dt$ (取倒取后逆时针变顺, 与 $d(\frac{1}{z}) = -\frac{1}{z^2} dz$ 负号抵消)

有: $I = \oint_{|t|=\frac{1}{r}<1} \frac{e^t}{t^2(1+t)} dt$, 由 $|t|=\frac{1}{r}<1$, 此时仅有一个奇点 $t=0$ 在 $t=0$ 处
故 $I = 2\pi i \operatorname{Res}\left[\frac{e^t}{t^2(1+t)}, 0\right]$, 由定义知 $\operatorname{Res}\left[\frac{e^t}{t^2(1+t)}, 0\right]$ 为展开式中 t^{-1} 系数
也即 $\frac{e^t}{1+t}$ 展开式中 t^1 的系数 (即 m 阶极点处留数 $= \frac{1}{(m-1)!} \frac{d}{dz}^{m-1} f(z) \Big|_{z=0}$)
由 $\frac{e^t}{1+t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot \sum_{k=0}^{\infty} (-1)^k \cdot t^k$ 知 t^1 系数为 $\frac{1}{1!} - \frac{1}{1!} + 1 - 1 = -\frac{1}{3}$
 $\Rightarrow I = -\frac{2\pi i}{3}$

$$2. J_n = \oint_{|z|=r>1} \frac{dz}{1+z^n} = \oint_{|z|=r>1} \frac{z^{2n}}{1+z^n} dz = \begin{cases} 2\pi i, & n=1 \\ 0, & n \geq 2 \end{cases} (n \in \mathbb{N})$$

(一阶极点 $\operatorname{Res} = \frac{P(z_0)}{Q'(z_0)}$)

① $n=1$ 时, $J_1 = \oint_{|z|=r>1} \frac{dz}{1+z} = 2\pi i \operatorname{Res}\left[\frac{1}{1+z}, -1\right] = 2\pi i \cdot \frac{1}{(1+z)'} \Big|_{z=-1} = 2\pi i$

② $n \geq 2$ 时, (另一: $J_1 = \oint_{|z|=r>1} \frac{z^2}{1+z} dz = 2\pi i \operatorname{Res}\left[\frac{z^2}{1+z}, -1\right] = 2\pi i$)

令 $z^n + 1 = 0$ 得奇点 $z = z_k = -\frac{1}{z_k^{n-1}}$ (变形) ($k=1, \dots, n$)

留数求和 $\sum_{k=1}^n \operatorname{Res}\left[\frac{1}{1+z^n}, z_k\right] = \sum_{k=1}^n \frac{1}{n z_k^{n-1}} = \sum_{k=1}^n \frac{1}{n} z_k$ (*)

又由 z_k 为分圆 n 次式零点, z^{n-1} 次系数为 $-\sum_{k=1}^n z_k = 0$ 则 (*) 式为 0.

$$\prod_{k=1}^n (z - z_k) = z^n - \left(\sum_{k=1}^n z_k\right) z^{n-1} + \dots + (-1)^n \prod_{k=1}^n z_k = z^n + 1$$

则 $J_n = 0$.

(另一个: $\operatorname{Res}\left[\frac{z^{2n}}{1+z^n}, z_k\right] = \frac{z_k^{2n}}{n z_k^{n-1}} = \frac{z_k^{n+1}}{n} = -\frac{z_k}{n} (z_k^n = -1)$)

△ 若分母改为 $r^n + z^n$, 范围为 $|z|=R>r$, 则: (奇点为 $z^n = -r^n \Rightarrow z = \dots$)

$n=1$ 时, $\operatorname{Res}\left[\frac{1}{r+z}, -r\right] = 1$, $\oint_{|z|=R>r} \frac{dz}{r^n + z^n} = 2\pi i$

而 $\operatorname{Res}\left[\frac{z^{2n}}{r^n + z^n}, -r\right] = r^2$, $\oint_{|z|=R>r} \frac{z^{2n}}{r^n + z^n} dz = 2\pi i r^2 = 2\pi r^2 i$

$n \geq 2$ 均为 0.

△ 证相号: $\oint_{|z|=R>r} \frac{z^{2n}}{1+z^n} dz = \oint_{|z|=R>r} \frac{(z^{2n}-1)+1}{1+z^n} dz = \oint_{|z|=R>r} (z^n - 1) dz + \oint_{|z|=R>r} \frac{1}{1+z^n} dz$

$= 0$ (Cauchy-Goursat) daollen

$$3. I = \oint_{m,n} \frac{\sin z^n}{z^m} dz \quad (m, n \in \mathbb{N}^*)$$

$m=0$ 时, $f(z) = \frac{\sin z^n}{z^m} = \sin z^n$ 处处解析, 故 $I_{0,n} = 0$.

$m>0$ 时, $I_n = 2\pi i \cdot \text{Res}[\frac{\sin z^n}{z^m}, 0]$, $f(z) = \frac{\sin z^n}{z^m} = \sum_{k=0}^{+\infty} (-1)^k \frac{(z^n)^{2k+1}}{(2k+1)!} \cdot \frac{1}{z^m}$

(由 $\sin x$ 的 Taylor (Maclaurin) 展开为 $x - \frac{1}{3!}x^3 + \dots = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$)

$$\text{令 } n(2k+1) - m = -1, \text{ 得 } k = \frac{1}{2}(\frac{m-1}{n} - 1)$$

为使 $k \in \mathbb{N}^*$, 需要 $m-1-n = 2t$ ($t \in \mathbb{N}^*$)

$$\text{此时 } \text{Res}[\frac{\sin z^n}{z^m}, 0] = (-1)^{\frac{m-1}{2}} \cdot \frac{1}{(\frac{m-1}{n})!}$$

$$\Rightarrow I = \begin{cases} 2\pi i \cdot (-1)^{\frac{m-1}{2}} \cdot \frac{1}{(\frac{m-1}{n})!}, & m = (2t+1)n + 1, t \in \mathbb{N}^* \\ 0, & \text{else.} \end{cases}$$

① 注意 $\sin, \cos$ 的级数

! 是 $z^{n(2k+1)-m}$, $-m$ 在求 n 之后!

$$4. I_n = \oint_{|z|=r>0} \frac{1 - \cos 4z^3}{z^n} dz \quad (n \in \mathbb{N}^*)$$

$n=0$ 时, $f(z) = 1 - \cos 4z^3$ 在复平面处处解析, 由 Cauchy-Goursat 知 $I_n = 0$.

$n>0$ 时, $I_n = 2\pi i \text{Res}[\frac{1 - \cos 4z^3}{z^n}, 0]$ (<0 也是)

留数即 $\frac{1 - \cos 4z^3}{z^n}$ 中 z^{-1} 的系数, 也即 $1 - \cos 4z^3$ 中 z^{n-1} 的系数.

由 $\cos x$ 的 Taylor 展开为 $1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$$\text{则 } 1 - \cos 4z^3 = 1 - \sum_{n=0}^{+\infty} (-1)^n \frac{(4z^3)^{2n}}{(2n)!} = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{(4z^3)^{2n}}{(2n)!} = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{4^{2n}}{(2n)!} z^{6n}$$

$$\text{令 } 6k = n-1 \text{ 得 } k = \frac{n-1}{6}$$

为使 $k \in \mathbb{N}^*$, 需要 $n-1 = 6m+1$ ($m \in \mathbb{N}^*$), 此时 $\text{Res}[\frac{1 - \cos 4z^3}{z^n}, 0] = (-1)^{\frac{n-1}{6}} \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}$

$$\Rightarrow I = \begin{cases} 2\pi i \cdot (-1)^{\frac{n-1}{6}} \cdot \frac{4^{\frac{n-1}{3}}}{(\frac{n-1}{3})!}, & n = 6k+1, k \in \mathbb{N}^* \\ 0, & \text{else} \end{cases}$$

[实积分-三角]

$$J = \int_{a,b}^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, \quad a>|b|\geq 0, \quad a,b\in\mathbb{R}$$

① 证相等: $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} \xrightarrow{t=\theta-\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{dt}{a+b\sin(t+\frac{\pi}{2})} = \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{dt}{a+b\cos t} \xrightarrow{\text{周期}} \int_0^{2\pi} \frac{dt}{a+b\cos t}$

② 求积分:

令 $z=e^{i\theta} (= \cos\theta + i\sin\theta)$, 则 $\cos\theta = \frac{z+z^{-1}}{2} \quad (|z|=1)$

将实积分转化为复积分. $d\theta = \frac{dz}{iz}$ (不好求)

$$\Rightarrow I_{a,b} = \oint_{|z|=1} \frac{\frac{1}{iz} dz}{a+b \cdot \frac{z+z^{-1}}{2}}$$

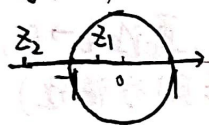
$$= \frac{2}{i} \oint_{|z|=1} \frac{dz}{bz^2+2az+b}$$

为用留数, 考虑 $bz^2+2az+b=0$ 的解.

I) $b=0$ 时, 解为 $z=0$, $I_{a,b} = 2\pi i \cdot \text{Res}[\frac{1}{2az}, 0] \cdot \frac{2}{i} = 2\pi i \cdot \frac{1}{2a} \cdot \frac{2}{i} = \frac{2\pi}{a}$

II) $b>0$ 时, 方程即 $z^2 + \frac{2a}{b}z + 1 = 0$, $\Delta = \frac{4a^2}{b^2} - 4 > 0$ (易漏①)

解为 $\begin{cases} z_1 = -\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b} < 0 \\ z_2 = -\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b} < -\frac{a}{b} < -1 \end{cases}$, 而 $z_1 z_2 = 1$ 知 $z_1 > -1$.



$$\Rightarrow I_{a,b} = 2\pi i \cdot \frac{2}{i} \text{Res}[\frac{1}{bz^2+2az+b}, z_1] = 4\pi \cdot \frac{1}{2bz_1+2a} = \frac{2\pi}{bz_1+a}$$

$$= \frac{2\pi}{-a+\sqrt{a^2-b^2}+a} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$b<0$ 同理 ($z_2 > 1 > z_1 > 0$), $I_{a,b} = 2\pi i \cdot \frac{2}{i} \text{Res}[\frac{1}{bz^2+2az+b}, z_1] = \frac{2\pi}{\sqrt{a^2-b^2}}$

① 注意 $b<0$ 时 $-\frac{\sqrt{a^2-b^2}}{b} > 0$!!!

$$\Rightarrow I_{a,b} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

b. $I = \int_0^{2\pi} \frac{d\theta}{A^2\cos^2\theta + B^2\sin^2\theta} = \frac{2\pi}{AB}, \quad A,B>0$

由 $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, 转化为 5: $I = \int_0^{2\pi} \frac{2d\theta}{(A^2+B^2) + (A^2-B^2)\cos 2\theta}$

令 $t=2\theta$, $I = \int_0^{2\pi} \frac{dt}{(A^2+B^2) + (A^2-B^2)\cos t} = 2 \int_0^{2\pi} \frac{dt}{(A^2+B^2) + (A^2-B^2)\cos t}$

$$= 2 \int_0^{2\pi} \frac{dt}{a + b\cos t} \quad \begin{matrix} a=A^2+B^2 \\ b=A^2-B^2 \end{matrix} = 2 \cdot \frac{2\pi}{\sqrt{(A^2+B^2)^2 - (A^2-B^2)^2}} = 2 \cdot \frac{2\pi}{\sqrt{4A^2B^2}} = \frac{2\pi}{AB}$$

① 要保证 $a>b$

(定理: 对有理分式 $f(z) = \frac{P_n(z)}{Q_m(z)}$, 若 ① $Q_m(z)$ 在 $\mathbb{R}$ 上无零点, ② $\deg P_n(z) - \deg Q_m(z) = -n-m \leq -2$ 则 $\lim_{R \rightarrow +\infty} \int_{C_R} \frac{P_n(z)}{Q_m(z)} dz = 0$ (上半平面))

7. $\int_0^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, a>0, b>0$

$I = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2} \lim_{R \rightarrow +\infty} \int_{-R}^R \frac{dx}{(x^2+a^2)(x^2+b^2)}$ (由 $f(x)$ 为偶函数)

作出 $\Gamma_R: [-R, R] \cup C_R, R > \max\{a, b\}$

$C_R: z = Re^{i\theta}, \theta \in [0, \pi]$
 $\oint_{\Gamma_R} f(z) dz = \int_{-R}^R f(z) dz + \int_{C_R} f(z) dz$
 令 $R \rightarrow +\infty$, 则 $\int_{C_R} f(z) dz \rightarrow 0$

$a \neq b \Rightarrow I = \frac{1}{2} \cdot 2\pi i \cdot (\text{Res}[f(z), ai] + \text{Res}[f(z), bi])$ (均为一阶极点)

$= \pi i \cdot \left(\frac{1}{(z^2+(a^2+b^2)z^2+a^2b^2)} \Big|_{z=ai} + \dots \right)$

$= \pi i \cdot \left(\frac{1}{4z^2+2(a^2+b^2)z} \Big|_{z=ai} + \dots \right)$

$= \pi i \cdot \left(\frac{1}{2ai(b^2-a^2)} + \frac{1}{2bi(a^2-b^2)} \right)$

$= \pi \cdot \frac{1}{2i(b^2-a^2)} \left(\frac{1}{a} - \frac{1}{b} \right)$

$= \pi \cdot \frac{1}{2(b+a)(b-a)} \cdot \frac{b-a}{ab}$

$= \frac{\pi}{2ab(a+b)}$

$a=b$ 时, ai 成为二阶极点, 实际上也可当作一阶做 (令 $b \neq a$, 再让 $b \rightarrow a$, 若存在则为该值)

$I = \frac{\pi}{4a^3}$

8. $\int_0^{+\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}, a>0, b>0$

取 $R > \max\{a, b\}$, 作出 $\Gamma_R: [-R, R] \cup C_R$, 其中 $C_R: z = Re^{i\theta}, \theta \in [0, \pi]$

有 $\oint_{\Gamma_R} f(z) dz = \int_{-R}^R f(z) dz + \int_{C_R} f(z) dz$, 令 $R \rightarrow +\infty$ 有 $\int_{C_R} f(z) dz = 0$.

$\Rightarrow I = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2} \lim_{R \rightarrow +\infty} \int_{-R}^R f(z) dz = \frac{1}{2} \lim_{R \rightarrow +\infty} \oint_{\Gamma_R} f(z) dz$

① ($a \neq b$) $= \frac{1}{2} \cdot 2\pi i \cdot [\text{Res}[f(z), ai] + \text{Res}[f(z), bi)]$

(求一阶极点) $\text{Res}[f(z), ai] = \frac{z^2}{4z^2+2(a^2+b^2)z} \Big|_{z=ai} = \frac{-a^2}{2ai(b^2-a^2)} = \frac{-a}{2i(b^2-a^2)}$

交换 a, b 得 $\text{Res}[f(z), bi] = \frac{-b}{2i(a^2-b^2)}$

则 $I = \pi i \cdot \frac{1}{2i(b^2-a^2)} (-a+b) = \frac{\pi}{2} (a+b)$

② $a=b, I = \frac{\pi}{4a} (b \rightarrow a)$

[定理: $\frac{P(x)}{Q(x)} e^{ikx}$ 中若 ① $Q(x)$ 在 $\mathbb{R}$ 上无零点, ② $\deg P - \deg Q \geq 1$, 则 $\lim_{R \rightarrow +\infty} \int_{\Gamma_R} \frac{P}{Q} e^{ikx} dx = 0$]

$$9. I = \int_0^{+\infty} \frac{x \sin kx}{x^2 + a^2} dx = \begin{cases} \frac{\pi}{2e^{ka}}, & a > 0, k > 0 \\ \frac{\pi}{2}, & a = 0, k > 0 \end{cases} \Rightarrow \int_0^{+\infty} \frac{\sin kx}{x} dx = \begin{cases} \frac{\pi}{2}, & k > 0 \\ 0, & k = 0 \text{ (令 } a \rightarrow 0) \\ -\frac{\pi}{2}, & k < 0 \text{ (类似一下符号)} \end{cases}$$

(注意不能直接将 $\sin kx$ 换为 $\sin kz$, 而要用 e^{ikz} 形式)

由 $e^{ikx} = \cos kx + i \sin kx$ 而 $\frac{x \cos kx}{x^2 + a^2}$ 为奇函数

$$\text{则 } I = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{x \sin kx}{x^2 + a^2} dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{i} \cdot \frac{x \cos kx + i x \sin kx}{x^2 + a^2} dx = \frac{1}{2i} \lim_{R \rightarrow +\infty} \int_{\Gamma_R} \frac{x e^{ikx}}{x^2 + a^2} dx$$

$$\lim_{R \rightarrow +\infty} \oint_{\Gamma_R} \frac{z e^{ikz}}{z^2 + a^2} dz = \frac{1}{2i} \cdot 2\pi i \operatorname{Res} \left[\frac{z e^{ikz}}{z^2 + a^2}, ai \right] = \pi \cdot \operatorname{Res} \left[\frac{z e^{ikz}}{z^2 + a^2}, ai \right]$$

($R > a$)

① $a = 0$, $\operatorname{Res} \left[\frac{z e^{ikz}}{z^2}, 0 \right] = \operatorname{Res} \left[\frac{e^{ikz}}{z}, 0 \right] = e^{ikz} \Big|_{z=0} = 1$

$$\Rightarrow I = \frac{\pi}{2}$$

② $a \neq 0$, $\operatorname{Res} \left[\frac{z e^{ikz}}{z^2 + a^2}, ai \right] = \frac{z e^{ikz}}{2z} \Big|_{z=ai} = \frac{e^{-ka}}{2} = \frac{1}{2e^{ka}}$

($a > 0$) $\Rightarrow I = \frac{\pi}{2e^{ka}}$

$$10. I = \int_0^{+\infty} \frac{x \sin kx}{(x^2 + a^2)(x^2 + b^2)} dx, \quad a, b, k > 0$$

取 $R > \max\{a, b\}$ 作 $\Gamma_R: [-R, R] \cup \Gamma_R$

$$I = \frac{1}{2i} \lim_{R \rightarrow +\infty} \oint_{\Gamma_R} \frac{z e^{ikz}}{(z^2 + a^2)(z^2 + b^2)} dz = \frac{1}{2i} \cdot 2\pi i \cdot \sum_k C_k^{(k)}$$

① $a \neq b$,

$$\operatorname{Res} \left[\frac{z e^{ikz}}{(z^2 + a^2)(z^2 + b^2)}, ai \right] = \frac{z e^{ikz}}{(z^2 + (a^2 + b^2)z + a^2 b^2)} \Big|_{z=ai} = \frac{z e^{ikz}}{4z^2 + 2z(a^2 + b^2)} \Big|_{z=ai}$$

$$= \frac{e^{ik(ai)}}{2(2a^2 + a^2 + b^2)} = \frac{1}{2(b^2 - a^2)} e^{-ka}$$

$$\operatorname{Res} [f(z), bi] = \frac{1}{2(a^2 - b^2)} e^{-kb}$$

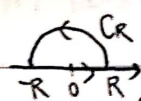
$$\Rightarrow I = \pi \cdot \frac{1}{2(b^2 - a^2)} \left(\frac{1}{e^{ka}} - \frac{1}{e^{kb}} \right) = \frac{\pi}{2(b^2 - a^2)} (e^{-ka} - e^{-kb})$$

② $a = b$, 令上式 $b \rightarrow a$ (a 当作 const), 由洛必达法则得 $\frac{k\pi e^{-kb}}{4b} \Big|_{b=a} = \frac{k\pi e^{-ka}}{4a}$

$$\Rightarrow I = \frac{\pi k e^{-ka}}{4a}$$

11. $\int_0^{+\infty} \frac{x^3 \sin kx}{(x^2+a^2)(x^2+b^2)} dx, a, b, k > 0$

$$I = \frac{1}{2i} \lim_{R \rightarrow +\infty} \oint_{\Gamma_R} f(z) dz = \frac{1}{2i} \cdot 2\pi i \sum_k C_{-1}^{(k)} = \pi \sum_k C_{-1}^{(k)}, \quad f(z) = \frac{z^3 e^{ikz}}{(z^2+a^2)(z^2+b^2)}$$



① $a \neq b$ 时, $\text{Res}\left[\frac{z^3 e^{ikz}}{(z^2+a^2)(z^2+b^2)}, ai\right] = \frac{z^3 e^{ikz}}{4z^3 + 2z(a^2+b^2)} \Big|_{ai} = \frac{(ai)^3 e^{-ka}}{2ai(b^2-a^2)} = \frac{-a^2}{2(b^2-a^2)e^{ka}}$

$$\text{Res}[f(z), bi] = \frac{-b^2}{2(a^2-b^2)e^{kb}}$$

$$\Rightarrow I = \pi \cdot \frac{1}{2(b^2-a^2)} \cdot (-a^2 e^{-ka} + b^2 e^{-kb})$$

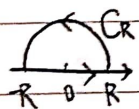
① 注意不要重复使用!

② $a=b$ 时, $\lim_{b \rightarrow a} \frac{-kb^2 e^{-kb} + 2be^{-kb}}{4b} = \frac{\pi}{4} \cdot \frac{-kae^{-ka} + 2e^{-ka}}{4} = \frac{\pi(2-ka)e^{-ka}}{4}$

$$\Rightarrow I = \frac{\pi(2-ka)e^{-ka}}{4}$$

12. $\int_0^{+\infty} \frac{x^2 \cos kx}{(x^2+a^2)(x^2+b^2)} dx, a, b, k > 0$

$$I = \frac{1}{2} \lim_{R \rightarrow +\infty} \oint_{\Gamma_R} \frac{z^2 e^{ikz}}{(z^2+a^2)(z^2+b^2)} dz = \frac{1}{2} \cdot 2\pi i \cdot \sum_k C_{-1}^{(k)}$$



⚠ $\cos kx$ 就没有 $\frac{1}{i}$!

① $a \neq b$ 时, $\text{Res}[f(z), ai] = \frac{z^2 e^{ikz}}{2z(2z^2+a^2+b^2)} \Big|_{z=ai} = \frac{ai e^{-ka}}{2(b^2-a^2)}$

$$\text{Res}[f(z), bi] = \frac{bi e^{-kb}}{2(a^2-b^2)} = -\frac{\pi(ae^{-ka} - be^{-kb})}{2(b^2-a^2)}$$

$$\Rightarrow I = \pi i \cdot \frac{i}{2(b^2-a^2)} (ae^{-ka} - be^{-kb}) = -\frac{\pi(ae^{-ka} - be^{-kb})}{2(b^2-a^2)}$$

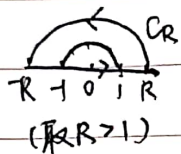
② $a=b$ 时, $\lim_{b \rightarrow a} -\frac{\pi}{2} \cdot \frac{ae^{-ka} - be^{-kb}}{b^2-a^2} = \lim_{b \rightarrow a} \left(-\frac{\pi}{2}\right) \cdot \frac{-e^{-kb} + kbe^{-kb}}{2b} = \left(-\frac{\pi}{2}\right) \cdot \frac{-e^{-ka} + kae^{-ka}}{2a}$

$$\Rightarrow I = -\frac{\pi(ka-1)e^{-ka}}{4a}$$

$$13. I_{r,n} = \int_0^{+\infty} \frac{dx}{r^{2n} + x^{2n}} = \frac{1}{r^{2n-1}} \frac{\pi}{\sin \frac{\pi}{2n}} \quad \left(\frac{1}{r^{2n-1}} I_n \right) \quad (r > 0, n \in \mathbb{Z}^+)$$

(转化):

$$I_{r,n} = \frac{1}{r^{2n}} \int_0^{+\infty} \frac{dx}{1 + (\frac{x}{r})^{2n}} = \frac{1}{r^{2n-1}} \int_0^{+\infty} \frac{d(\frac{x}{r})}{1 + (\frac{x}{r})^{2n}} \stackrel{t=\frac{x}{r}}{=} \frac{1}{r^{2n-1}} \int_0^{+\infty} \frac{dt}{1+t^{2n}}, \quad \text{令 } I_n = \int_0^{+\infty} \frac{dx}{1+x^{2n}}$$



$$\oint_{I_R} \frac{dz}{1+z^{2n}} = 2\pi i \sum_{k=1}^n \text{Res} \left[\frac{1}{1+z^{2n}}, z_k \right] = 2\pi i \cdot \sum_{k=1}^n \frac{1}{2n z_k^{2n-1}}, \quad \text{代入 } z_k^{2n} = -1 \text{ 即 } z_k^{2n-1} = \frac{-1}{z_k}$$

得 $\sum_{k=1}^n \text{Res} \left[\frac{1}{1+z^{2n}}, z_k \right] = \sum_{k=1}^n \frac{1}{2n z_k^{2n-1}} = \sum_{k=1}^n \frac{-z_k}{2n}$, 而 $z_k = (-1)^{\frac{k-1}{2n}} = \left(e^{\pi i + 2k\pi} \right)^{\frac{k-1}{2n}} = e^{\frac{(2k-1)\pi i}{2n}}$

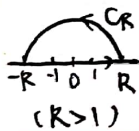
$$\sum_{k=1}^n z_k = \sum_{k=1}^n e^{-\frac{\pi i}{2n}} \cdot e^{\frac{k\pi i}{n}} = e^{-\frac{\pi i}{2n}} \cdot \frac{e^{\frac{\pi i}{n}} (1 - e^{\frac{\pi i}{n}})}{1 - e^{\frac{\pi i}{n}}} = \frac{2e^{\frac{\pi i}{2n}}}{1 - e^{\frac{\pi i}{n}}} \stackrel{\theta = \frac{\pi}{2n}}{=} \frac{2e^{i\theta}}{1 - e^{2i\theta}} = \frac{2}{e^{-i\theta} - e^{i\theta}}$$

(由 $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$)

$$\Rightarrow I_n = \frac{1}{2} \oint_{I_R} \frac{dz}{1+z^{2n}} = \frac{1}{2} \cdot 2\pi i \cdot \left(-\frac{1}{2n} \right) \cdot \left(-\frac{1}{i \sin \frac{\pi}{2n}} \right) = \frac{\pi}{\sin \frac{\pi}{2n}}$$

$$\Rightarrow I_{r,n} = \frac{1}{r^{2n-1}} I_n = \frac{1}{r^{2n-1}} \frac{\pi}{\sin \frac{\pi}{2n}}$$

$$14. J_{r,n} = \int_0^{+\infty} \frac{dx}{(r^2 + x^2)^n} = \frac{1}{r^{2n-1}} J_n = \frac{1}{r^{2n-1}} \cdot \frac{\pi C_{2n-2}^{n-1}}{2^{2n-1}} \quad (r > 0, n \in \mathbb{Z}^+)$$



(转化)

$$J_{r,n} = \frac{1}{r^{2n}} \int_0^{+\infty} \frac{dx}{(1 + (\frac{x}{r})^2)^n} = \frac{1}{r^{2n-1}} \int_0^{+\infty} \frac{dx}{(1+x^2)^n} = \frac{1}{r^{2n-1}} J_n, \quad \text{令 } J_n = \int_0^{+\infty} \frac{dx}{(1+x^2)^n}$$

$$J_n = \frac{1}{2} \lim_{R \rightarrow +\infty} \oint_{I_R} \frac{dz}{(1+z^2)^n} = \frac{1}{2} \cdot 2\pi i \cdot \text{Res} \left[\frac{1}{(1+z^2)^n}, i \right]$$

[引理: 推广的二项式] $(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} z^k + \dots$

令 $\alpha = -n$ 得 $\frac{1}{(1+z)^n} = 1 - nz + \frac{n(n+1)}{2!} z^2 + \dots + \frac{(-1)^k \frac{n(n+1)\dots(n+k-1)}{k!}}{(-1)^k \frac{n(n+1)\dots(n+k-1)}{k!}} z^k + \dots$

$\nearrow C_{n+k-1}^k$

为使用 $\frac{1}{(1+z^2)^n}$, 对 z^2+1 因式分解为 $(z-i)(z+i)$, 即

$$f(z) = \frac{1}{(1+z^2)^n} = \frac{1}{(z-i)^n (z+i)^n}$$

$$= \frac{1}{(z-i)^n (z-i+i)^n} = \frac{1}{(z-i)^n (2i)^n \left(1 + \frac{z-i}{2i}\right)^n} = \frac{1}{(z-i)^n (2i)^n} \sum_{k=0}^{+\infty} (-1)^k C_{n+k-1}^k \left(\frac{z-i}{2i}\right)^k$$

令 $k = n-1$ 得 $\frac{1}{z-i}$ 的系数 $C_{-1} = \frac{1}{(2i)} (-1)^{n-1} C_{2n-2}^{n-1} \cdot \frac{1}{(2i)^{n-1}} = \frac{(-1)^{n-1}}{2^{2n-1} i^{2n-1}} C_{2n-2}^{n-1} = -\frac{1}{2^{2n-1} i}$

$$\Rightarrow J_n = \pi i \cdot \frac{-1}{2^{2n-1} i} C_{2n-2}^{n-1} = \frac{\pi C_{2n-2}^{n-1}}{2^{2n-1}}$$

$$\Rightarrow J_{r,n} = \frac{1}{r^{2n-1}} \cdot \frac{\pi C_{2n-2}^{n-1}}{2^{2n-1}} = \frac{\pi C_{2n-2}^{n-1}}{(2r)^{2n-1}}$$

Chap 6. 共形映射

$$w = \frac{az+b}{cz+d}$$

[分式: 圆盘 $\rightarrow$ 圆盘]

1. 求将单位圆盘 $|z| < 1$ 映为 $|w| < 1$, 给定点 z_1 映为原点的分式线性映射

令 $\alpha = \frac{a}{c}$, 则 $w = \alpha \frac{z-z_1}{z-\bar{z}_1}$ (点 z_1 映为原点, 故 z_1 的对称点 z_1' 映为无穷远, 则 $z_2 = z_1'$)

对称点满足 $z_1' = z_0 + \frac{r^2}{\bar{z}_1 - z_0}$ (z_0 为圆心, r 为半径), 对单位圆盘 $z_1' = \frac{1}{\bar{z}_1}$

$$\Rightarrow w = \alpha \cdot \frac{z-z_1}{z-\frac{1}{\bar{z}_1}} = \alpha \cdot (-\bar{z}_1) \cdot \frac{z-z_1}{1-z\bar{z}_1} \triangleq \beta \cdot \frac{z-z_1}{1-z\bar{z}_1}$$

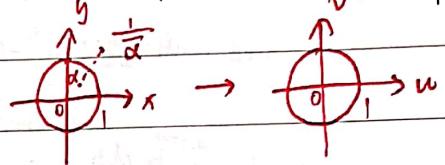
由最大模原理, $|z|=1 \Leftrightarrow |w|=1$, 建立等式求 β :

$$1 = |\beta| \cdot \left| \frac{z-z_1}{1-z\bar{z}_1} \right|_{|z|=1}, \text{ 而 } |z|=z\bar{z}=1 \text{ 时 } \left| \frac{z-z_1}{1-z\bar{z}_1} \right| = \left| \frac{z-z_1}{z\bar{z}-z\bar{z}_1} \right| = \left| \frac{z-z_1}{\bar{z}-\bar{z}_1} \right| = \left| \frac{z-z_1}{\overline{z-z_1}} \right| = 1$$

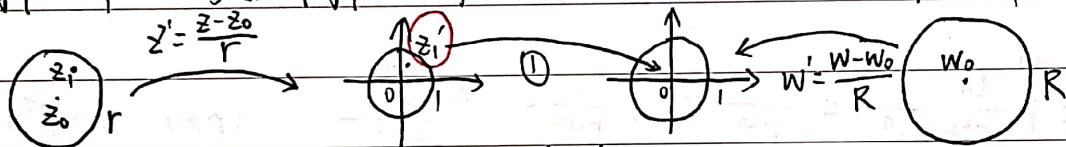
$$\Rightarrow |\beta| = 1$$

$$\text{设 } \beta = e^{i\theta}, \theta \in [0, 2\pi)$$

$$\Rightarrow w = e^{i\theta} \cdot \frac{z-z_1}{1-z\bar{z}_1}, \text{ 其中 } |z_1| < 1, \theta \in [0, 2\pi)$$



2. 圆 $|z-z_0| < r$ 映成圆 $|w-w_0| < R$, 且使 $w(z_1) = w_0$ ($|z_1-z_0| < r$)



由 1. 知 ① 处映射为 $w' = e^{i\theta} \cdot \frac{z'-z_1'}{1-z'\bar{z}_1'}$! 全部加'!

$$\text{即 } w' = \frac{w-w_0}{R} = e^{i\theta} \cdot \frac{z'-z_1'}{1-z'\bar{z}_1'}, \text{ 代入 } z' = \frac{z-z_0}{r} \text{ 得 } \frac{w-w_0}{R} = e^{i\theta} \cdot \frac{\frac{z-z_0}{r} - \frac{z_1-z_0}{r}}{1 - \frac{z-z_0}{r} \cdot \frac{\bar{z}_1-z_0}{r}}$$

$$\Rightarrow w = w_0 + e^{i\theta} \cdot R \cdot r \cdot \frac{z-z_1}{r^2(z-z_0)(\bar{z}_1-\bar{z}_0)}, \text{ 其中 } |z_1-z_0| < r, \theta \in [0, 2\pi)$$

3. $|z|$ 的不变式 $\frac{|dw|}{1-|w|^2} = \frac{|dz|}{1-|z|^2}$

证明: 原式 $\Leftrightarrow \frac{|dw|}{|dz|} (1-|z|^2) = 1-|w|^2$, 可知 $1-|z|^2$ 与 $1-|w|^2$ 同号

① 注意 $e^{i\theta}$ 在分母!

对于 $w = \frac{az+b}{cz+d}$, (即 $\frac{dw}{dz}$) $\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2}$, 代入 1 中 $a = e^{i\theta}, b = -e^{i\theta}z_1, c = -\bar{z}_1, d = 1$

$$\Rightarrow \frac{dw}{dz} = \frac{e^{i\theta} - e^{i\theta}z_1\bar{z}_1}{(-\bar{z}_1z + 1)^2} = e^{i\theta} \cdot \frac{1-|z_1|^2}{(1-z\bar{z}_1)^2}, \text{ 原式 } \Leftrightarrow \frac{1-|z|^2}{|1-z\bar{z}_1|^2} (1-|z|^2) = 1-|w|^2$$

$$\begin{aligned} \text{RHS} &= 1-|w|^2 = 1-w\bar{w} = 1 - e^{i\theta} \cdot \frac{z-z_1}{1-z\bar{z}_1} \cdot e^{-i\theta} \cdot \frac{\bar{z}-\bar{z}_1}{1-\bar{z}\bar{z}_1} \\ &= 1 - \frac{(z-z_1)(\bar{z}-\bar{z}_1)}{|1-z\bar{z}_1|^2} = \dots = \text{LHS} \end{aligned}$$

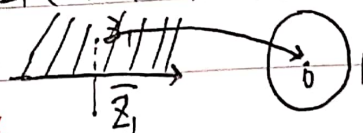
(共轭与+、-可交换)

4. $|z|$ 的不变式 $\frac{R|dw|}{R^2-|w-w_0|^2} = \frac{r|dz|}{r^2-|z-z_0|^2}$, 将 $z' = \frac{z-z_0}{r}, w' = \frac{w-w_0}{R}$ 代入 $\frac{|dw|}{1-|w|^2} = \frac{|dz|}{1-|z|^2}$

[半平面 $\rightarrow$ 圆盘]

5. 上半平面 ($\text{Im } z > 0$) $\rightarrow$ 圆盘, 某点 z_1 ($\text{Im } z_1 > 0$) 到圆心:

$$w = \frac{a(z-z_1)}{c(z-\bar{z}_1)} = \alpha \cdot \frac{z-z_1}{z-\bar{z}_1},$$



① 注意是
关于边界(实轴)的对称点.

由最大模原理,

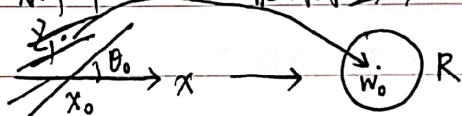
$$\forall z = x \in \mathbb{R} \Leftrightarrow |w| = 1, \text{ 即 } |w| = 1 = |\alpha| \left| \frac{x-z_1}{x-\bar{z}_1} \right|, \text{ 取 } x=0 \text{ 得 } |\alpha| = 1$$

$$\Rightarrow \alpha = e^{i\theta}, \theta \in [0, 2\pi)$$

$$\Rightarrow w = e^{i\theta} \cdot \frac{z-z_1}{z-\bar{z}_1}, \text{Im } z_1 > 0, \theta \in [0, 2\pi)$$

若令 $\theta=0, z_1=i$ 得到最简单的上半平面 $\rightarrow$ 圆盘: $w = \frac{z-i}{z+i}$

6. 斜平面 $\rightarrow$ 一般圆盘, ...



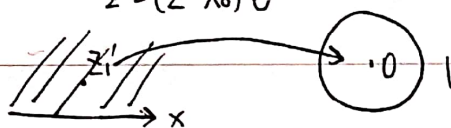
由 5. $w' = e^{i\theta} \cdot \frac{z'-z_1'}{z'-\bar{z}_1'}$ 代入 $z' = (z-x_0)e^{-i\theta_0}$ 得

先平移

$$z' = (z-x_0)e^{-i\theta_0}$$

$$w' = \frac{w-w_0}{R}$$

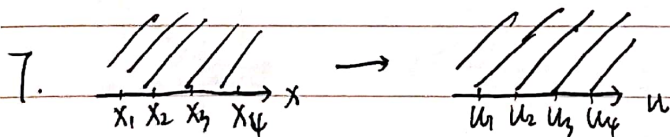
$$w' = \frac{w-w_0}{R} = e^{i\theta} \cdot \frac{(z-x_0)e^{-i\theta_0} - (z_1-x_0)e^{-i\theta_0}}{(z-x_0)e^{-i\theta_0} - (\bar{z}-x_0)e^{-i\theta_0}}$$



$$\Rightarrow w = w_0 + R e^{i\theta} \cdot \frac{z-z_1}{(z-x_0) - (\bar{z}-x_0)e^{2i\theta_0}},$$

其中 $\theta \in [0, 2\pi)$, $\text{Re } z_1 > x_0$

△ 圆盘 $\rightarrow$ 半平面: 反解 $z = f(w)$



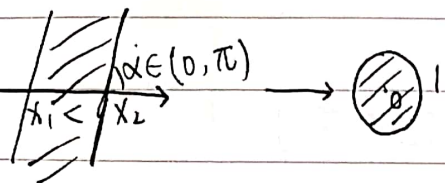
$$\text{求 } w = \frac{az+b}{cz+d}$$

证明若 $a, b, c, d \in \mathbb{R}$, 则 $\text{Im } z > 0 \Leftrightarrow \text{Im } w > 0$ 的充要条件是 $ad - bc > 0$.

方法: 将 $z = x + iy$ 代入 $w = \frac{az+b}{cz+d}$, 与 $w = u + iv$ 对比得 $v = f(a, b, c, d, x, y)$

$[e^z]$ 长条 (π) 映到上半平面.

8. 平行长条映成单位圆盘的单值解析映射.



① 先映成上半平面的长条, 宽 $h = (x_2 - x_1) \sin \alpha$.

$$z' = (z - x_2) e^{-i\alpha} \quad (x_2 \text{ 为原点, 否则 } x_1 \rightarrow \text{下半})$$

② 宽 $h \rightarrow$ 宽为 π .

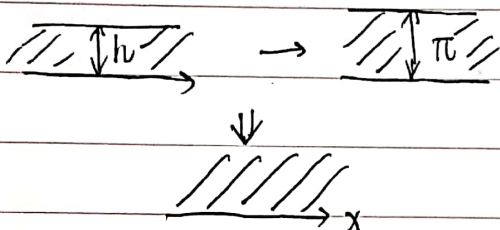
$$z'' = \frac{\pi}{h} z' = \frac{\pi}{(x_2 - x_1) \sin \alpha} (z - x_2) e^{-i\alpha}$$

③ 宽为 $\pi \rightarrow$ 上半平面.

$$z''' = e^{z''}$$

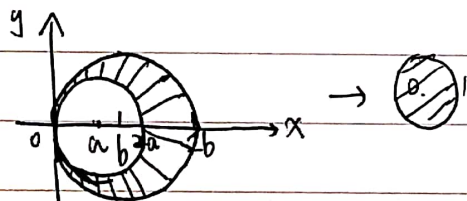
④ 上半平面 $\rightarrow$ 单位圆盘.

$$W = \frac{z''' - i}{z''' + i} = \frac{e^{\frac{\pi}{(x_2 - x_1) \sin \alpha} (z - x_2) e^{-i\alpha}} - i}{e^{\frac{\pi}{(x_2 - x_1) \sin \alpha} (z - x_2) e^{-i\alpha}} + i}$$



$[z^\alpha]$ $\alpha > 1$, 不保角(放大): 从 $z = re^{i\theta} \Rightarrow w = r^\alpha e^{i\alpha\theta}$

9. 区域 $D = \{z \mid |z-a| > a, |z-b| > b, 0 < a < b\}$ 映成 $D' = \{z \mid |z| < 1\}$ 单位圆盘.



① 圆弧拉直, 变为竖直长条.

$$\text{令 } z' = \frac{z-2a}{z} \quad (\text{原点})$$

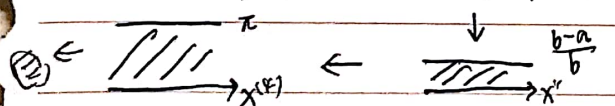
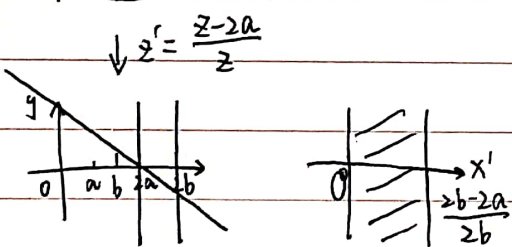
② 逆时针 $\frac{\pi}{2}$: (原来的 0 变为无穷远)

$$z'' = e^{i\frac{\pi}{2}} z' = iz' \quad (\text{关于实轴对称点})$$

③ 宽为 π : $z''' = \frac{\pi}{\frac{b-a}{b}} z'' = \frac{b\pi i}{b-a} \cdot \frac{z-2a}{z}$

④ 上半平面: $z^{(4)} = e^{z'''} = \dots$

$$\text{⑤ 单位圆盘: } W = \frac{z^{(4)} - i}{z^{(4)} + i} = \frac{e^{\frac{b\pi i}{b-a} \cdot \frac{z-2a}{z}} - i}{e^{\frac{b\pi i}{b-a} \cdot \frac{z-2a}{z}} + i}$$



10. 弓形 $\angle A$, $\theta_0 \in (0, \pi)$ 映到 $|w| < 1$.

① 圆弧拉直 + 保角 ($A \rightarrow$ 原点, $B \rightarrow$ 无穷远)

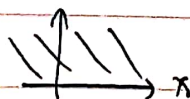
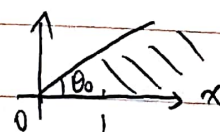
$$z' = \frac{z-A}{z-B} \quad (z = \frac{A+B}{2} \text{ 时 } z' = -1, \text{ 修正}) \quad \frac{z-A}{B-z}$$

② 扩角, 从 $[0, \theta_0)$ 到 $[0, \pi)$

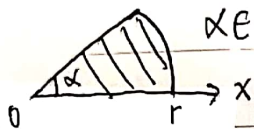
$$z'' = (z')^{\frac{\pi}{\theta_0}} = \left(\frac{z-A}{B-z} \right)^{\frac{\pi}{\theta_0}}$$

③ 单位圆盘

$$W = \frac{z'' - i}{z'' + i} = \frac{\left(\frac{z-A}{B-z} \right)^{\frac{\pi}{\theta_0}} - i}{\left(\frac{z-A}{B-z} \right)^{\frac{\pi}{\theta_0}} + i}$$



11. $\alpha \in (0, \pi), r > 0$ 单值解析映射到 $|w| < 1$.



① 打角为上半圆 ($\theta_0 = \frac{\pi}{2}$ 的弓形)

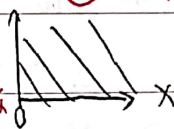
$$z' = z^{\frac{\pi}{\alpha}}$$



① 注意半径也变了!!!

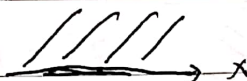
② 圆弧拉直 + 保角

$$z'' = \frac{z' - (-r)}{r - z'} = \frac{z' + r}{r - z'}$$



③ 打角为上半平面

$$z''' = (z'')^2 = \left(\frac{z' + r}{r - z'} \right)^2$$



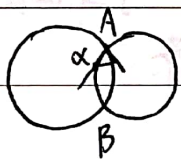
④ 单位圆盘

$$w = \frac{z''' - i}{z''' + i} = \frac{\left(\frac{z' + r}{r - z'} \right)^2 - i}{\left(\frac{z' + r}{r - z'} \right)^2 + i}$$



检验: 代入 α, r

12. $\alpha \in (0, \pi)$ $\rightarrow |w| < 1$ (???)



①