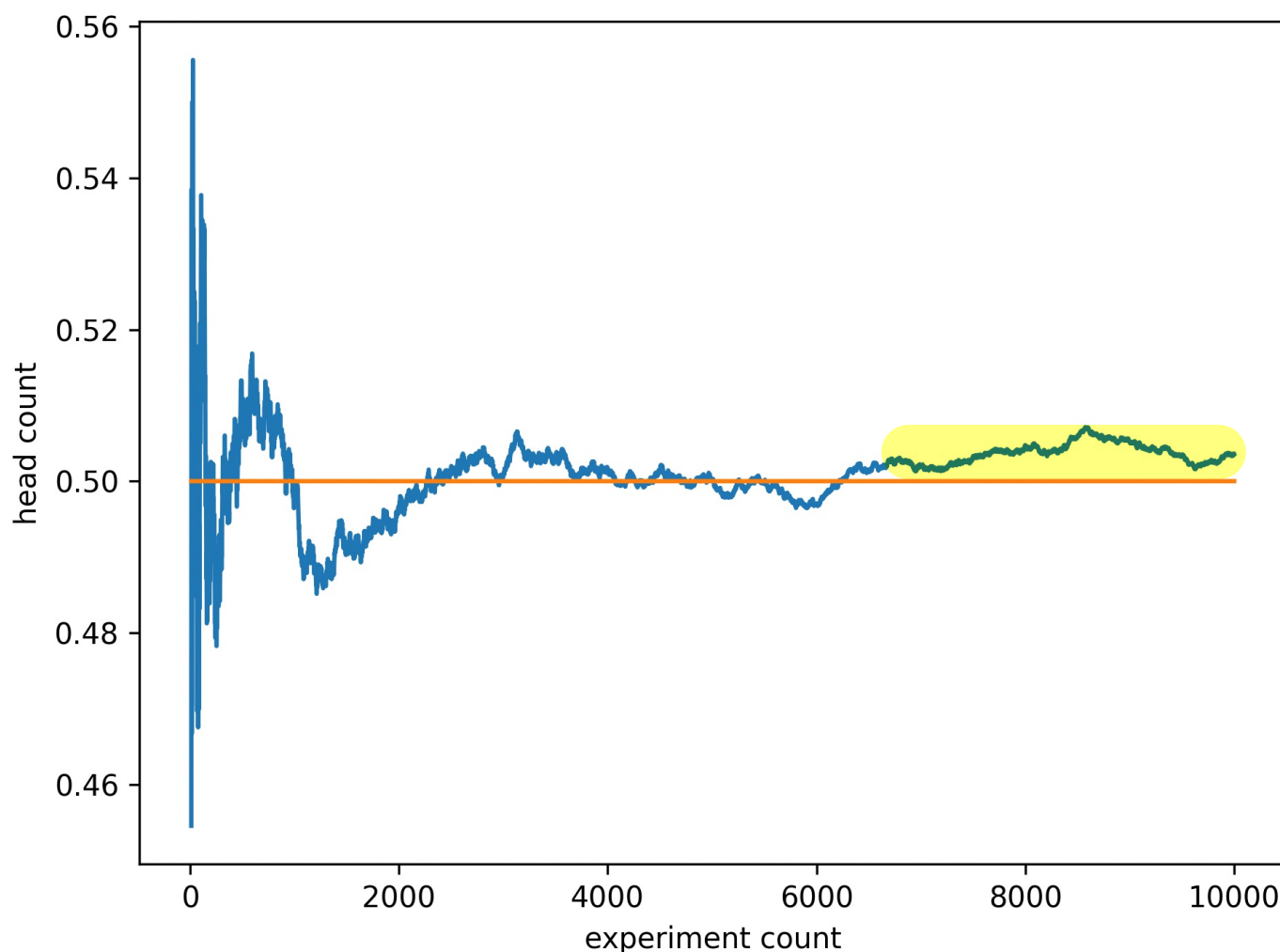


# HW1

1. 概率论的产生起源于解决实际问题,虽然在概率一词的含义上有理解的差异,但数学和分析的方法是一致的,也是学好概率论的关键.
2. Bertrand Paradox表明,"随机"的选取方式对计算概率十分重要.只有随机的方式明确,事件概率的定义才是良好的.
3. 设他认为A队获胜的概率为 $p$ . 他的收益为 $X$ 元, 有
$$E(x) = 5p - 20(1-p) = 25p - 20 \geq 0$$
$$\therefore A \text{队获胜的主观概率至少为} \frac{8}{5}.$$
4. 用Python模拟掷 $10^4$ 次硬币. 代码, 结果如图. 为了更好地展示频率变化, 图中略去了前10次的频率.

```
1 from functools import reduce
2 from matplotlib import pyplot as plt
3 from random import randint as ri
4 cnt = 10000
5 offset = 10
6 result = []
7 for i in range(1, cnt + 1):
8     result += [ri(0, 1)]
9 frequency = [reduce(lambda x, y: x + y, result[:i])/i for i in range(1, cnt + 1)]
10 plt.figure(dpi=300)
11 rg = range(1 + offset, cnt + 1)
12 plt.plot(rg, frequency[offset:])
13 plt.plot(range(1 + offset, cnt + 1), [0.5 for ii in rg])
14 plt.xlabel('experiment count')
15 plt.ylabel('head count')
16 plt.show()
```



(1) 橙色线 (2) 蓝色线

从总体来说, 频数呈现稳定趋势, 但我们并没有信心认为  $\lim_{n \rightarrow \infty} f(\text{Head}) = p(\text{Head})$ , 如图中黄色荧光所示.

两者存在实质性差异, 因为  $f$  与  $p$  间关系不满足

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, |f(\text{Head} | n \text{ 次实验}) - p(\text{Head} | n \text{ 次实验})| < \varepsilon.$$

5. <sup>(a)</sup>  $A = \{\text{病严重且有保险}, \text{病严重且无保险}\}$

<sup>(b)</sup>  $B = \{\text{良好且无保险}, \text{一般且无保险}, \text{严重且无保险}\}$

<sup>(c)</sup>  $B + A = \{\text{良好且有保险}, \text{一般且有保险}, \text{严重且有保险}, \text{严重且无保险}\}$

6. 证明: 对  $\forall x \in \Omega$ , 有  $x \in B$  或  $x \in B^c$ . 此时:

$$x \in A \Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in B^c) \Leftrightarrow x \in (AB + AB^c)$$

$$\text{所以 } A = AB^c + AB$$

7. 证明: 对  $\forall x \in (A+B)$ ,

假设  $x \in A$ , 则  $x \in A$

假设  $x \notin A$ , 则  $x \in B \wedge x \in (B-A)$

所以  $(A+B) \subseteq A+(B-A)$

同样, 对  $\forall x \in (A+(B-A))$

无论  $x \in A$ , 还是  $x \in (B-A)$ , 均有  $x \in (A+B)$

所以  $A+(B-A) \subseteq A+B$

即  $A+B = A+(B-A)$ .

由定义, 右边两事件互斥

(b) 容易由定义说明  $A+B = (B-A) + (A-B) + AB$ .

由 Venn 图也能看出右侧三事件互斥.

$$8. (A+B) - (A-B)$$

$$= (A+B) \cdot (A-B)^c$$

$$= (A+B) \cdot (A \cdot B^c)^c$$

$$= (A+B) \cdot (A^c + B)$$

$$= A \cdot A^c + AB + A^c B + B$$

$$= \emptyset + B + B = B$$

$$9. \text{令 } B_i = A_i - \bigcup_{j=1}^{i-1} A_j.$$

先证互斥性, 设  $i > j > 0$ .

$$B_i \cdot B_j = (A_i - \bigcup_{\alpha=1}^{i-1} A_\alpha) \cap (A_j - \bigcup_{\beta=1}^{j-1} A_\beta)$$

$$= A_i \cdot A_j \cdot \bigcap_{\alpha=1}^{i-1} A_\alpha^c$$

注意到对  $\forall s \subseteq A_j^c$ ,  $s \cdot A_j = \emptyset$

由  $i > j$ :  $A_j^c \subseteq \bigcap_{\alpha=1}^{i-1} A_\alpha^c$ , 所以  $A_j \cdot \bigcap_{\alpha=1}^{i-1} A_\alpha^c = \emptyset$ , 即  $B_i \cdot B_j = \emptyset$ .

再证和相等: 即证  $\bigcup_{i=1}^n B_i = \bigcup_{i=1}^n A_i$ .

对  $n$  进行归纳: 当  $n=1$  时, 显然成立.

$$\text{设 } n=k \text{ 时: } \bigcup_{i=1}^k B_i = \bigcup_{i=1}^k A_i,$$

$$\text{当 } n=k+1 \text{ 时: } \bigcup_{i=1}^{k+1} B_i = \bigcup_{i=1}^k B_i \cup (A_{k+1} - \bigcup_{i=1}^k A_i)$$

$$= (\bigcup_{i=1}^k A_i) \cup (A_{k+1} \cap (\bigcap_{i=1}^k A_i^c))$$

$$= \bigcup_{i=1}^k A_i \cup (\bigcup_{i=1}^k (A_i \cap \bigcap_{j=1}^k A_j^c))$$

$$= \bigcup_{i=1}^{k+1} A_i \cup \emptyset = \bigcup_{i=1}^{k+1} A_i.$$

10. 已装 MATLAB2020a 及 Python.

# HW2

1. 证明:  $P(B) = P(B \cap \Omega) = P(BA + BA^c) = P(BA) + P(BA^c)$   
 $\Rightarrow P(BA^c) = P(B) - P(AB)$

$$\begin{aligned} P(A+B) &= P(A + (B-A)) = P(A) + P(BA^c) = P(B) + P(A) - P(AB) \\ \Rightarrow P(A+B+C) &= P(A) + P(B-A) + P(C-A-B) \\ &= P(A) + P(BA^c) + P(C(A+B)^c) \\ &= P(A) + P(B) - P(AB) + P(C) - P(AC+BC) \\ &= P(A) + P(B) + P(C) - P(AB) - (P(AC) + P(BC) - P(BC \cdot AC)) \\ &= P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) \end{aligned}$$

2. 对  $\forall A \subset \Omega$ .

1°  $P(A|B) = \frac{P(AB)}{P(B)} \geq 0$ .

2°  $P(\Omega|B) = \frac{P(\Omega B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

3° 若  $\forall i \neq j, A_i \cdot A_j = \emptyset$ , 那么

$$\begin{aligned} P(\sum_{i=1}^{\infty} A_i | B) &= \frac{P(B \sum_{i=1}^{\infty} A_i)}{P(B)} \\ &= \frac{P(\sum_{i=1}^{\infty} BA_i)}{P(B)} \\ &= \frac{\sum_{i=1}^{\infty} P(BA_i)}{P(B)} \\ &= \sum_{i=1}^{\infty} \frac{P(BA_i)}{P(B)} \\ &= \sum_{i=1}^{\infty} P(A_i | B). \end{aligned}$$

所以  $P(\cdot|B)$  是概率函数

3. (1) 不正确. 反例: 掷 1 枚均匀 6 面色子,  $A = \{1\}$ ,  $B = \{1, 2, 3\}$

那么  $P(A) = \frac{1}{6}$ ,  $P(A|B) = \frac{1}{3}$ ,  $P(A) < P(A|B)$

(2) 不正确. 设  $P(B) = 0$ , 则  $P(B^c) = 1$ , 有  $P(BB^c) = P(B)P(B^c) = 0$ .

此时,  $B, B^c$  即互斥又独立.

(3) 不正确. 反例: 掷 1 枚均匀 6 面色子 2 次.  $A = \{\text{第 1 次为 } 1, 2 \text{ 或 } 3\}$ ,  $B = \{\text{第 1 次为 } 3, 4 \text{ 或 } 5\}$

$C = \{\text{两次点数和} = 9\}$ . 有  $P(A) = P(B) = \frac{1}{2}$ ,  $P(C) = \frac{4}{36}$ ,  $P(ABC) = \frac{1}{36} = P(A)P(B)P(C)$   
 但是  $P(A)P(B) = \frac{1}{4}$ ,  $P(AB) = \frac{1}{6}$ ,  $P(A)P(B) \neq P(AB)$



4. 使用古典概型容易计算:

$$P(A_2) = \frac{18}{36}, P(A_3) = \frac{12}{36}$$

$$P(A_5) = \frac{7}{36}, P(A_2 A_3) = \frac{6}{36}$$

$$P(A_2 A_5) = \frac{3}{36}$$

$$P(A_2) P(A_5) \neq P(A_2 A_5)$$

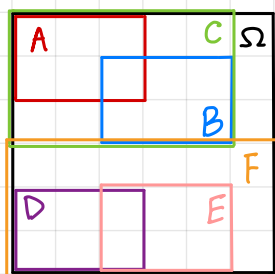
$$P(A_2) P(A_3) = \frac{1}{2} \times \frac{1}{3} = P(A_2 A_3)$$

所以  $A_2, A_3$  独立,  $A_2, A_5$  不独立.

|   | 1            | 2            | 3            | 4            | 5             | 6             |
|---|--------------|--------------|--------------|--------------|---------------|---------------|
| 1 | <del>1</del> | <del>2</del> | <del>3</del> | <del>4</del> | <del>5</del>  | <del>6</del>  |
| 2 | <del>2</del> | 3            | <del>4</del> | 5            | <del>6</del>  | 7             |
| 3 | <del>3</del> | <del>4</del> | 5            | <del>6</del> | 7             | <del>8</del>  |
| 4 | <del>4</del> | 5            | <del>6</del> | 7            | <del>8</del>  | 9             |
| 5 | <del>5</del> | <del>6</del> | 7            | <del>8</del> | 9             | <del>10</del> |
| 6 | <del>6</del> | 7            | <del>8</del> | 9            | <del>10</del> | <del>11</del> |

5. 考虑从下述36个格子中随机取一个格子

用以下各色区域表示相应区域中的格子被取到的事件.



先看  $A, B, C$ :

$$\text{有 } P(A) = P(B) = \frac{1}{6}, P(AB) = \frac{1}{36}$$

$$\Rightarrow P(A)P(B) = P(AB)$$

$A, B$  独立

$$\text{但 } P(A|C) = P(B|C) = \frac{2}{5}, P(AB|C) = \frac{1}{15}$$

$$\Rightarrow P(A|C) \cdot P(B|C) \neq P(AB|C)$$

$A, B$  关于  $C$  没有条件独立.

这说明独立不蕴涵条件独立.

再看  $D, E, F$ :

$$\text{有 } P(D) = P(E) = \frac{1}{6}, P(DE) = \frac{1}{18}$$

$$P(D)P(E) \neq P(DE)$$

$D, E$  不独立

$$\text{但 } P(D|F) = P(E|F) = \frac{1}{3}, P(DE|F) = \frac{1}{9}$$

$$P(D|F) \cdot P(E|F) = P(DE|F)$$

$D, E$  关于  $F$  条件独立

这说明条件独立不蕴涵独立.

6. 设重复该事件  $n$  次,  $i \in \{1, 2, \dots, n\}$ .

$A_i$  为第  $i$  次该事件发生,  $B$ : 该事件至少发生一次,  $\forall i \neq j, A_i A_j$  独立.

$$P(B) = 1 - P\left(\prod_{i=1}^n A_i^c\right) = 1 - \prod_{i=1}^n P(A_i^c) = 1 - (1 - \varepsilon)^n$$

$$\lim_{n \rightarrow +\infty} P(B) = 1$$

所以, 该事件迟早要发生的概率为 1.

7. 设朝上面为红色为  $A$ , 朝下面为黑色为  $B$ .

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{1}{3} + \frac{1}{3} \times \frac{1}{2}} = \frac{1}{3}$$

8. 两种方法公平且等价. 只须说明两种方式 "中" 的概率均为  $\frac{1}{n}$ .

$$(1) P(\text{"中"}) = \frac{1}{n} \text{ 显然.}$$

(2) 设第  $i \leq n$  个人抓 "中" 为  $A_i$ .

$$\begin{aligned} P(A_i) &= P\left(A_i \cdot \prod_{j=1}^{i-1} A_j^c\right) = P(A_i^c) \cdot P(A_2^c | A_1^c) \cdot P(A_3^c | A_1^c A_2^c) \cdot \dots \cdot P(A_i | \prod_{j=1}^{i-1} A_j^c) \\ &= \frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \dots \cdot \frac{n-i+1}{n-i+2} \cdot \frac{1}{n-i+1} = \frac{1}{n}. \end{aligned}$$

这说明两种方法公平且等价.

$$\begin{aligned} 9. \quad p &= \frac{\left(\binom{5}{3} + \frac{\binom{5}{2}\binom{3}{2}}{p_2^2}\right) \cdot p_3^3}{3^5} \\ &= \frac{(10 + \frac{10 \times 3}{2}) \cdot 3 \times 2 \times 1}{3^5} \\ &= \frac{1}{24} (10 + 15) \times 2 \\ &= \frac{50}{81} \end{aligned}$$

考虑反面

令  $A_i$  为第  $i$  个电梯为空

$$1 - p = P\left(\sum_{i=1}^3 A_i\right)$$

某两个确定的

$$= \sum_i P(A_i) - \sum_{i \neq j} P(A_i A_j) + P(A_1 A_2 A_3)$$

$$= 3 \times \frac{2^5}{3^5} - 3 \times \frac{1}{3^5} + 0$$

$$= \frac{31}{81}$$

$$\Rightarrow p = \frac{50}{81}$$

10. 设小明患病为  $A$ , 结果阳性为  $B$

$$P(B|A^c) = 0.3, P(A) = 0.6, P(B|A) = 1$$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B|A)}{P(B|A) \cdot P(A) + P(B|A^c) \cdot P(A^c)} = \frac{0.6}{0.6 + 0.4 \times 0.3} \approx 0.833$$

$\therefore$  医生仍应建议手术.

11 (1) 设这一事件为  $A_L$  (左空),  $A_R$  (右空)

$$P(A_L) = (1-p) \cdot \binom{2n-m}{n} (1-p)^n p^{n-m}$$

$$P(A_R) = p \binom{2n-m}{n} p^n (1-p)^{n-m}$$

$$P(A_L + A_R) = \binom{2n-m}{n} p^{n-m} (1-p)^{n-m} (p^{n+m} + (1-p)^{n+m})$$

(2) 设用完某一盒时, 另一盒恰有  $m$  支为  $B_L$  (左亮),  $B_R$  (右亮)

$$P(B_L) = (1-p) \binom{2n-m-1}{n-1} (1-p)^{n-1} p^{n-m}$$

$$P(B_R) = p \binom{2n-m-1}{n-1} p^{n-1} (1-p)^{n-m}$$

$$P(B_L + B_R) = \binom{2n-m-1}{n-1} p^{n-m} (1-p)^{n-m} (p^m + (1-p)^m)$$

12. 毫无思路 qwq...

解方程法: 记  $A$ : 生物最终灭亡

$B_1$ : 生物一代后灭亡

$B_2$ : 生物一代后存活

$B_3$ : 生物一代后  $\times 2$

利用全概率公式, 设  $P(A) = p$

$$P(A) = P(B_1)P(A|B_1)$$

$$+ P(B_2)P(A|B_2)$$

$$+ P(B_3)P(A|B_3)$$

$$= \frac{1}{3} \cdot 1 + \frac{1}{3}p + \frac{1}{3}p^2$$

$$\Rightarrow p^2 - 2p + 1 = 0 \Rightarrow p = 1$$

13. 设患  $A, B, C$  分别为事件  $A, B, C$

假设用甲方案:

基本假设: 一代不影响无穷

$$\text{治愈的概率 } p_1 = 0.8P(A) + 0.05P(B) + 0.1P(C) = 0.655$$

假设用乙方案:

$$\text{治愈的概率 } p_2 = 0.6P(A) + 0.9P(B) + 0.9P(C) = 0.660$$

$p_2 > p_1$ , 所以建议用乙方案.

另一方案, emmmmmmm.....

14. 设他用完 1h 为  $A$ , 第 45 min 仍在答题为  $B$ ,  $P(B) = \frac{5}{8}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{1 - \frac{1}{2} \times 1}{\frac{5}{8}} = \frac{4}{5}$$

$$15. (1) P(B_1) = \frac{1}{2} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{5} = 0.6$$

$$(2) \text{放回: } P(B_2) = P(B_1) = 0.6$$

因为系统和观测都毫无变化

$$\text{不放回: } P(B_2) = \frac{1}{2} \cdot \left( \frac{4}{5} \cdot \frac{3}{4} + \frac{1}{5} \cdot 1 \right) + \frac{1}{2} \cdot \left( \frac{2}{5} \cdot \frac{1}{4} + \frac{3}{5} \cdot \frac{1}{2} \right)$$

正解: 用全概率展开  $P(B_2) = P(A_1)P(B_2|A_1) + P(A_2)P(B_2|A_2)$

$$= \frac{1+4}{2+5} + \frac{2+3 \times 2}{2+5} = \frac{20+4}{40} = 0.6 = P(B_1)$$

依赖于第1次  
用  $B_1$  再做全概率拆分

$$P(B_2|A_1) = P(B_1|A_1)P(B_2|B_1A_1) + P(B_1^c|A_1)P(B_2|B_1^cA_1) = \frac{4}{5} \cdot \frac{3}{4} + \frac{1}{5} \cdot \frac{4}{4}$$

因为没有引入新观测信息。

(3) 记  $A_i$  为选中的袋子是  $i$  号 ( $i=1,2$ )

$$P(A_1|B_1) = \frac{P(A_1B_1)}{P(B_1)} = \frac{\frac{1}{2} \times \frac{4}{5}}{\frac{3}{5}} = \frac{2}{3}$$

$$P(A_2|B_1) = \frac{P(A_2B_1)}{P(B_1)} = \frac{\frac{1}{2} \times \frac{2}{5}}{\frac{3}{5}} = \frac{1}{3}$$

$$P(B_2|B_1) = \frac{P(B_1B_2)}{P(B_1)} = \frac{\frac{1}{2} \times \frac{4}{5} \times \frac{4}{5} + \frac{1}{2} \times \frac{2}{5} \times \frac{2}{5}}{\frac{3}{5}} = \frac{2}{3} > P(B_2)$$

实际上就是做了全概率拆分

∴ 第1次摸出黑球, 增大了是1号袋子的概率

$$(4) P(B_{n+1} | \prod_{i=1}^n B_i) = \frac{P(\prod_{i=1}^{n+1} B_i)}{P(\prod_{i=1}^n B_i)}$$

$$\Rightarrow P(\prod_{i=1}^k B_i) = \frac{1}{2} \cdot \left(\frac{4}{5}\right)^k + \frac{1}{2} \cdot \left(\frac{2}{5}\right)^k$$

$$\Rightarrow P(B_{n+1} | \prod_{i=1}^n B_i) = \frac{\left(\frac{4}{5}\right)^{n+1} + \left(\frac{2}{5}\right)^{n+1}}{\left(\frac{4}{5}\right)^n + \left(\frac{2}{5}\right)^n} = \frac{2^{2n+2} + 2^{n+1}}{2^{2n} + 2^n} \cdot \frac{1}{5} = \frac{1}{5} \cdot \frac{2^{n+2} + 2}{2^n + 1}$$

$$\lim_{n \rightarrow \infty} P(B_{n+1} | \prod_{i=1}^n B_i) = \lim_{n \rightarrow \infty} \frac{1}{5} \left( \frac{4 \cdot 2^n + 4 - 2}{2^n + 1} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{5} \left( 4 - \frac{2}{2^n + 1} \right)$$

$$= \frac{4}{5}$$

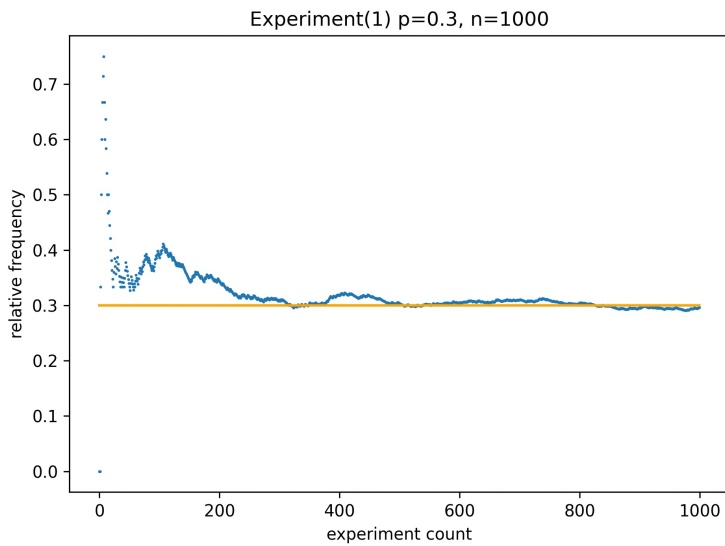
总摸黑球使得取1号袋子的概率  $\rightarrow 1$ ,  
事件渐渐趋近于总在1号袋子中摸球.

$$(5) P(A_1 | \prod_{i=1}^n B_i) = \frac{P(A_1 \prod_{i=1}^n B_i)}{P(\prod_{i=1}^n B_i)} = \frac{\frac{1}{2} \times \left(\frac{4}{5}\right)^n}{\frac{1}{2} \left(\left(\frac{4}{5}\right)^n + \left(\frac{2}{5}\right)^n\right)} = 1 - \frac{1}{2^n + 1}$$

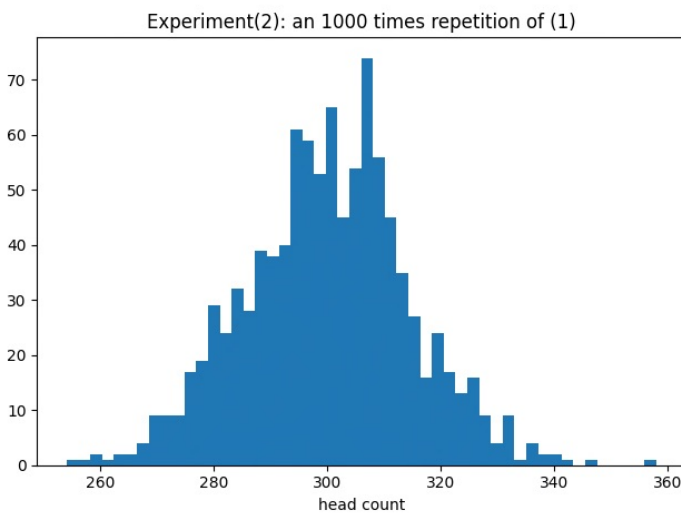
$$\lim_{n \rightarrow \infty} P(A_1 | \prod_{i=1}^n B_i) = \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{2^n + 1} \right) = 1$$

解释同上.

16.  
(11)



(12)

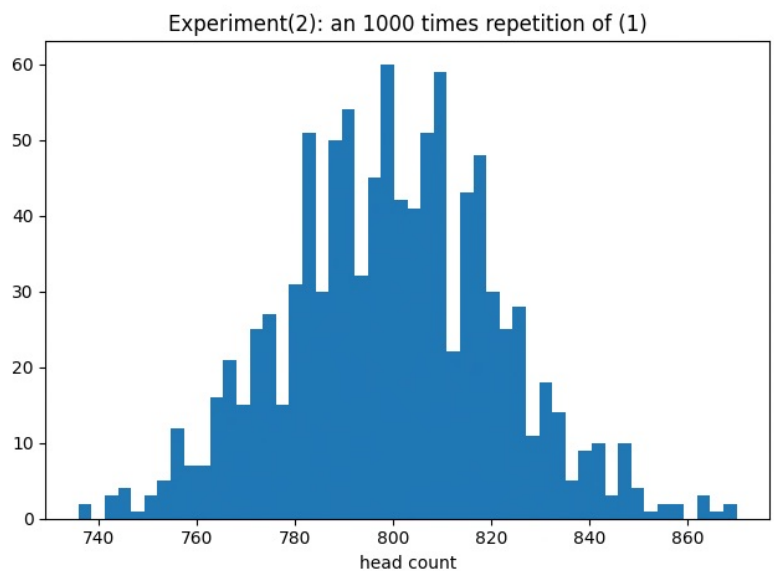
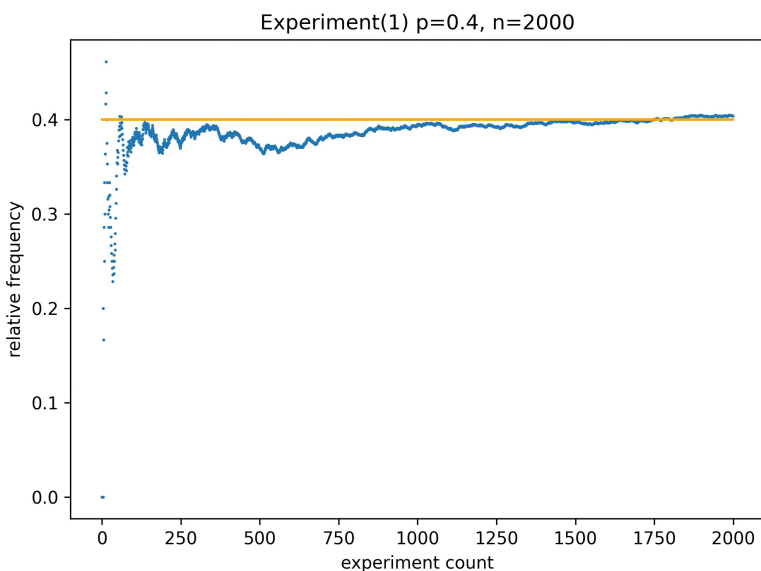


感觉100次有点少,重复了1000次

(13)  $np=300,$

平均值300.125

(14) 令  $n=2000, p=0.4$ , 有



## 本题代码:

```
1 from functools import reduce
2 from matplotlib import pyplot as plt
3 from scipy.stats import bernoulli
4
5
6 def experiment():
7     d = bernoulli.rvs(size=sz, p=p)
8     f = [reduce(lambda x, y: x + y, d[:j + 1]) / (j + 1) for j in range(sz)]
9     return d, f
10
11
12 sz = 1000
13 p = 0.3
14
15 plt.figure(dpi=300)
16
17 # (1)
18 data, frequency = experiment()
19 plt.figure(dpi=300)
20 plt.scatter(range(sz), frequency, s=0.5)
21 plt.plot(range(sz), [p for ii in range(sz)], color="orange")
22 plt.xlabel("experiment count")
23 plt.ylabel("relative frequency")
24 plt.title(f"Experiment(1) p={p}, n={sz}")
25 plt.show()
26
27 # (2)
28 heads = []
29 for i in range(1000):
30     heads += [sum(experiment()[0])]
31 plt.title("Experiment(2): an 1000 times repetition of (1)")
32 plt.xlabel("head count")
33 plt.hist(heads, bins=50)
34 plt.show()
35
36 # (3)
37 print(sum(heads)/1000, sz*p)
38
```





# HW3

1. ① 掷100次硬币, 正面向上的次数  $X_1$

离散型

样本空间  $\Omega = \{(x_1, \dots, x_{100}) \mid x_i = 0, 1, i \in \mathbb{Z} \cap [1, 100]\}$

② 从THU抓100位同学, 支持国庆放假政策的人数  $X_2$

离散型

样本空间  $\Omega = \{100 \text{ 位 THU 同学的意见的所有可能的集合}\}$

③ 从THU任抓一名同学, 其身高  $X_3$  (cm)

连续型

样本空间  $\Omega = \{\text{所有 THU 同学的集合}\}$

④ 概统课上某名同学的期末成绩  $X_4$

离散型

样本空间  $\Omega = \{\text{所有选修概统的同学的集合}\}$

⑤ 今天手机电用完的时刻  $X_5$  (从 0:00 计, 秒)

连续型

样本空间  $\Omega = \{\text{今天所有的手机使用情况}\}$

2. (1) 先证  $F(x)$  单调:  $\forall x > y, F(y) = P(X \leq y) = P(X \leq x) + P(x < X \leq y)$   
 $= F(x) + P(x < X \leq y)$

$\Rightarrow F(y) - F(x) = P(x < X \leq y) \geq 0$  即  $F(x)$  (非严格) 单增

而  $F(x)$  有界  $\Rightarrow \lim_{x \rightarrow +\infty} F(x), \lim_{x \rightarrow -\infty} F(x)$  存在. (单调收敛原理)

$P(\Omega) = 1, P(\emptyset) = 0$  且  $-\infty < X(\Omega), X(\emptyset) < +\infty$

$\therefore \lim_{x \rightarrow +\infty} F(x) = 1, \lim_{x \rightarrow -\infty} F(x) = 0$

$$(2) \forall \varepsilon > 0. \exists \delta, 0 \leq x - x_0 < \delta, F(x) - F(x_0) < \varepsilon \dots$$

不会39w9...

$$(3) P(a < X \leq b) = F(b) - F(a)$$

$$P(X=a) = F(a) - \lim_{x \rightarrow a^-} F(x)$$

$$\therefore P(a \leq X \leq b) = F(b) - \lim_{x \rightarrow a^-} F(x)$$

11)  
3. 设  $X, Y$  cdf 分别为  $F, G$ .  $X, Y$  同分布  $\Leftrightarrow \forall x, F(x) = G(x)$

$$\forall i \in \{1, 2, 3\}, P(\omega_i) = \frac{1}{3}.$$

$$\therefore \forall i, P(X=i) = P(Y=i) = \frac{1}{3}$$

$$\Rightarrow F(x) = G(x) \quad (\forall x \in \mathbb{R})$$

$$(2) (X+Y)(\omega_1) = 3, (X+Y)(\omega_2) = 5, (X+Y)(\omega_3) = 4.$$

$$\text{故 } P(X+Y=3) = P(X+Y=5) = P(X+Y=4) = \frac{1}{3}.$$

$$(X-Y)(\omega_1) = -1, (X-Y)(\omega_2) = -1, (X-Y)(\omega_3) = 2$$

$$\text{故 } P(X-Y=-1) = \frac{2}{3}, P(X-Y=2) = \frac{1}{3}.$$

$$4. \text{证明: } \text{Var}(X) = \sum_i (x_i - E(X))^2 p_i$$

$$= \sum_i x_i^2 p_i - 2E(X) \sum_i x_i p_i + E^2(X) \sum_i p_i$$

$$= E(X^2) - 2E(X)E(X) + E^2(X)$$

$$= E(X^2) - E^2(X)$$

完全一致

$$5. (1) X \in [1, a+1] \cap \mathbb{N}$$

$$P(X=k) = \frac{\binom{k-1}{a} \cdot \binom{1}{b}}{\binom{a+b}{k}}$$

$$(2) P(X=k) = \left(\frac{a}{a+b}\right)^{k-1} \frac{b}{a+b}, \text{ 记 } \alpha = \frac{a}{a+b}$$

$$E(X) = \sum_{k=1}^{\infty} k P(X=k) = \frac{b}{a+b} \sum_{k=1}^{\infty} k \alpha^{k-1}$$

$$\text{令 } S = 1 \cdot \alpha^0 + 2 \cdot \alpha^1 + \dots + k \alpha^{k-1}$$

$$\alpha S = 1 \cdot \alpha^1 + \dots + (k-1) \alpha^{k-1} + k \alpha^k$$

$$(1-\alpha)S_k = 1 + \alpha + \dots + \alpha^{k-1} - k \alpha^k = \frac{1-\alpha^k}{1-\alpha} - k \alpha^k$$

$$\Rightarrow S_k = \frac{1}{(1-\alpha)^2} - \left(\frac{1}{(1-\alpha)^2} + k\right) \alpha^k$$

$$E(X) = \frac{b}{a+b} \lim_{k \rightarrow \infty} S_k$$

$$= \frac{b}{a+b} \cdot \frac{1}{\left(\frac{b}{a+b}\right)^2} = 1 + \frac{a}{b}$$

6. 存在, 构造如下:

$$P(X=0) = P(Y=1) = 0.999$$

$$P(X=10^5) = P(Y=0) = 0.001$$

$$\text{有 } P(Y > X) = 0.998001 \geq 0.99$$

$$E(X) = 100$$

$$E(Y) = 0.999$$

$$E(X) > 100E(Y) = 99.9$$

$$7. 11) P(X=k) = (1-p)^{k-1} p$$

$$\begin{aligned} (a) E(X) &= \sum_{k=1}^{\infty} k P(X=k) \\ &= p \sum_{k=1}^{\infty} k (1-p)^{k-1} \\ &= p \cdot \frac{1}{p^2} = \frac{1}{p} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \sum_{k=1}^{\infty} k^2 P(X=k) \\ &= p \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} \end{aligned}$$

$$\sum_{k=1}^n S_n = 1^2 \alpha^0 + 2^2 \alpha^1 + \dots + n^2 \alpha^{n-1}$$

$$\alpha S_n = 1^2 \alpha^1 + \dots + (n-1)^2 \alpha^{n-1} + n^2 \alpha^n$$

$$(1-\alpha) S_n = 1\alpha^0 + 3\alpha^1 + \dots + (2n-1)\alpha^{n-1} - n^2 \alpha^n$$

$$= \sum_{k=1}^n (2k-1) \alpha^{k-1} - n^2 \alpha^n$$

$$\sum_{k=1}^n T_n = \sum_{k=1}^n (2k-1) \alpha^{k-1} = 1 + \sum_{k=2}^n (2k-1) \alpha^{k-1}$$

$$\alpha T_n = \sum_{k=1}^n (2k-1) \alpha^k$$

$$= \sum_{k=2}^n (2k-3) \alpha^{k-1} + (2n-1) \alpha^n$$

$$\Rightarrow (1-\alpha) T_n = 1 + 2 \sum_{k=2}^n \alpha^{k-1} + (1-2n) \alpha^n$$

$$= 1 + 2\alpha \frac{1-\alpha^{n-1}}{1-\alpha} + (1-2n) \alpha^n$$

$$\therefore (1-\alpha) S_n = \frac{1}{1-\alpha} \left( 1 + 2\alpha \frac{1-\alpha^{n-1}}{1-\alpha} + (1-2n) \alpha^n \right) - n^2 \alpha^n$$

$$S_n = \frac{1}{(1-\alpha)^2} \left( 1 + 2\alpha \frac{1-\alpha^{n-1}}{1-\alpha} + (1-2n) \alpha^n \right) - \frac{n^2 \alpha^n}{1-\alpha}$$

$$\lim_{n \rightarrow \infty} S_n = \frac{1}{(1-\alpha)^2} \left( 1 + \frac{2\alpha}{1-\alpha} \right) = \frac{1}{(1-\alpha)^2} \cdot \frac{1-\alpha+2\alpha}{1-\alpha} = \frac{1+\alpha}{(1-\alpha)^3}$$

$$\sum \alpha = 1-p \Rightarrow E(X^2) = p \cdot \frac{2-p}{p^3} = \frac{2-p}{p^2}$$

$$\Rightarrow \text{Var}(X) = E(X^2) - E^2(X)$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

8. 记  $X$  为这 25 人中曾通过该电商平台购买商品的人数

$X \sim B(0.6, 25)$ , 下面用计算机计算:

$$P(X \geq 15) = 0.5858$$

$$P(X > 20) = 0.0095$$

$$P(X < 10) = 0.0132$$

9. 设  $X \sim B(n, p)$ ,  $Y \sim B(n-1, p)$ ,  $Z \sim B(n-2, p)$

$$E(X) = \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^n k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^n \frac{k(n-1)!}{k!(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=1}^n \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} p^k (1-p)^{(n-1)-k}$$

$$= np \sum_{k=0}^{n-1} P(Y=k)$$

$$= np$$

$$E(X^2) = \sum_{k=0}^n k^2 \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^n k(k-1) \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} + \sum_{k=0}^n k P(X=k)$$

$$= n(n-1)p^2 \sum_{k=2}^n \frac{(n-2)!}{(k-2)!(n-2-(k-2))!} p^{k-2} (1-p)^{(n-2)-(k-2)} + np$$

$$= n(n-1)p^2 \sum_{k=0}^{n-2} P(Z=k) + np$$

$$= n^2 p^2 - np^2 + np$$

$$\begin{aligned}\Rightarrow \text{Var}(X) &= E(X^2) - E^2(X) \\ &= n^2 p^2 - np^2 + np - n^2 p^2 \\ &= np(1-p).\end{aligned}$$

10. 设  $X$  为 6 个色子, 掷出 1 点的个数, 有  
 $X \sim B(6, \frac{1}{6})$

$$\begin{aligned}P(X=2) &= C_6^2 \left(\frac{1}{6}\right)^2 \cdot \left(1 - \frac{1}{6}\right)^4 \\ &= 15 \times \frac{625}{6^6} \\ &= \frac{3125}{15552} \approx 0.2009\end{aligned}$$

令  $\lambda = np = 1$ ,  $Y \sim \text{Poisson}(\lambda)$

$$P(Y=2) = \frac{1^k}{2!} e^{-1} \approx 0.1839$$

11. 设  $X$  为  $10^6$  人中访问该网站的人数

$$X \sim B(10^6, 2 \times 10^{-6})$$

$$P(X \leq 2) = 1 - \left[ (1 - 2 \times 10^{-6})^{10^6} + 10^6 \times 2 \times 10^{-6} \times (1 - 2 \times 10^{-6})^{10^6 - 1} + \frac{10^6(10^6 - 1)}{2} 4 \times 10^{-12} \times (1 - 2 \times 10^{-6})^{10^6 - 2} \right] = 0.6767$$

令  $\lambda = np = 2$ ,  $Y \sim \text{Poisson}(2)$

$$P(Y \geq 3) = 1 - P(Y=0) - P(Y=1) - P(Y=2)$$

$$\begin{aligned}&= \frac{1}{e^2} \left( \frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} \right) \\ &= 0.6767\end{aligned}$$

12. 设昆虫产卵数为  $X$ , 有  $Y$  个后代发育成虫

则  $X \sim \text{Poisson}(\lambda)$ ,  $Y \sim B(X, p)$

$$P(X=m) = \frac{\lambda^m}{m!} e^{-\lambda}$$

$$P(Y=k) = \sum_{i=k}^{\infty} P(X=i) \cdot \binom{i}{k} p^k (1-p)^{i-k}$$

$$= \sum_{i=k}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} \frac{i!}{k!(i-k)!} p^k (1-p)^{i-k}$$

$$= \frac{e^{-\lambda} p^k}{k!} \sum_{n=k}^{\infty} \frac{\lambda^n}{(n-k)!} (1-p)^{n-k}$$

$$= \frac{\lambda^k e^{-\lambda} p^k}{k!} \sum_{m=0}^{\infty} \frac{(\lambda(1-p))^m}{m!}$$

$$= \frac{\lambda^k e^{-\lambda} p^k}{k!} \cdot e^{\lambda} e^{-\lambda p}$$

$$= \frac{(\lambda p)^k}{k!} e^{-\lambda p}$$

$Y \sim \text{Poisson}(\lambda p)$

13. (1)  $P(X=m) = \frac{C_M^m C_{N-M}^{n-m}}{C_N^n}$

(2)  $\frac{M}{\hat{N}} = \frac{m}{n} \Rightarrow \hat{N} = \frac{nM}{m}$

(3) 
$$b_N = P(X=m) = \frac{\frac{M!}{m!(M-m)!} \cdot \frac{(N-M)!}{(n-m)!(N-M-n+m)!}}{\frac{N!}{n!(N-n)!}}$$

$$\propto \frac{(N-n)! \cdot (N-M)!}{N! \cdot (N-M-n+m)!} := b_N$$

由  $\begin{cases} b_N \geq b_{N+1} \\ b_N \geq b_{N-1} \end{cases}$ , 有:

$$\textcircled{1} \quad \frac{N-n+1}{N+1} \cdot \frac{N-M+1}{(N-M+m-n+1)} \leq 1, \quad \text{令 } k=N+1, \text{ 有}$$

$$(k-n)(k-M) \leq k(k-M+m-n)$$

$$\Rightarrow k^2 - (n+M)k + nM \leq k^2 - (m+n-M)k$$

$$\Rightarrow mk \geq nM$$

$$\Rightarrow N+1 \geq \frac{nM}{m}$$

$$\textcircled{2} \quad \frac{(N-n)(N-M)}{N \cdot (N-M+m-n)} \geq 1, \quad \text{有}$$

$$(N-n)(N-M) \geq N(N-M+m-n)$$

$$\Rightarrow N \leq \frac{nM}{m}$$

$$\textcircled{1}, \textcircled{2} \Rightarrow \frac{nM}{m} - 1 \leq N \leq \frac{nM}{m}$$

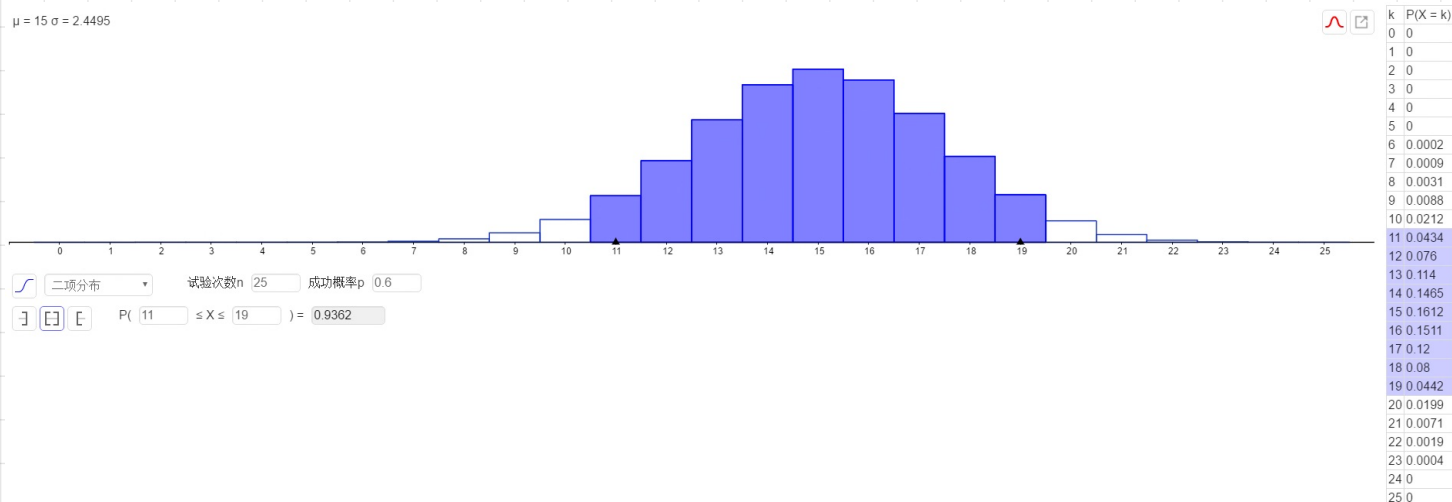
$$\therefore \text{最大值于 } N = \frac{nM}{m} \text{ 或 } \frac{nM}{m} - 1 \text{ 处取得}$$

(4) 给定  $M, n, m$  这些观测值,  $N$  最可能的值将是  $\frac{n}{m}M$ .



# 14. 用 Geo Gebra 统计.

$\mu = 15 \sigma = 2.4495$



(1)  $X = 15$  时, 有最大概率

(2)  $\mu = X$

(3)  $\sigma^2 = 5.978$

(4)  $P(\mu - 2\sigma < X \leq \mu + 2\sigma) = P(11 \leq X \leq 19) = 0.9362$



# HW4

$$1. \begin{cases} E(X) = \int_{\mathbb{R}} x f(x) dx = \int_0^1 x(a+bx^2) dx = \int_0^1 (ax+bx^3) dx = \left(\frac{a}{2}x^2 + \frac{b}{4}x^4\right)\bigg|_0^1 = \frac{a}{2} + \frac{b}{4} = \frac{2}{3} \\ P(\Omega) = \int_{\mathbb{R}} f(x) dx = \int_0^1 (a+bx^2) dx = \left(ax + \frac{b}{3}x^3\right)\bigg|_0^1 = a + \frac{b}{3} = 1 \end{cases}$$
$$\Rightarrow \begin{cases} a = \frac{1}{3} \\ b = 2 \end{cases}$$

2. 设  $X \sim U(10, 11)$ ,  $X$  的 PDF:  $f(x) = \begin{cases} 1, & x \in (10, 11) \\ 0, & x \notin (10, 11) \end{cases}$

(1) 记该事件为  $A$ .

$$P(A) = P\left(X \in \left[10\frac{2}{6}, 10\frac{3}{6}\right] \text{ 或 } X \in \left[10\frac{5}{6}, 11\right]\right)$$
$$= \frac{1}{3}$$

(2) 记该事件为  $B$ .

$$P(B) = P\left(X \in \left[10, 10\frac{1}{6}\right] \text{ 或 } X \in \left[10.5, 10\frac{4}{6}\right]\right)$$
$$= \frac{1}{3}.$$

3. 设  $X \sim N(270, 100)$ .

$$P(X \leq 240 \text{ 或 } X \geq 290) = 0.0241$$

4. 设  $X \sim \text{Exp}(\frac{1}{3})$   $X$  (万米)

$$P(X \geq 2.5 | X \geq 1.5) = \frac{e^{-\frac{1}{3} \times 2.5}}{e^{-\frac{1}{3} \times 1.5}} = e^{-\frac{1}{3}}$$

设  $X$  的 cdf 为  $F(x)$

$$P(X \geq 2.5 | X \geq 1.5) = \frac{1 - F(2.5)}{1 - F(1.5)}$$

5. 设  $X \sim \text{Exp}(\frac{1}{\mu})$

(1) 令  $\mu=1, X \sim \text{Exp}(1)$

$$P(X \leq c) = 95\% = 1 - e^{-c} \Rightarrow c = \ln 20 \approx 2.9957$$

(2) 令  $\mu=2, X \sim \text{Exp}(\frac{1}{2})$

$$P(X > c) = e^{-\frac{1}{2} \ln 20} = \frac{1}{\sqrt{20}} = \frac{1}{2\sqrt{5}} \approx 0.2236$$

6. 该“死亡率”类似瞬时失效率, 是单位时间内的死亡率.

设  $X$  是吸烟者的寿命,  $Y$  是非吸烟者的寿命

$$\therefore X-50 \sim \text{Exp}(\frac{2}{30}),$$

$$Y-50 \sim \text{Exp}(\frac{1}{30})$$

$$P(X \geq 60) = e^{-\frac{1}{15} \times 10} = 0.5134$$

$$P(Y \geq 60) = e^{-\frac{1}{30} \times 10} = 0.7165$$

$\Rightarrow$  50岁吸烟者活到60岁概率是 51.34%

非吸烟者为 71.65%

7. 令  $F(x) = \int_{-\infty}^x f(t) dt = P(X \leq x)$

$G(x)$  为  $g(X)$  的累积分布函数, 设  $g'(x) > 0$ .

$$G(x) = P(g(X) \leq x)$$

$$= P(X \leq g^{-1}(x))$$

$$= F(g^{-1}(x))$$

$$= \int_{-\infty}^{g^{-1}(x)} f(t) dt$$

$$\frac{1}{x\sqrt{\pi} \cdot \sigma} \cdot e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}$$

$$\therefore g(X) \text{ 的概率密度函数 } G'(x) = \begin{cases} f \circ g^{-1}(x) \cdot \frac{d(g^{-1}(x))}{dx}, & \text{if } g'(x) > 0 \\ -f \circ g^{-1}(x) \cdot \frac{d(g^{-1}(x))}{dx}, & \text{if } g'(x) < 0 \end{cases}$$

8. 设  $X \sim B(\alpha, \beta)$

则  $X$  的 PDF 为  $f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}$

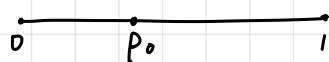
$$E(X) = \int_0^1 x f(x) dx = \frac{\alpha}{\alpha + \beta}$$

$$\text{Var}(X) = E(X^2) - E^2(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

(不太会算特殊函数 qwq...)

9. 设断点坐标为  $X$ ,  $X \sim U(0, 1)$  其 pdf:  $f(x) = 1, x \in [0, 1]$

$X$  所诱导的含  $p_0$  的那一段长度为  $Y$



当  $X < p_0$  时:  $Y = 1 - X$

当  $X > p_0$  时:  $Y = X$

$$E(Y) = \int_0^{p_0} (1-x) f(x) dx + \int_{p_0}^1 x f(x) dx$$

$$= (x - \frac{1}{2}x^2) \Big|_0^{p_0} + \frac{1}{2}x^2 \Big|_{p_0}^1$$

$$= p_0 - \frac{1}{2}p_0^2 + \frac{1}{2} - \frac{1}{2}p_0^2 = -p_0^2 + p_0 + \frac{1}{2}$$

10. 记  $X$  的 PDF 为  $f(x) = \begin{cases} \frac{1}{2}, & x \in [0, 1] \\ \frac{1}{2}, & x \in [3, 4] \\ 0, & \text{其他} \end{cases}$

$$\begin{aligned} E(X) &= \int_{\mathbb{R}} x f(x) dx \\ &= \frac{1}{2} \int_0^1 x dx + \frac{1}{2} \int_3^4 x dx \\ &= \frac{1}{4} (1^2 - 0^2 + 4^2 - 3^2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_{\mathbb{R}} x^2 f(x) dx \\ &= \frac{1}{2} \int_0^1 x^2 dx + \frac{1}{2} \int_3^4 x^2 dx \\ &= \frac{1}{6} (1 + 64 - 27) \\ &= \frac{19}{3} \end{aligned}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - E(X)^2 \\ &= \frac{361}{9} - 4 \\ &= \frac{325}{9} \end{aligned}$$

11. 用第 7 题结论, 令  $f(x) = \begin{cases} 1, & x \in (0, 1) \\ 0, & x \notin (0, 1) \end{cases}$ ,  $g(x) = \frac{1}{x}$ ,  $g'(x) = -\frac{1}{x^2}$

则  $Y$  的 PDF:  $G'(x) = -f \circ g^{-1}(x) \cdot g^{-1}'(x) = \begin{cases} \frac{1}{x^2}, & x > 1 \\ 0, & x \leq 1 \end{cases}$

12. 易知:  $P(Y=i) = p_i$ ,  $i \in \{1, \dots, n\}$

对于离散型变量  $P(X=x) = p_i$ , 也可以有同样效果.

这样, 就有了用随机数生成特定分布的离散型随机变量的方法.

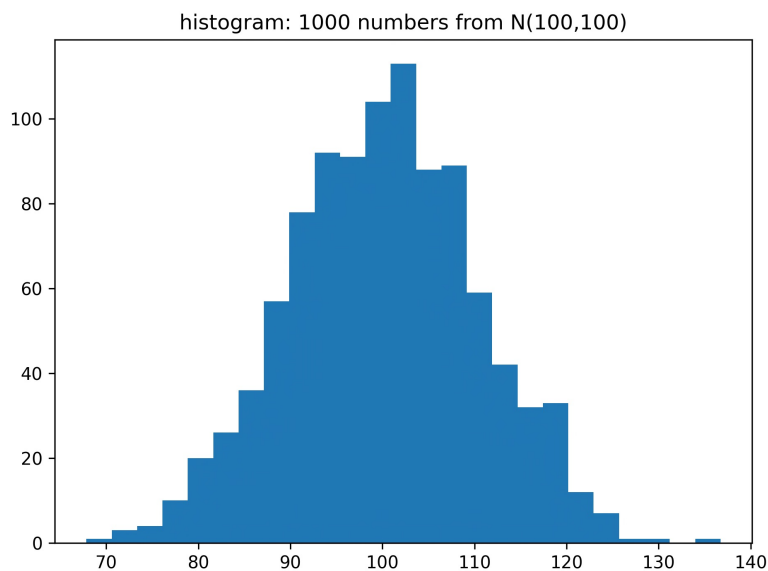
13. 用第7题的结论: 设  $Y$  的 PDF 为  $f$ .  $\ln y$  的 PDF 为  $G'(x)$ ,  $g(x) = \ln x$ .

$$\text{有 } G'(x) = f(e^x) e^x = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

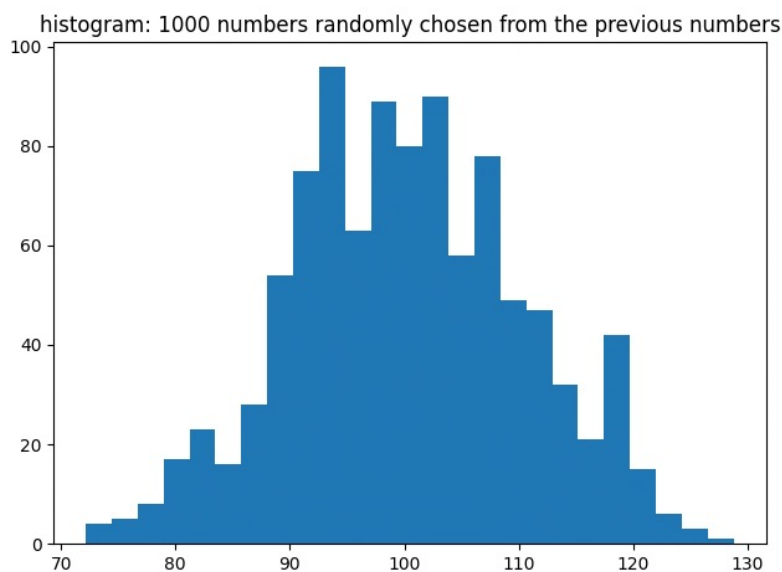
$$\text{令 } t = e^x, \text{ 有 } x = \ln t \Rightarrow f(t) \cdot t = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{(\ln t - \mu)^2}{2\sigma^2}}$$

$$\Rightarrow f(t) = \frac{1}{t \sqrt{2\pi} \cdot \sigma} e^{-\frac{(\ln t - \mu)^2}{2\sigma^2}}$$

14. (1)



(2)



(3)

```
Python 3.8.11 (default, Aug 6 2021, 09:57:55) [MSC v.1916 64 bit (AMD64)] on win32
>>> runfile('C:/Users/joshu/PycharmProjects/scientificProject/hw4.py', wdir='C:/User
Mean of (1) 100.17537798738171 Variance of (1) 105.47617446191789
Mean of (2) 100.27476837154087 Variance of (2) 105.19289984278363
```

本题代码:

```
1 import random
2 from matplotlib import pyplot as plt
3 import numpy as np
4
5 plt.figure(dpi=300)
6
7 # (1)
8 data = [random.normalvariate(100, 10) for i in range(1000)]
9 plt.hist(data, bins=25)
10 plt.title("histogram: 1000 numbers from N(100,100)")
11 plt.show()
12
13 # (2)
14 choices = [random.choice(data) for ii in range(1000)]
15 plt.hist(choices, bins=25)
16 plt.title("histogram: 1000 numbers randomly chosen from the previous numbers")
17 plt.show()
18
19 # (3)
20 data = np.array(data)
21 choices = np.array(choices)
22 print("Mean of (1)", data.mean(), "Variance of (1)", data.var())
23 print("Mean of (2)", choices.mean(), "Variance of (2)", choices.var())
24
```





# HW5

1. (1) 设出现红球  $R$  次, 白球  $W$  次, 黑球  $B$  次

$$P(R=2, W=3, B=1) = \binom{6}{2 \ 3 \ 1} \left(\frac{3}{12}\right)^2 \left(\frac{4}{12}\right)^3 \left(\frac{1}{12}\right)^1$$

$$= \frac{6 \times 4 \times 3 \times 2 \times 1}{6 \times 2} \cdot \frac{1}{16} \times \frac{1}{27} \times \frac{1}{12}$$

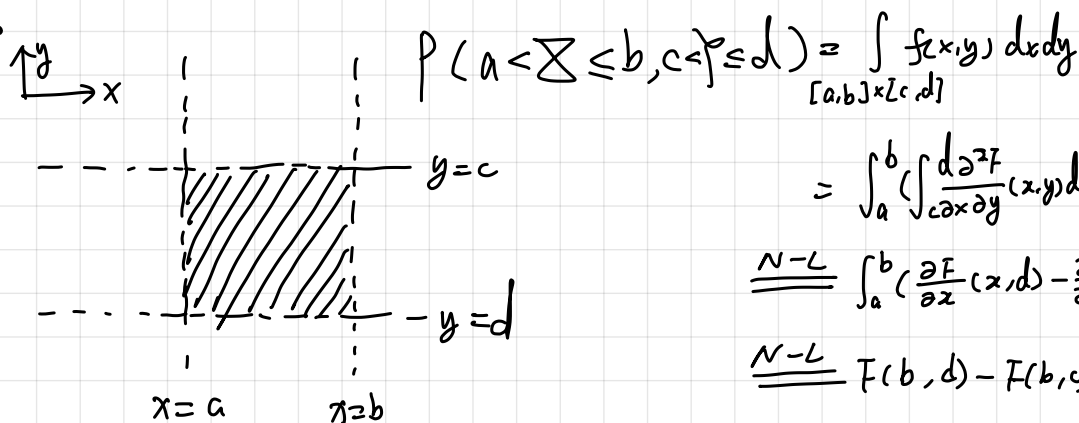
$$= \frac{5}{432}$$

$$(2) P(X=x, Y=y) = \frac{C_3^x \cdot C_4^y \cdot C_5^{3-x-y}}{C_{12}^3}$$

| $Y \backslash X$ | 0                | 1                | 2                | 3               |
|------------------|------------------|------------------|------------------|-----------------|
| 0                | $\frac{10}{220}$ | $\frac{30}{220}$ | $\frac{15}{220}$ | $\frac{1}{220}$ |
| 1                | $\frac{40}{220}$ | $\frac{60}{220}$ | $\frac{12}{220}$ |                 |
| 2                | $\frac{30}{220}$ | $\frac{18}{220}$ |                  |                 |
| 3                | $\frac{4}{220}$  |                  |                  |                 |

$$(3) P(X=1) = \sum_{y=0}^2 P(X=1, Y=y) = \frac{108}{220} = \frac{27}{55}$$

2.



$$P(a < X \leq b, c < Y \leq d) = \int_{[a,b] \times [c,d]} f(x,y) dx dy$$

$$= \int_a^b \left( \int_c^d \frac{d^2 F}{dx dy}(x,y) dy \right) dx$$

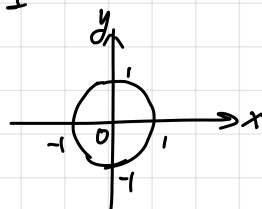
$$\stackrel{N-L}{=} \int_a^b \left( \frac{\partial F}{\partial x}(x,d) - \frac{\partial F}{\partial x}(x,c) \right) dx$$

$$\stackrel{N-L}{=} F(b,d) - F(b,c) - F(a,d) + F(a,c)$$

3. (1) 设 pdf 为  $f(x,y)$

由于  $\forall$  可测  $I \subseteq \mathbb{R}^2$ ,  $P((X,Y) \in I) = \int_I f(x,y) dx dy$

$$f(x,y) = \begin{cases} \frac{1}{\pi} & , x^2 + y^2 < 1 \\ 0 & , x^2 + y^2 \geq 1 \end{cases}$$



$$(2) f_X(x) = \int_{\mathbb{R}} f(x,y) dy = \begin{cases} \frac{2\sqrt{1-x^2}}{\pi}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

$$f_Y(y) = \int_{\mathbb{R}} f(x,y) dx = \begin{cases} \frac{2\sqrt{1-y^2}}{\pi}, & |y| < 1 \\ 0, & |y| \geq 1. \end{cases}$$

$$(3) P(R \leq r) = P(X^2 + Y^2 \leq r^2) = \int_{x^2+y^2 \leq r^2} f(x,y) dx dy = r^2$$

$$(4) \text{ 设 } R \text{ 的 pdf 为 } \varphi(r), \text{ 则 } \varphi(r) = \frac{dr^2}{dr} = 2r$$

$$E(R) = \int_{\mathbb{R}} \varphi(r) r dr = \int_0^1 2r^2 dr = \frac{2}{3} r^3 \Big|_0^1 = \frac{2}{3}$$

4. 设  $(X, Y) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$$\begin{aligned} f_X(x) &= \int_{\mathbb{R}} f(x,y) dy \\ &= \int_{\mathbb{R}} \frac{1}{2\pi\sigma_1\sigma_2} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\frac{x-\mu_1}{\sigma_1} \cdot \frac{y-\mu_2}{\sigma_2}\right]\right\} dy \\ &= \frac{1}{2\pi\sigma_1} \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \int_{\mathbb{R}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_2}{\sigma_2}\right)^2} \cdot e^{\frac{\rho}{1-\rho^2} \cdot \frac{x-\mu_1}{\sigma_1} \cdot \frac{y-\mu_2}{\sigma_2}} d\frac{y-\mu_2}{\sigma_2} \\ &= \frac{1}{2\pi\sigma_1} \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \int_{\mathbb{R}} e^{-\frac{1}{2(1-\rho^2)}\left(y^2 - 2\rho\frac{x-\mu_1}{\sigma_1}y + \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right) + \frac{\rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2}{2(1-\rho^2)}} dy \\ &= \frac{1}{2\pi\sigma_1} \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(1-\rho^2)\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \int_{\mathbb{R}} e^{-\frac{1}{2(1-\rho^2)}\left(y - \rho\frac{x-\mu_1}{\sigma_1}\right)^2} dy \\ &= \frac{1}{2\pi\sigma_1} \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \int_{\mathbb{R}} e^{-\frac{1}{2(1-\rho^2)}y^2} dy \\ &= \frac{1}{2\pi\sigma_1} \frac{1}{\sqrt{1-\rho^2}} \cdot e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \cdot \sqrt{2(1-\rho^2)}\pi \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}, \text{ 即 } X \sim N(\mu_1, \sigma_1) \end{aligned}$$

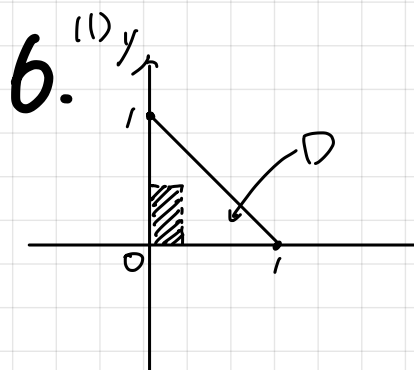
同理,  $Y \sim N(\mu_2, \sigma_2)$

5. 设  $(X, Y) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$$f_Y(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{2\pi\sigma_1\sigma_2} \cdot \frac{1}{\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2} \cdot \frac{1}{1-\rho^2} \left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1} \cdot \frac{y-\mu_2}{\sigma_2}\right)\right]\right\}}{\frac{1}{\sqrt{2\pi}\sigma_1} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_2} \cdot \frac{1}{\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2} \cdot \frac{1}{1-\rho^2} \left(\frac{y - (\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x-\mu_1))}{\sigma_2}\right)^2\right\}$$

此时  $Y_{X=x} \sim (\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x-\mu_1), \sigma_2^2(1-\rho^2))$

6. <sup>(1)</sup>   $P(X \leq x, Y \leq y) = \frac{xy}{\frac{1}{2}} \quad ((x,y) \in D)$

其 pdf 为  $f(x,y) = \begin{cases} 2 - x - y, & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$

(2)  $f_Y(y) = \int_{\mathbb{R}} f(x,y) dx = \int_0^{1-y} 2 - x - y dx = \begin{cases} 2(1-y), & y \in (0,1) \\ 0, & y \notin (0,1) \end{cases}$

(3)  $f_X(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{1-y}, \quad y \in (0,1), x \in (0, 1-y)$

7. (1)  $F_X(x) = \lim_{y \rightarrow +\infty} H(x,y) = F(x)$

$F_Y(y) = \lim_{x \rightarrow +\infty} H(x,y) = G(y)$

(2) 由于边缘分布与  $\alpha$  无关, 故令

$$F(x) = G(x) = \begin{cases} 0, & x \leq 0 \\ x, & x \in (0,1) \\ 1, & x \geq 1 \end{cases}$$

有  $X \sim U(0,1), Y \sim U(0,1)$

8. 令  $H(x, y) = C(F(x), G(y))$  是  $(X, Y)$  的 cdf.

$$F_X(x) = \lim_{y \rightarrow +\infty} H(x, y) = C(F(x), 1) = \lim_{\alpha \rightarrow +\infty} C(F(x), \alpha) = F(x)$$

同理  $F_Y(y) = G(y)$

故  $H(x, y)$  是满足条件的一个构造.

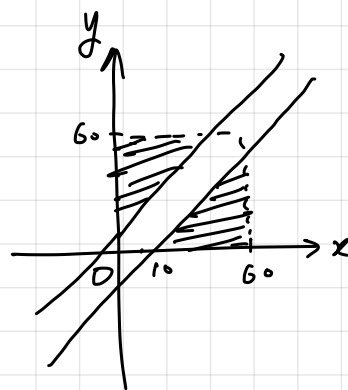
9. 平移至  $D = [0, 60]^2$ , 以“分”为单位

记甲于  $X$  时到达, 乙于  $Y$  时到达

设  $(X, Y)$  的 pdf 为  $f$ , cdf 为  $F$

当  $(x, y) \in D$  时:  $F(x, y) = \frac{xy}{3600}$

$$f(x, y) = \frac{\partial^2 F}{\partial x \partial y} = \begin{cases} \frac{1}{3600} & , (x, y) \in D \\ 0 & , (x, y) \notin D \end{cases}$$



设先到等后到 10 分钟以上为事件  $A$

$$P(A) = \int_{|x-y| \geq 10} f dS = \frac{50 \times 50}{3600} = \frac{25}{36}$$

10. (1) 由  $\int_{\mathbb{R}^2} f dx dy = 1$  有  $\int_{\mathbb{R}^2} f dx dy = \int_{x^2+y^2 \leq 1} \frac{c}{1+x^2+y^2} dx dy$

$$= \int_{0 \leq r \leq 1} \int_{0 \leq \theta < 2\pi} \frac{c}{1+r^2} r dr d\theta$$

$$= c \cdot \frac{2\pi}{2} \int_0^1 \frac{1}{1+r^2} dr^2$$

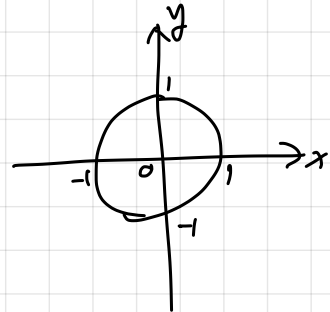
$$= c\pi \int_0^1 \frac{1}{1+x} dx$$

$$= c\pi \ln(1+x) \Big|_0^1$$

$$= c\pi \ln 2$$

$$\Rightarrow c = \frac{1}{\pi \ln 2}$$

$$\begin{aligned}
 (2) f_X(x) &= \int_{\mathbb{R}} f dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{c}{1+x^2+y^2} dy \\
 &= c \ln(y + \sqrt{1+x^2+y^2}) \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \\
 &= c \ln \frac{\sqrt{2} + \sqrt{1-x^2}}{\sqrt{2} - \sqrt{1-x^2}}
 \end{aligned}$$



$$\text{同理 } f_Y(y) = \int_{\mathbb{R}} f dx = c \ln \frac{\sqrt{2} + \sqrt{1-y^2}}{\sqrt{2} - \sqrt{1-y^2}}$$

$$\text{显然 } f(x,y) \neq f_X(x) \cdot f_Y(y)$$

$\therefore X, Y$  不独立

11 (1) 由独立性  $f(x,y) = f_X(x) \cdot f_Y(y)$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$$\text{所以 } \int_{\mathbb{R}^2} g(x,y) dx dy = \int_{x^2+y^2 \leq 1} \frac{xy}{100} dx dy + \int_{\mathbb{R}^2} f(x,y) dx dy$$

$$= \int_0^{+\infty} e^{-\frac{1}{2}r^2} d\frac{r^2}{2}$$

$$= \int_0^{+\infty} e^{-x} dx$$

$$= -e^{-x} \Big|_0^{+\infty} = 1$$

$\therefore g(x,y)$  是二维概率密度函数

$$(2) f_U(x) = \int_{\mathbb{R}} g(x,y) dy = \int_{\mathbb{R}} f(x,y) dy + \int_{x^2+y^2 \leq 1} \frac{xy}{100} dy \text{ (odd func.)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\therefore U \sim N(0,1), \text{ 同理 } V \sim N(0,1)$$

但在  $E = \{(x,y) \mid x^2+y^2 \leq 1, y \leq 0\}$  中: 不存在  $\rho \in (-1,1)$  使得

$$\int_E g dx dy = \int_E \frac{1}{2\pi\sigma_1\sigma_2} \frac{1}{\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right)\right\} dx dy$$

$\therefore (U,V)$  不服从二元正态分布.

## 12. 1° $X, Y$ 均连续

$$\begin{aligned}\text{全概率 } f_X(x) &= \int_{\mathbb{R}} f(x, y) dy \\ &= \int_{\mathbb{R}} f_Y(y) \cdot f_X(x|y) dy\end{aligned}$$

$$\text{Bayes 公式: } f_X(x|y) = \frac{f_X(x) \cdot f_Y(y|x)}{\int_{\mathbb{R}} f_Y(y|x) \cdot f_X(x) dx}$$

## 2° $X, Y$ 均离散

$$\text{全概率: } f_X(x) = \sum_{y \in \mathbb{R}} f(x, y)$$

$$\text{Bayes: } f_X(x|y) = \frac{f_X(x) \cdot f_Y(y|x)}{\sum_{x \in \mathbb{R}} f_X(x) \cdot f_Y(y|x)}$$

## 3° $X$ 离散, $Y$ 连续:

$$\text{全概率: } f_X(x) = \int_{\mathbb{R}} f(x, y) dy$$

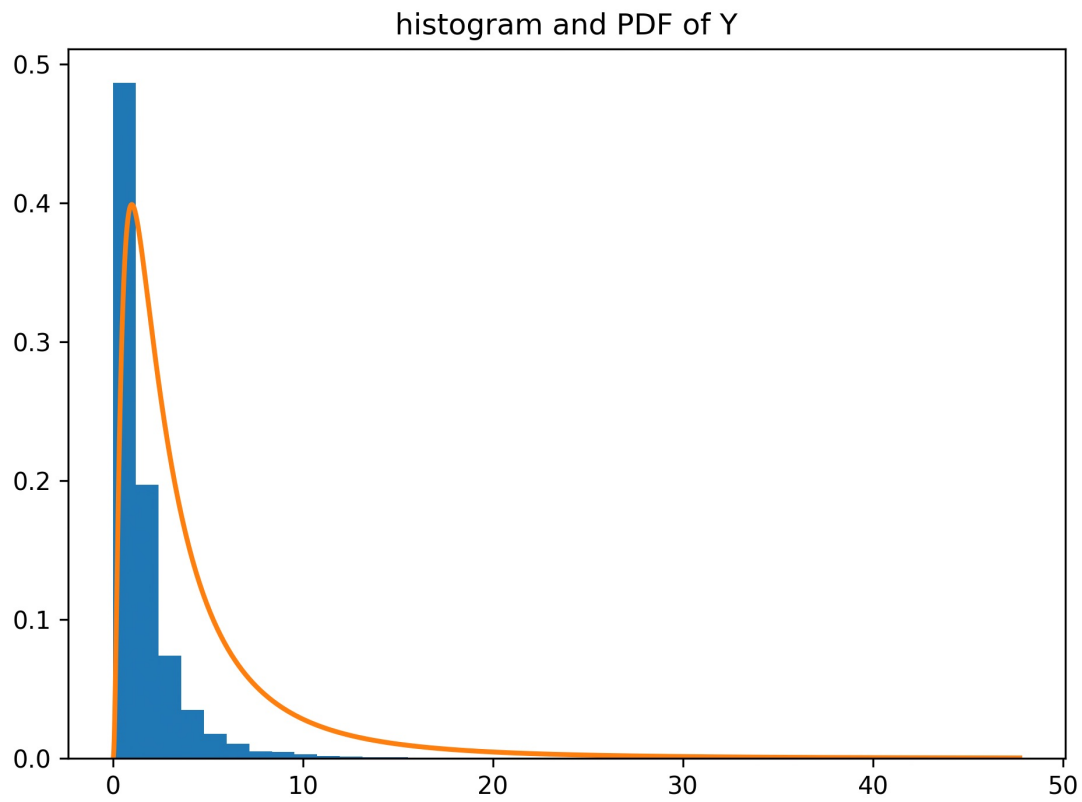
$$\text{Bayes: } f_Y(y|x) = \frac{f_Y(y) f_X(x|y)}{\int_{\mathbb{R}} f_Y(y) f_X(x|y) dx}$$

## 4° $X$ 连续, $Y$ 离散:

$$\text{全概率: } f_X(x) = \sum_{y \in \mathbb{R}} f(x, y)$$

$$\text{Bayes: } f_Y(y|x) = \frac{f_Y(y) f_X(x|y)}{\sum_{y \in \mathbb{R}} f_Y(y) f_X(x|y)}$$

# 13.



```
1 import random
2 from matplotlib import pyplot as plt
3 from math import e, log, sqrt, pi
4 import numpy as np
5
6 plt.figure(dpi=300)
7
8 x = [random.normalvariate(0, 1) for i in range(10000)]
9 y = [pow(e, t) for t in x]
10
11 space = np.linspace(min(y), max(y), 10000)
12 pdf = [pow(e, -log(t, e)*log(t, e)/2)/sqrt(pi*2) for t in space]
13
14 plt.hist(y, bins=40, density=True)
15 plt.plot(space, pdf, linewidth=2.0)
16 plt.title("histogram and PDF of Y")
17 plt.show()
```





# HW 6

$$\begin{aligned}
 1 \quad P(Y=y) &= P(X_1 + X_2 = y) \\
 &= \sum_{k=0}^y P(X_1=k, X_2=y-k) \\
 &= \sum_{k=0}^y P(X_1=k) P(X_2=y-k) \\
 &= \sum_{k=0}^y \frac{\lambda_1^k}{k!} e^{-\lambda_1} \frac{\lambda_2^{y-k}}{(y-k)!} e^{-\lambda_2} \\
 &= \sum_{k=0}^y \frac{\lambda_1^k \lambda_2^{y-k}}{k!(y-k)!} e^{-(\lambda_1+\lambda_2)} \\
 &= \frac{e^{-(\lambda_1+\lambda_2)}}{y!} \sum_{k=0}^y \binom{y}{k} \lambda_1^k \lambda_2^{y-k} \\
 &= \frac{(\lambda_1+\lambda_2)^y}{y!} e^{-(\lambda_1+\lambda_2)}
 \end{aligned}$$

即  $Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$

直观上, 单位时间内两件事发生的次数服从 Poisson 分布.

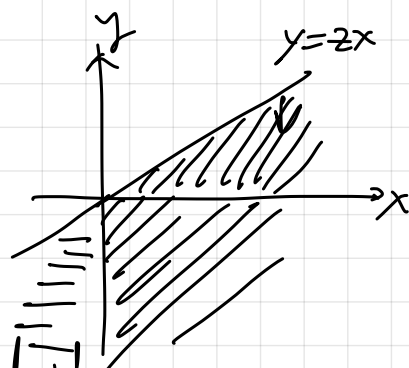
则其发生次数之和仍服从 Poisson 分布, 即两事件期望相加的 Poisson 分布

$$2 \quad (1) \quad X \sim N(0, 1), Y \sim (0, 1), f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, f(x, y) = f(x)f(y)$$

$$P(Z \leq z) = P\left(\frac{Y}{X} \leq z\right)$$

$$= P(Y \leq zX \wedge X > 0 \text{ 或 } Y \leq zX \wedge X < 0)$$

$$= \int_0^{+\infty} \left( \int_{-\infty}^{zx} f(x, y) dy \right) dx + \int_{-\infty}^0 \left( \int_{-\infty}^{zx} f(x, y) dy \right) dx$$



$$\begin{aligned}
 \text{pdf. } \frac{d}{dz} P(Z \leq z) &= \int_0^{+\infty} x f(x, zx) dx + \int_{-\infty}^0 x f(x, zx) dx \\
 &= \int_{\mathbb{R}} x f(x, zx) dx.
 \end{aligned}$$

(2) 用密度函数变换法: 设  $R, \theta$  pdf 为  $h(r, \theta)$ .

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

$$h(r, \theta) = f(r \cos \theta) f(r \sin \theta) r$$

$$= \frac{1}{2\pi} e^{-\frac{r^2 \cos^2 \theta}{2}} \cdot e^{-\frac{r^2 \sin^2 \theta}{2}} \cdot r$$

$$= \frac{1}{2\pi} \cdot r \cdot e^{-\frac{r^2}{2}}$$

pdf 是变量分离的, 所以,  $R, \theta$  独立.

$$(3) \begin{cases} U = X + Y \\ V = X - Y \end{cases} \Rightarrow \begin{cases} X = \frac{1}{2}(U + V) \\ Y = \frac{1}{2}(U - V) \end{cases} \Rightarrow \frac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$|\det(J)| = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

设  $U, V$  pdf 为  $k(u, v)$

$$k(u, v) = f\left(\frac{1}{2}(u+v)\right) \cdot f\left(\frac{1}{2}(u-v)\right) \cdot \frac{1}{2}$$

$$= \frac{1}{4\pi} e^{-\frac{1}{8}((u+v)^2 + (u-v)^2)}$$

$$= \frac{1}{4\pi} e^{-\frac{1}{4}(u^2 + v^2)}$$

$$= \frac{1}{4\pi} e^{-\frac{1}{4}u^2} \cdot e^{-\frac{1}{4}v^2}$$

pdf 变量分离,  $U, V$  独立.

3. 设  $y$  的 pdf 为  $g(y)$ .  $(X_1, X_2)$  pdf 为  $f = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x-\mu_2}{\sigma_2}\right)^2 - 2\frac{x-\mu_1}{\sigma_1} \cdot \frac{x-\mu_2}{\sigma_2}\right\}$

$$g(y) = \int_{\mathbb{R}} f(y-x, x) dx$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{\mathbb{R}} \exp\left\{\frac{1}{2(1-\rho^2)}\left(\frac{y-x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x-\mu_2}{\sigma_2}\right)^2 - 2\frac{y-x-\mu_1}{\sigma_1} \cdot \frac{x-\mu_2}{\sigma_2}\right\} dx$$

= ...

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{(y-\mu_1-\mu_2)^2}{2(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)}\right\} \int_{\mathbb{R}} e^{-\frac{1}{2(1-\rho^2)}(ax-b)^2} dx$$

其中:  $a = \frac{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}}{\sigma_1\sigma_2}$ ,  $b = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}} \left(\frac{y-\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2} + \frac{2\rho(y-\mu_1-\mu_2)}{\sigma_1\sigma_2}\right)$

$$\text{故 } g(y) = \frac{1}{\sqrt{2\pi} \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2} \exp\left\{-\frac{(y-\mu_1-\mu_2)^2}{2(\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)}\right\}$$

$$\text{即 } Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$$

#### 4<sup>(1)</sup> 密度函数变换.

$$\begin{cases} z = XY \\ W = X \end{cases} \Rightarrow \begin{cases} X = W \\ Y = \frac{z}{W} \end{cases}$$

$$\frac{\partial(x, Y)}{\partial(z, W)} = \begin{pmatrix} 0 & 1 \\ \frac{1}{W} & -\frac{z}{W^2} \end{pmatrix}$$

$$\det(J) = \left| \frac{1}{W} \right| = \frac{1}{W}$$

设  $z, w$  的 pdf 为  $g$ .

$$\text{有: } g(z, w) = f(w)g\left(\frac{z}{w}\right) \cdot \frac{1}{w}$$

$$g_z(z) = \int_{\mathbb{R}} f(w)g\left(\frac{z}{w}\right) \frac{1}{w} dw$$

$$(2) \text{ 设 } X^{-1} \text{ 的 pdf 为 } h(x) = -f\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) \\ = \frac{1}{x^2} f\left(\frac{1}{x}\right)$$

$$Z = XY = \frac{Y}{X^{-1}}, \text{ 设 pdf 为 } k(z)$$

$$k(z) = \int_0^{+\infty} x h(x) g(xz) dx$$

$$= \int_0^{+\infty} x \cdot \frac{1}{x^2} f\left(\frac{1}{x}\right) g(xz) dx$$

$$k(z) = \int_0^{+\infty} \frac{1}{x} f\left(\frac{1}{x}\right) g(xz) dx$$

对 (2) 的结果令  $W = \frac{1}{x}$ , 即与 (1) 一致.

$$\begin{aligned}
 5 \quad P(Y \leq y) &= P(\max_i \{Z_i\} \leq y) \\
 &= P(Z_1 \leq y \wedge Z_2 \leq y \wedge \dots \wedge Z_n \leq y) \\
 &= (F(y))^n
 \end{aligned}$$

$$\begin{aligned}
 P(Z \geq z) &= P(\min_i (Z_i) \geq z) \\
 &= (1 - F(z))^n \\
 \therefore P(Z \leq z) &= 1 - (1 - F(z))^n
 \end{aligned}$$

6. ① 卡方分布: 设  $Z_1, \dots, Z_k$  独立,  $Z_i \sim N(0, 1)$   
 称  $Y = \sum_{i=1}^k Z_i^2$  服从自由度为  $k$  的卡方分布

② t分布: 设  $X_1, \dots, X_n$  独立,  $X_i \sim N(\mu, \sigma^2)$

$$\text{记 } \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

$$\text{设 } T = \frac{\bar{X} - \mu}{\frac{S_n}{\sqrt{n}}}$$

则  $T$  服从的分布为 t 分布

③  $Z \sim F(d_1, d_2)$

$$\Leftrightarrow Z \text{ 的 pdf: } f(x)_{d_1, d_2} = \frac{\sqrt{\frac{(d_1 x)^{d_1} d_2^{d_2}}{(d_1 x + d_2)^{d_1 + d_2}}}}{x B(\frac{d_1}{2}, \frac{d_2}{2})}.$$

7. <sup>(1)</sup> 正确, 设  $\mu = E(X)$ .

$$\begin{aligned} E((X-c)^2) &= \text{Var}(X) \\ &= E((X-c)^2) - E((X-\mu)^2) \\ &= E(-2(c-\mu)X + (c-\mu)(c+\mu)) \\ &= (c-\mu)E(-2X + c + \mu) \\ &= (c-\mu)^2 \geq 0 \quad \forall c \in \mathbb{R} \end{aligned}$$

即  $E((X-c)^2) \geq \text{Var}(X)$  当且仅当  $c = \mu$  时, 等号成立.

(2) 不正确

$$\text{Var}(XY) = E(X^2Y^2) - E(XY)^2$$

$X, Y$  独立  $\Rightarrow X^2, Y^2$  独立

$$\Rightarrow \text{Var}(XY) = E(X^2)E(Y^2) - E^2(X)E^2(Y) \neq (E(X^2) - E^2(X))(E(Y^2) - E^2(Y))$$

(3) 不正确, 设  $X$  :

|     |     |     |
|-----|-----|-----|
| $X$ | 0   | 1   |
| $P$ | 0.1 | 0.9 |

中位数是 1.

$$E(X) = 0.9$$

8. 令  $\alpha(c) = E(|X-c|)$

$$= \int_{\mathbb{R}} |x-c| f(x) dx$$

$$= \int_{-\infty}^c (c-x) f(x) dx + \int_c^{+\infty} (x-c) f(x) dx$$

$$\alpha'(c) = \int_{-\infty}^c f(x) dx - \int_c^{+\infty} f(x) dx$$

$$= P(X \leq c) - P(X \geq c)$$

$$= P(X \leq c) - (1 - P(X \leq c))$$

$$= 2P(X \leq c) - 1$$

$\therefore$  当  $P(X \leq c) = \frac{1}{2}$  时,  $\alpha(c)$  取最小值

依定义可知, 这时  $c$  是中位数  $m$ .

$$\text{即 } E(|X-c|) = \alpha(c) \geq \alpha(m) = E(|X-m|).$$

9

令  $\log X \sim \mathcal{N}(\mu, \sigma^2)$ .

有  $X$  的 pdf:  $f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2}$

$$E(X) = \int_0^{+\infty} x f(x) dx$$

$$= \int_0^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2} dx$$

$$\stackrel{t=\ln x}{=} \int_{\mathbb{R}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}(t^2 - 2\mu t + \mu^2 - 2\sigma^2 t)} dt$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}(t^2 - 2(\mu + \sigma^2)t + (\mu + \sigma^2)^2 - 2\mu\sigma^2 - \sigma^4)} dt$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{\sigma^2}{2} + \mu} \int_{\mathbb{R}} e^{\frac{(t - \mu - \sigma^2)^2}{-2\sigma^2}} dt$$

Lemma  $\int_{\mathbb{R}} e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}}$ , 有  $\int_{\mathbb{R}} e^{-\frac{x^2}{2}} = \sigma\sqrt{2\pi}$

$$\Rightarrow E(X) = e^{\mu + \frac{\sigma^2}{2}}$$

$$E(X^2) = \int_0^{+\infty} x^2 f(x) dx$$

$$= \int_0^{+\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} e^{2x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}(x^2 - 2(\mu + 2\sigma^2)x + (\mu + 2\sigma^2)^2 + \mu^2 - (\mu + 2\sigma^2)^2)} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{4\sigma^4 + 4\sigma^2\mu}{2\sigma^2}} \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}(x - \mu - 2\sigma^2)^2} dx$$

$$= e^{2\sigma^2 + 2\mu}$$

$$\text{Var}(X) = E(X^2) - E(X)^2$$

$$= e^{2\sigma^2 + 2\mu} - e^{2\mu + \sigma^2}$$

$$= e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$$



10

$$\text{Var}(\bar{X}) = \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^n X_i\right)$$

$$= \frac{1}{n^2} \cdot n\sigma^2$$

$$= \frac{\sigma^2}{n}$$

$$E(S^2) = \frac{1}{n-1} \sum_{i=1}^n E((X_i - \bar{X})^2)$$

$$= \frac{1}{n-1} \sum_{i=1}^n (E(X_i^2) - 2E(X_i \bar{X}) + E(\bar{X}^2))$$

$$\text{Var}(\bar{X}) = E(\bar{X}^2) - (E(\bar{X}))^2$$

$$= E(\bar{X}^2) - \mu^2 \Rightarrow E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2$$

$$E(X_i \bar{X}) = E\left(\frac{1}{n} \sum_{j=1}^n X_i X_j\right)$$

$$= \frac{1}{n} \left[ (n-1)\mu^2 + \mu^2 + \sigma^2 \right] = \mu^2 + \frac{\sigma^2}{n}$$

$$\Rightarrow E(S^2) = \frac{1}{n-1} \cdot \sum_{i=1}^n \left( \mu^2 + \sigma^2 - 2\mu^2 - 2\frac{\sigma^2}{n} + \frac{\sigma^2}{n} + \mu^2 \right)$$

$$= \frac{1}{n-1} \cdot n \cdot \left( \sigma^2 - \frac{\sigma^2}{n} \right)$$

$$= \sigma^2$$

$$E(\bar{X}) = \mu,$$

$$\text{Var}(X_i) = \sigma^2$$

$$= E(X_i^2) - \mu^2$$

$$\Rightarrow E(X_i^2) = \mu^2 + \sigma^2$$

11 等价.

(1)  $\Leftrightarrow$  (2) 这是不相关的定义

$$(1) \Leftrightarrow (3) \text{Corr}(X, Y) = E((X - \mu_1)(Y - \mu_2))$$

$$= E(XY) - E(X)E(Y) = 0$$

$$\Leftrightarrow E(XY) = E(X)E(Y)$$

$$(1) \Leftrightarrow (4) \text{Var}(X+Y) = E((X - \mu_1) + (Y - \mu_2))^2$$

$$= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y).$$

$$12. \text{Corr}(X, Y) = E\left(\frac{X-\mu_1}{\sigma_1} \cdot \frac{Y-\mu_2}{\sigma_2}\right)$$

$$= \int_{\mathbb{R}^2} \frac{x-\mu_1}{\sigma_1} \frac{y-\mu_2}{\sigma_2} \cdot \frac{1}{2\pi\sigma_1\sigma_2} \cdot \frac{1}{\sqrt{1-\rho^2}} \cdot \exp\left\{\frac{-1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \frac{y-\mu_2}{\sigma_2}\right\} dx dy$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{\mathbb{R}^2} xy e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)} dx dy$$

$$\text{令 } \vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}, W = \frac{1}{1-\rho^2} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix} = \frac{1}{1-\rho^2} A^T A, A = \begin{pmatrix} 1 & -\rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix} \text{ 则 } \text{Corr}(X, Y)$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{\mathbb{R}^2} xy e^{-\frac{1}{2(1-\rho^2)}(A\vec{v})^T(A\vec{v})} dx dy$$

$$\text{令 } u = A\vec{v} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}u = \frac{1}{\sqrt{1-\rho^2}} \begin{bmatrix} \sqrt{1-\rho^2}\rho \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{\sqrt{1-\rho^2}} \begin{pmatrix} \sqrt{1-\rho^2}\rho u_1 + u_2 \\ u_2 \end{pmatrix}$$

$$\text{Corr}(X, Y)$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{\mathbb{R}^2} \frac{1}{1-\rho^2} \cdot (\sqrt{1-\rho^2}u_1u_2 + u_2^2) e^{-\frac{1}{2} \cdot \frac{1}{(1-\rho^2)}(u_1^2+u_2^2)} du_1 du_2$$

$$= \frac{\rho}{2\pi(1-\rho^2)^{\frac{3}{2}}} \int_{\mathbb{R}^2} u_2^2 e^{-\frac{1}{2} \cdot \frac{1}{1-\rho^2}(u_1^2+u_2^2)} du_1 du_2$$

$$= \frac{\rho}{4\pi(1-\rho^2)^{\frac{3}{2}}} \int_{\mathbb{R}^2} (u_1^2+u_2^2) e^{-\frac{1}{2} \cdot \frac{1}{1-\rho^2}(u_1^2+u_2^2)} du_1 du_2$$

$$= \frac{\rho}{4\pi} \cdot \frac{1}{(1-\rho^2)^{\frac{3}{2}}} \cdot 2\pi \cdot \int_0^{+\infty} r^3 e^{-\frac{1}{2(1-\rho^2)}r^2} dr$$

$$= \frac{-\rho}{2} \cdot \frac{1}{\sqrt{1-\rho^2}} \int_0^{+\infty} r^2 d e^{-\frac{r^2}{2(1-\rho^2)}}$$

$$= \frac{\rho}{2} \frac{1}{\sqrt{1-\rho^2}} \int_0^{+\infty} e^{-\frac{r^2}{2(1-\rho^2)}} dr^2$$

$$= \rho$$

13 设第  $i$  个人取到帽子为  $A_i = \begin{cases} 0, & \text{未发生} \\ 1, & \text{发生} \end{cases}$

$$\text{有 } \bar{X} = \sum_{i=1}^n A_i.$$

$$E(\bar{X}) = \sum_{i=1}^n E(A_i)$$

$$= n \cdot \frac{1}{n}$$

$$= 1$$

$$\text{Var}(\bar{X}) = \text{Cov}(\bar{X}, \bar{X}) = \sum_{i,j} \text{Cov}(A_i, A_j)$$

$$\text{Cov}(A_i, A_j) = E\left(\left(A_i - \frac{1}{n}\right)\left(A_j - \frac{1}{n}\right)\right)$$

$$= E(A_i A_j) - \frac{1}{n} E(A_i + A_j) + \frac{1}{n^2}$$

$$= -\frac{1}{n^2} + E(A_i A_j).$$

$$i=j \text{ 时: } E(A_i A_j) = \frac{1}{n}$$

$$i \neq j \text{ 时: } E(A_i A_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$$\Rightarrow \text{Var}(\bar{X}) = -\frac{n^2}{n^2} + (n^2 - n) \cdot \frac{1}{n(n-1)} + n \cdot \frac{1}{n} = 1$$

14. <sup>(1)</sup> 注意到  $E((V - t_0 U)^2) \geq 0$ .

$$\text{即 } t_0^2 E(U^2) - 2E(UV)t_0 + E(V^2) \geq 0$$

$$\text{故 } \Delta \leq 0 \Rightarrow 4E(UV)^2 - 4E(U^2)E(V^2) \leq 0$$

$$\Rightarrow E(UV)^2 \leq E(U^2)E(V^2), \text{ 当且仅当 } \exists t_0 \text{ s.t. } V = t_0 U \text{ 取等.}$$

$$^{(2)} |\text{Corr}(\bar{X}, Y)| = \left| \frac{\text{Cov}(\bar{X}, Y)}{\sqrt{\text{Var}(\bar{X})\text{Var}(Y)}} \right| = \left| \frac{E((\bar{X} - \mu_1)(Y - \mu_2))}{\sqrt{E(\bar{X} - \mu_1)E(Y - \mu_2)}} \right| \leq 1$$

$$\text{当且仅当 } \frac{Y - \mu_2}{\sigma_2} = t_0 \frac{\bar{X} - \mu_1}{\sigma_1} \text{ 时: " = " 成立.}$$

$$\text{此时: } Y = \frac{\sigma_2}{\sigma_1} t_0 \bar{X} - t_0 \frac{\sigma_2}{\sigma_1} \mu_1 + \mu_2$$

$$\text{即 } Y = aX + b$$

15 (1)  $Cov(X_i - \bar{X}, \bar{X})$

$$= Cov(X_i, \bar{X}) - Var(\bar{X})$$

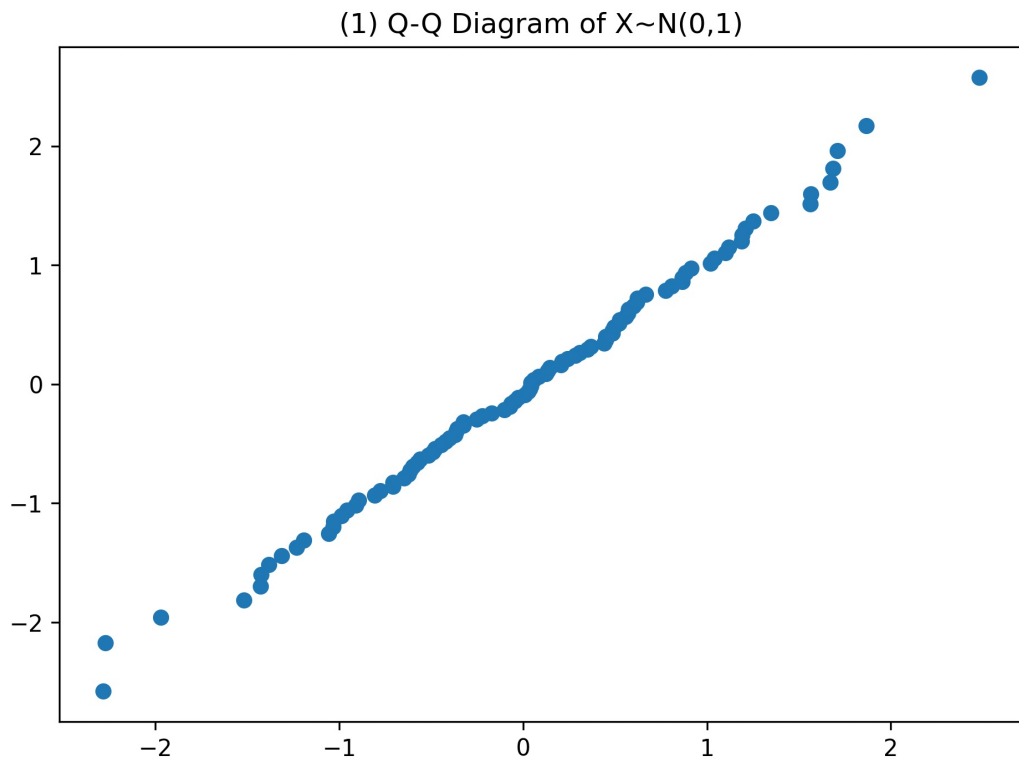
$$= -\frac{\sigma^2}{n} + E((X_i - \mu)(\bar{X} - \mu))$$

$$= -\frac{\sigma^2}{n} + E(X_i \bar{X}) - 2\mu^2 + \mu^2$$

$$= -\frac{\sigma^2}{n} + \mu^2 + \frac{\sigma^2}{n} - \mu^2 = 0.$$

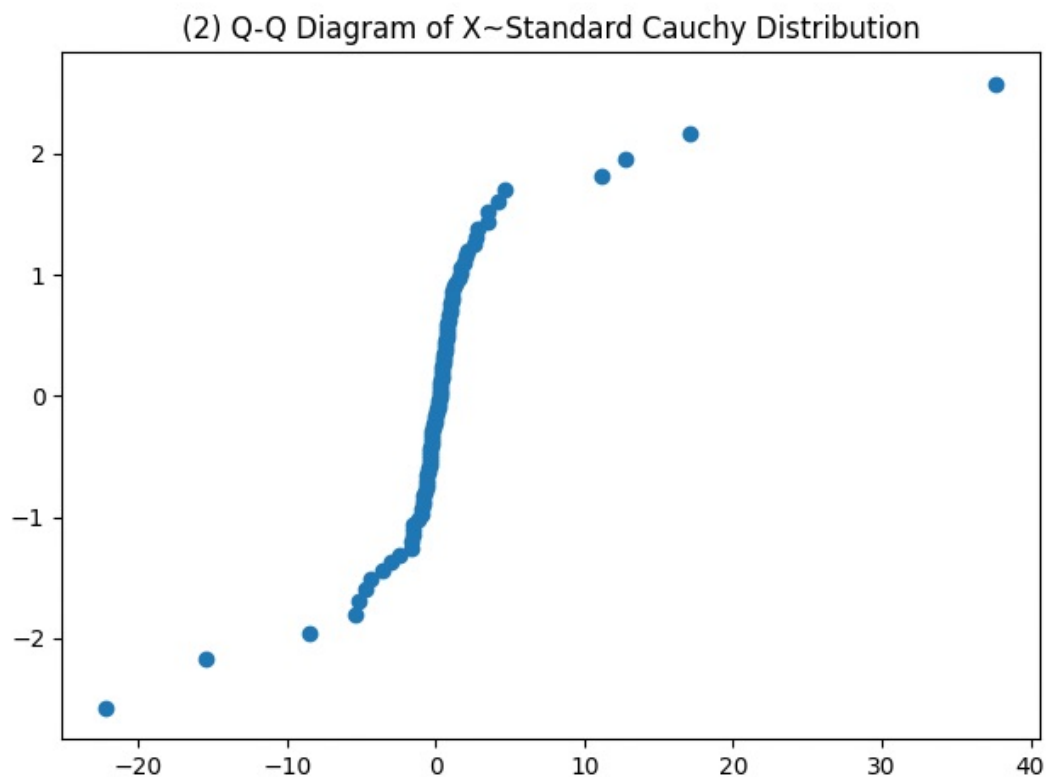
(2) 不一定独立.  $Corr = 0$  只表明没有线性相关性.

16. ①



确定近似成直线

②



明显不成直线。

③ 可以，只需将  $\Phi^{-1}$  换成假设分布 cdf 的逆。

④ 可以，将两数据集分别排序，置于  $X, Y$  轴，观察散点，图是否集中于一直线附近。

```
1 import numpy as np
2 from matplotlib import pyplot as plt
3 from scipy.stats import norm
4 import random
5
6 plt.figure(dpi=300)
7 cnt = 100
8
9 # (1)
10 data = sorted([random.normalvariate(0, 1) for ii in range(cnt)])
11 inv = [norm.ppf((i-.5)/cnt) for i in range(1, cnt + 1)]
12 plt.title("(1) Q-Q Diagram of  $X \sim N(0,1)$ ")
13 plt.scatter(data, inv)
14 plt.show()
15
16 # (2)
17 cauchy = sorted(np.random.standard_cauchy(cnt))
18 plt.title("(2) Q-Q Diagram of  $X \sim \text{Standard Cauchy Distribution}$ ")
19 plt.scatter(cauchy, inv)
20 plt.show()
21
```



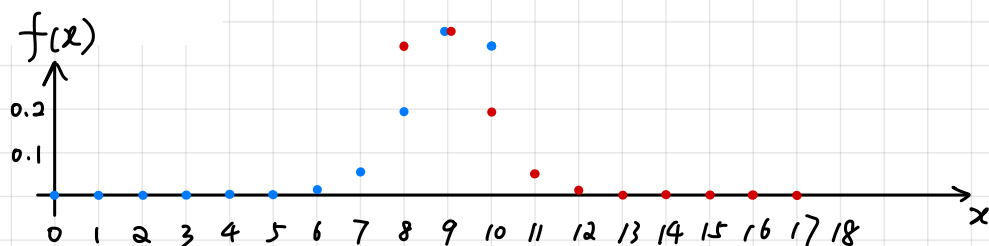
# HW 7

1. 设  $X \sim B(10, 0.9)$ ,  $Y-8 \sim B(10, 0.1)$ 。

- (1) 分别画出  $X, Y$  的概率质量函数图。
- (2) 计算并比较  $X, Y$  的均值、方差、中位数和众数。
- (3) 计算  $X, Y$  的偏度系数。

$$(1) P(X=k) = C_{10}^k 0.9^k 0.1^{10-k} \quad k \in \{0, \dots, 10\}$$

$$P(Y=k) = C_{10}^{k-8} 0.1^{k-8} 0.9^{18-k} \quad k \in \{8, \dots, 18\}$$



$$(2) E(X) = 10 \times 0.9 = 9.$$

$$\text{Var}(X) = 10 \times 0.9 \times 0.1 = 0.9$$

$$\text{Median}(X) = 9$$

$$\text{Mode}(X) = 9$$

$$E(Y) = E(Y-8) + 8 = 9$$

$$\text{Var}(Y) = \text{Var}(Y-8) = 0.9$$

$$\text{Median}(Y) = 9$$

$$\text{Mode}(Y) = 9$$

$$(3) E(X^3) = \sum_{k=0}^{10} k^3 C_{10}^k 0.9^k 0.1^{10-k} \quad \underline{\text{Wolfram Alpha}} \quad 752.58$$

$$SK(X) = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$$

$$= \sigma^{-3} E(X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3)$$

$$= \sigma^{-3} (E(X^3) - 3\mu(\mu^2 + \sigma^2) + 3\mu^3 - \mu^3)$$

$$= \sigma^{-3} (E(X^3) - 3\mu\sigma^2 - \mu^3)$$

$$\Rightarrow SK(X) \approx -0.8433$$

$$\text{同理 } SK(Y) = +0.8433$$



2. (1) 分别计算  $\text{Exp}(1)$ ,  $P(4)$  和  $U(0,1)$  的偏度系数和峰度系数。

(2) 分别计算  $\text{Exp}(1)$ ,  $P(4)$  和  $U(0,1)$  的矩母函数

(3) 利用矩母函数计算上述三个分布的偏度系数和峰度系数。

(1) 设  $X \sim \text{Exp}(1)$ , 有  $f(x) = e^{-x} (x > 0)$

$$SK(X) = \frac{E(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3}$$

$$\text{Lemma: } \int_0^{+\infty} x^n e^{-x} dx = n!$$

$$E(X^3) = \int_0^{+\infty} x^3 e^{-x} dx = 6, \mu = 1, \sigma = 1$$

$$SK(X) = \frac{6 - 3 \times 1 \times 1 - 1}{1} = 2$$

$$\begin{aligned} \text{Kurtosis}(X) &= E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \sigma^{-4} E(X^4 - 4\mu X^3 + 6\mu^2 X^2 - 4\mu^3 X + \mu^4) \\ &= \sigma^{-4} (E(X^4) - 4\mu E(X^3) + 6\mu^2(\mu^2 + \sigma^2) - 4\mu^4 + \mu^4) \\ &= \sigma^{-4} (E(X^4) - 4\mu E(X^3) + 3\mu^4 + 6\mu^2\sigma^2) \\ &= 24 - 4 \times 6 + 3 + 6 = 9 \end{aligned}$$

设  $Y \sim \text{Poisson}(4)$ ,  $P(Y=k) = \frac{4^k}{k!} e^{-4}$

$$E(Y^3) = \sum_{k=0}^{\infty} k^3 \frac{4^k}{k!} e^{-4} \quad \text{Wolfram Alpha} \quad 116; \mu = 4, \sigma = 2$$

$$SK(Y) = \frac{E(Y^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3} = \frac{116 - 48 - 64}{8} = \frac{1}{2}$$

$$\text{Kurtosis}(Y) = E\left(\frac{Y-\mu}{\sigma}\right)^4 = \frac{13}{4}$$

设  $Z \sim U(0,1)$ ,  $f(x) = 1, x \in (0,1)$

$$E(Z^3) = \int_0^1 z^3 dx = \frac{1}{4}, \mu = \frac{1}{2}, \sigma^2 = \frac{1}{12} \Rightarrow \sigma = \frac{1}{2\sqrt{3}}$$

$$SK(Z) = \frac{E(Z^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3} = 24\sqrt{3} \left( \frac{1}{4} - 3 \times \frac{1}{2} \times \frac{1}{12} - \frac{1}{8} \right) = 0$$

$$\text{Kurtosis}(Z) = \frac{9}{5}$$

$$(2) \text{Exp}(1): \text{MGF: } M_X(t) = E(e^{tx}) = \int_0^{+\infty} e^{tx} e^{-x} dx = \frac{\lambda}{\lambda - t} \quad (t < \lambda)$$

$$P(4): \text{MGF: } M_Y(t) = E(e^{tY}) = \sum_{k=0}^{\infty} e^{tk} \frac{4^k}{k!} e^{-4} = e^{-4} \sum_{k=0}^{\infty} \frac{(4e^t)^k}{k!} = e^{4e^t - 4}$$

$$U(0,1): \text{MGF: } M_Z(t) = E(e^{tZ}) = \int_0^1 e^{tx} dx = \frac{1}{t} e^{tx} \Big|_0^1 = \frac{e^t - 1}{t}$$

(3) • Exp(1)

$$M_X(t) = -\lambda(t-\lambda)^{-1}, M'_X(t) = \lambda(t-\lambda)^{-2}, \dots, M_X^{(n)}(t) = (-1)^{n-1} n! (t-\lambda)^{-(n+1)}$$

$$\text{令 } \lambda=1, t=0. M_X^{(n)}(0) = n! \Rightarrow E(X^3) = 6, E(X^4) = 24$$

$$\Rightarrow \text{偏度} = 2,$$

$$\text{峰度} = 9$$

$$\bullet P(4) M(t) = e^{\lambda(e^t-1)}, M'(t) = e^{\lambda(e^t-1)} \cdot \lambda e^t = \lambda e^{(\lambda e^t - \lambda + t)}$$

$$, M''(t) = \lambda e^{(\lambda e^t - \lambda + t)} \cdot (\lambda e^t + 1)$$

$$= \lambda^2 e^{(\lambda e^t - \lambda + t)} + M'(t)$$

$$, M'''(t) = M''(t) + \lambda^2 e^{(\lambda e^t - \lambda + t)} \cdot (\lambda e^t + 2)$$

$$= M''(t) + \lambda^3 e^{\lambda e^t - \lambda + 3t} + 2(M''(t) - M'(t))$$

$$, M^{(4)}(t) = 3M'''(t) - 2M''(t) + \lambda^3 e^{(\lambda e^t - \lambda + 3t)} (\lambda e^t + 3).$$

$$M^{(1)}(0) = \lambda, M^{(2)}(0) = \lambda^2 + \lambda, M^{(3)}(0) = \lambda^3 + 3(\lambda^2 + \lambda) - 2\lambda = \lambda^3 + 3\lambda^2 + \lambda$$

$$M^{(4)}(0) = 3(\lambda^3 + 3\lambda^2 + \lambda) - 2(\lambda^2 + \lambda) + \lambda^3(\lambda + 3)$$

$$= 3\lambda^3 + 9\lambda^2 + 3\lambda - 2\lambda^2 - 2\lambda + \lambda^4 + 3\lambda^3$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

$$\text{故: } E(Y^3) = 14 + 48 + 4 = 116$$

$$E(Y^4) = 256 + 6 \times 64 + 7 \times 16 + 4 = 756.$$

$$\Rightarrow \text{偏度} = \frac{1}{2}, \text{峰度} = \frac{756 - 4 \times 4 \times 116 + 3 \times 256 + 6 \times 16 \times 4}{\sigma^4}$$

$$= 3.25$$

$$\bullet U(0,1). M(t) = \frac{e^t - 1}{t} = \frac{1}{t} \sum_{k=1}^{\infty} \frac{t^k}{k!} = \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{t^{k-1}}{(k-1)!}$$

$$\Rightarrow M^{(k)}(t) = \frac{1}{k+1}$$

$$\Rightarrow M^{(3)}(t) = \frac{1}{4}, M^{(4)}(t) = \frac{1}{5}$$

$$\begin{aligned} \Rightarrow \text{偏度} &= 0, \text{峰度} = \sigma^{-4} (E(X^4) - 4\mu E(X^3) + 3\mu^4 + 6\mu^2 \sigma^2) \\ &= 144 \left( \frac{1}{5} - 4 \times \frac{1}{2} \times \frac{1}{4} + 3 \times \frac{1}{16} + 6 \times \frac{1}{4} \times \frac{1}{12} \right) \\ &= 144 \times \frac{1}{80} = \frac{9}{5} \end{aligned}$$

3. 令  $X \sim \text{Exp}(2)$ ,  $Y \sim \text{Exp}(3)$

设  $f(x) = \frac{1}{3}2e^{-2x} + \frac{2}{3} \cdot 3e^{-3x}$  是  $Z$  的 pdf

$$\text{则 } E(e^{tx}) = \int_0^{+\infty} e^{tx} f(x) = \frac{1}{3} M_X(t) + \frac{2}{3} M_Y(t) \quad \forall t < 2.$$

由于矩母函数确定分布,

$$\text{故 pdf 为 } f(x) = \frac{2}{3}e^{-2x} + 2e^{-3x}.$$

4. 1)  $E(e^{ty}) = \int_0^{+\infty} e^{ty} f(y) dy$

$$= \int_0^{+\infty} e^{ty} \cdot \frac{1}{y\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{1}{2}(\frac{\ln y - \mu}{\sigma})^2} dy$$

$$\stackrel{x=\ln y}{=} \int_{\mathbb{R}} e^{te^x} \cdot \frac{1}{e^x \sigma \sqrt{2\pi}} \cdot e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} e^x dx$$

$$= \frac{1}{\sigma \sqrt{2\pi}} \int_{\mathbb{R}} \exp\{te^x - \frac{1}{2}(\frac{x-\mu}{\sigma})^2\} dx$$

$$\triangleq \varphi(x) = \exp\{te^x - \frac{1}{2}(\frac{x-\mu}{\sigma})^2\}$$

$$\forall t > 0, \lim_{x \rightarrow +\infty} \varphi(x) = +\infty$$

$\therefore$  矩母函数不存在.

2)  $E(Y^n) = E(e^{nY}) = M_Y(n) = e^{\frac{\sigma^2}{2}n^2 + \mu n}$

5. 设  $X_i$  的矩母函数为  $M_i(t) = e^{\lambda_i(e^t-1)}$ ,  $t < \min\{\lambda_1, \lambda_2\}$

$$\begin{aligned} M_Y(t) &= M_1(t) \cdot M_2(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)} \\ &= e^{(\lambda_1+\lambda_2)(e^t-1)} \end{aligned}$$

$$\therefore Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

6. 设  $X \sim U(0,1)$  为第一次剩下的长度

$Y$  为第二次剩下的长度, 则

$$\text{当 } X=x \text{ 时 } f_Y(y|x) = \begin{cases} \frac{1}{x} & , y \in (0, x) \\ 0 & , y \notin (0, x) \end{cases}$$

$$E(Y|x) = \int_0^x y \cdot \frac{1}{x} dy = \frac{x}{2}$$

$$E(Y) = E(E(Y|X)) = \int_0^1 \frac{x}{2} \cdot 1 dx = \frac{1}{2} \cdot \frac{x^2}{2} \Big|_0^1 = \frac{1}{4}.$$

7. 设他选择第  $X$  个门 ( $X=1, 2, 3$ )  
到达安全之处用时  $Y$

$$E(Y|X=1) = 2$$

$$E(Y|X=2) = 3 + E(Y)$$

$$E(Y|X=3) = 1 + E(Y)$$

$$E(Y) = E(E(Y|X)) = \frac{1}{3} \times 2 + \frac{1}{3} (3 + E(Y)) + \frac{1}{3} (1 + E(Y))$$

$$\Rightarrow E(Y) = 6$$

即到达安全之处用时的期望是 6 小时.

8. 由全期望公式:  $E(Y) = E(E(Y|X)) = E(X)$  用全期望展开后,  $E(Y|X)$  的  $Y$  中  $X$  的函数均是常数, 可以轻易地变量分离.

$$\begin{aligned} E(XY) &= E(E(XY|X)) \\ &= E(X E(Y|X)) \\ &= E(X^2) \end{aligned}$$

$$\begin{aligned} \text{故 } \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\ &= E(X^2) - E(X)^2 = \text{Var}(X) \end{aligned}$$

9. (1) 假设  $X, Y$  是连续型,  $E(Y)$  存在

$$X, Y \text{ 独立} \Rightarrow \forall x: f_Y(y|x) = f_Y(y)$$

$$\begin{aligned} \forall x, E(Y|x) &= \int_{\mathbb{R}} y f_Y(y|x) dy \\ &= \int_{\mathbb{R}} y f_Y(y) dy \\ &= E(Y) \end{aligned}$$

$$\Rightarrow E(Y|X) = E(Y). \text{ 其他类型同理}$$

(2) 反之不成立

设  $X: \begin{array}{|c|c|c|} \hline X & 1 & 2 \\ \hline P & \frac{1}{2} & \frac{1}{2} \\ \hline \end{array}, Y \sim N(0, X).$

$$E(Y) = E(Y|X) = 0.$$

但  $X, Y$  不独立

10. (1)  $\text{Var}(Y|X) = E((Y - E(Y|X))^2 | X)$

$$= E(Y^2 - 2YE(Y|X) + E(Y|X)^2 | X)$$

$$= E(Y^2 | X) - 2E(Y|X)^2 + E(Y|X)^2$$

$$= E(Y^2 | X) - E(Y|X)^2$$

(2)  $E(\text{Var}(Y|X)) = E(E(Y^2|X)) - E(E(Y|X)^2)$

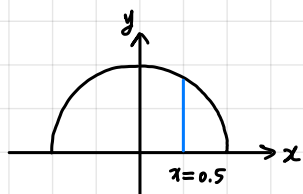
$$= E(Y^2) - E(E(Y|X)^2)$$

$$\text{Var}(E(Y|X)) = E(E(Y|X)^2) - E(E(Y|X))^2$$

$$= E(E(Y|X)^2) - E(Y)^2$$

故  $\text{Var}(Y) = E(\text{Var}(Y|X)) + \text{Var}(E(Y|X))$

11.



$$E(Y|X=0.5) = \frac{\sqrt{3}}{4}$$

由不等式  $E((Y - g(X))^2) \geq E((Y - E(Y|X))^2)$

有最佳估计值为  $E(Y|X=0.5) = \frac{\sqrt{3}}{4}$ .

12. (1) 设观测数据为  $(\vec{x}, \vec{y})$ , 其中  $\vec{x}, \vec{y} \in \mathbb{R}^n$ ,  $\vec{\alpha} = \begin{bmatrix} a \\ b \end{bmatrix}$ ,  $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{n \times 1}$

则  $\min_{a, b \in \mathbb{R}} E((Y - (aX + b))^2) \iff \min_{\vec{\alpha} \in \mathbb{R}^2} |\vec{y} - [\vec{x} \ \vec{v}] \vec{\alpha}|$ .

令  $B = [\vec{x} \ \vec{v}]$  注意到  $B^T B \vec{\alpha} = B^T \vec{y}$ .

而  $B^T B = \begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{bmatrix} \Rightarrow \det(B^T B) = n \sum x_i^2 - (\sum x_i)^2 = n^2 \text{Var}(X) \neq 0$ . 故  $B^T B$  可逆.

$$B^T \vec{y} = \begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix} = \begin{bmatrix} \sum x_i y_i \\ \sum y_i \end{bmatrix} = n \begin{bmatrix} E(XY) \\ E(Y) \end{bmatrix}$$

有  $\vec{\alpha} = (B^T B)^{-1} B^T \vec{y} = \frac{n^2}{n^2 \text{Var}(X)} \begin{pmatrix} E(XY) - E(X)E(Y) \\ E(Y)E(X^2) - E(XY)E(X) \end{pmatrix}$

$$= \text{Var}(X)^{-1} \begin{bmatrix} \text{Cov}(X, Y) \\ E(Y)(E(X^2) - E(X)^2) + E(X)(E(Y)E(X) - E(XY)) \end{bmatrix}$$

$$= \text{Var}(X)^{-1} \begin{bmatrix} \text{Corr}(X, Y) \sqrt{\text{Var}(X) \text{Var}(Y)} \\ E(Y) \text{Var}(X) - E(X) \text{Corr}(X, Y) \sqrt{\text{Var}(X) \text{Var}(Y)} \end{bmatrix}$$

故  $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \rho \frac{\sigma_2}{\sigma_1} \\ \mu_2 - \mu_1 \rho \frac{\sigma_2}{\sigma_1} \end{bmatrix}$

$$(2) E(XY) = \text{Cov}(X, Y) + E(X)E(Y) \\ = \sigma_1 \sigma_2 \rho + \mu_1 \mu_2$$

将上同  $a, b$  代入, 得

$$E((Y - aX - b)^2) \\ = E(Y^2 + a^2 X^2 + b^2 + 2abX - 2aXY - 2bY) \\ = \mu_2^2 + \sigma_2^2 + a^2(\mu_1^2 + \sigma_1^2) + b^2 + 2ab\mu_1 - 2aE(XY) - 2b\mu_2 \\ = \mu_2^2 + \sigma_2^2 + \rho \frac{\sigma_2^2}{\sigma_1^2} (\mu_1^2 + \sigma_1^2) + \mu_2^2 + \mu_1^2 \rho^2 \frac{\sigma_2^2}{\sigma_1^2} - 2\mu_1 \mu_2 \rho \frac{\sigma_2}{\sigma_1} - 2\rho \frac{\sigma_2}{\sigma_1} (\sigma_1 \sigma_2 \rho + \mu_1 \mu_2) \\ - 2(\mu_2 - \mu_1 \rho \frac{\sigma_2}{\sigma_1}) \mu_2 \\ = (1 - \rho^2) \sigma_2^2 \\ \text{当 } |\rho| \rightarrow 1, \sigma_2 \rightarrow 0 \text{ 时: } E((Y - aX - b)^2) \rightarrow 0.$$

(3) 已知: 当  $X = x$  时:  $Y \sim N(\mu_2 + \rho \frac{\sigma_2}{\sigma_1} (x - \mu_1), (1 - \rho^2) \sigma_2^2)$

$$\Rightarrow E(Y|x) = \mu_2 + \rho \frac{\sigma_2}{\sigma_1} (x - \mu_1) = \rho \frac{\sigma_2}{\sigma_1} x + (\mu_2 - \rho \frac{\sigma_2}{\sigma_1} \mu_1) = ax + b \\ \text{故 } E(Y|X) = aX + b.$$

13. (1)  $E(Y|n) = E(\sum_{k=1}^n X_k) = n\mu$

$$E(Y) = E(E(Y|N)) \\ = E(\mu N) = \mu a$$

$$(2) M_N(t) = E(e^{tN}) = \sum_n e^{tn} p_n$$

$$(3) M_Y(t) = E(e^{tY}) = \sum_n e^{t \sum_{k=1}^n X_k} p_n = \sum_n M_X^n(t) p_n$$

$$(4) X_i \sim \text{Exp}(\lambda) \Rightarrow M_X(t) = \frac{\lambda}{\lambda - t} \quad (t < \lambda)$$

$$M_Y(t) = \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda - t}\right)^n (1-p)^{n-1} p$$

$$= \frac{p}{1-p} \sum_{n=1}^{\infty} \left(\frac{\lambda(1-p)}{\lambda - t}\right)^n$$

$$\alpha := \frac{\lambda(1-p)}{\lambda - t} \\ \stackrel{\text{当 } n \rightarrow \infty}{=} \frac{p}{1-p} \frac{\alpha(1-p)^n}{1-\alpha} \quad (|t| < \lambda p)$$

$$= \frac{p}{1-p} \frac{\frac{\lambda}{\lambda - t} (1-p)}{1 - \frac{\lambda(1-p)}{\lambda - t}}$$

$$= \frac{p\lambda}{\lambda - t} \frac{1}{\frac{\lambda - t - \lambda(1-p)}{\lambda - t}} = \frac{\lambda p}{\lambda p - t}$$

$$\Rightarrow Y \sim P(\lambda p)$$

(5) 不同.

$$M_{Z_1+Z_2}(t) = M_Z^2(t)$$

$$M_Y(t) = \sum_{n=0}^{\infty} M_Z^n(t) p_n$$

不存在  $\rho > 0$  s.t.  $\forall t \in (-\rho, \rho)$ ,  $M_{Z_1+Z_2}(t) = M_Y(t)$ .

14. 利用13中的结论,

设  $Z_i \sim N(0, 1)$

$$N \sim G(p).$$

$$Y = \sum_{k=1}^N Z_k$$

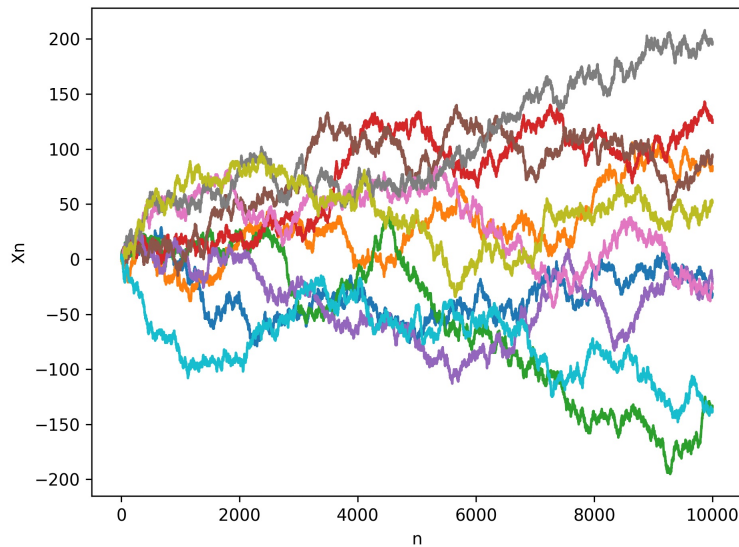
$$\begin{aligned} M_Y(t) &= E(e^{tY}) = \sum_{n=1}^{\infty} e^{t \sum_{i=1}^n Z_i} p_n \\ &= \sum_{n=1}^{\infty} \prod_{i=1}^n M_{Z_i}(t) p_n \\ &= \sum_{n=1}^{\infty} (e^{\frac{t^2}{2}})^n (1-p)^{n-1} p \\ &= \frac{p}{1-p} \sum_{n=1}^{\infty} ((1-p)e^{\frac{t^2}{2}})^n \\ &= \lim_{n \rightarrow \infty} \frac{p}{1-p} \cdot \frac{1 - ((1-p)e^{\frac{t^2}{2}})^n}{1 - (1-p)e^{\frac{t^2}{2}}} \\ &= \frac{p}{1-p} \cdot \frac{1}{1 - (1-p)e^{\frac{t^2}{2}}} \quad (|t| < \sqrt{2 \ln(1-p)}) \end{aligned}$$

$M_Y(t)$  不是  $e^{\frac{\sigma^2}{2}t^2 + \mu t}$  的函数, 所以  $Y$  不呈正态分布

15. (1)  $E(\sum_n) = \sum_{k=1}^n E(Y_k) = 0$

$$\text{Var}(\sum_n) = \sum_{k=1}^n \text{Var}(Y_i) = n$$

(2)



序列未呈现特定趋势，图形差异显著

由  $\text{Var}(X_n) = n$  知，随  $n$  增大， $X_n$  方差越来越大，  
这表明确实随实验次数增加， $X_n$  的分布越发分散。

```

1  from random import randint
2  from matplotlib import pyplot as plt
3
4  plt.figure(dpi=300)
5  plt.ylabel("Xn")
6  plt.xlabel("n")
7
8  def generate():
9      y = [1 if randint(0, 1) else -1 for i in range(0, 10000)]
10     x = [sum(y[0:ii + 1]) for ii in range(0, 10000)]
11     plt.plot(range(0, 10000), x)
12
13
14     for i in range(0, 10):
15         generate()
16
17     plt.show()
18

```





# HW8

1. 不准确. 设示性变量  $X_i = \begin{cases} 1, & \text{第 } i \text{ 次事件 } A \text{ 发生} \\ 0, & \text{第 } i \text{ 次事件 } A \text{ 不发生.} \end{cases}$   
 则  $X_i \sim B(p(A))$  设  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$

由 Kolmogorov 强大数定律 知:  $P(\lim_{n \rightarrow \infty} \bar{X} = E(X_i)) = 1$

即  $P(\lim_{n \rightarrow \infty} \frac{m}{n} = p(A)) = 1$ ,  $\frac{m}{n}$  在  $n \rightarrow \infty$  时以概率 1 收敛到  $p(A)$ .

但不是  $\lim_{n \rightarrow \infty} \frac{m}{n} = p(A)$

2. 由  $E(\frac{\sum_{i=1}^n X_i}{n}) = \frac{\sum_{i=1}^n \mu_i}{n}$  和 Chebyshev 不等式

$$\text{有 } P\left(\left|\frac{\sum_{i=1}^n X_i}{n} - \frac{\sum_{i=1}^n \mu_i}{n}\right| \geq \varepsilon\right) = \frac{\text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right)}{\varepsilon^2}$$

$$= \frac{\sum_{i=1}^n \sigma_i^2}{n^2 \varepsilon^2}$$

$$< \frac{C}{n \varepsilon^2}$$

$$\text{故 } 0 \leq P\left(\left|\frac{\sum_{i=1}^n X_i}{n} - \frac{\sum_{i=1}^n \mu_i}{n}\right| \geq \varepsilon\right) < \frac{C}{n \varepsilon^2} \rightarrow 0 \quad (n \rightarrow \infty)$$

$$\text{故 } \lim_{n \rightarrow \infty} P\left(\left|\frac{\sum_{i=1}^n X_i}{n} - \frac{\sum_{i=1}^n \mu_i}{n}\right| \geq \varepsilon\right) = 0$$

3. 证明:

由 CLT:

$$\lim_{n \rightarrow \infty} P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq x\right) = \Phi(x) \iff \lim_{n \rightarrow \infty} P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \geq x\right) = 1 - \Phi(x)$$

$$\text{而 } P(|\bar{X} - \mu| \geq \varepsilon)$$

$$= P(\bar{X} - \mu \geq \varepsilon \text{ 或 } \bar{X} - \mu \leq -\varepsilon)$$

$$= P(\bar{X} - \mu \geq \varepsilon) + P(\bar{X} - \mu \leq -\varepsilon) = P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \geq \frac{\sqrt{n}\varepsilon}{\sigma}\right) + P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq -\frac{\sqrt{n}\varepsilon}{\sigma}\right)$$

$$\text{故 } \lim_{n \rightarrow \infty} P(|\bar{X} - \mu| \geq \varepsilon) = \lim_{n \rightarrow \infty} P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \geq \frac{\sqrt{n}\varepsilon}{\sigma}\right) + \lim_{n \rightarrow \infty} P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq -\frac{\sqrt{n}\varepsilon}{\sigma}\right).$$

$$\stackrel{\text{CLT}}{=} \lim_{n \rightarrow \infty} (1 - \Phi\left(\frac{\sqrt{n}\varepsilon}{\sigma}\right) + \Phi\left(-\frac{\sqrt{n}\varepsilon}{\sigma}\right))$$

$$\text{而 } \lim_{n \rightarrow \infty} \Phi\left(\frac{\sqrt{n}\varepsilon}{\sigma}\right) = \int_{-\infty}^{+\infty} f(x) dx = 1.$$

$$\lim_{n \rightarrow \infty} \Phi\left(\frac{\sqrt{n}\varepsilon}{\sigma}\right) = \lim_{x \rightarrow -\infty} \int_{-\infty}^x f(x) dx = 0$$

$$\text{故 } \lim_{n \rightarrow \infty} P(|\bar{X} - \mu| \geq \varepsilon) = 0$$

4. 之前已证:  $E(S^2) = \sigma^2$

由 Chebyshev 不等式.

$$0 \leq P(|S^2 - \sigma^2| \geq \varepsilon) \leq \frac{\text{Var}(S^2)}{\varepsilon^2} = \frac{1}{\varepsilon^2} \cdot \frac{2\sigma^4}{n-1} \rightarrow 0 \quad (n \rightarrow \infty)$$

$$\text{由夹挤原理: } \lim_{n \rightarrow \infty} P(|S^2 - \sigma^2| \geq \varepsilon) = 0$$

$$\text{即 } S^2 \xrightarrow{P} \sigma^2$$

5. 设  $A: \lim_{n \rightarrow \infty} X_n = a, P(A) = 1$

$B: \lim_{n \rightarrow \infty} Y_n = b, P(B) = 1$

$C: \lim_{n \rightarrow \infty} \frac{X_n}{Y_n} = c$

$$\text{由 } AB \subset C \quad P((AB)^c) = P(A^c + B^c)$$

$$\text{有 } C^c \subset (AB)^c = A^c + B^c$$

$$0 \leq P(C^c) \leq P(A^c + B^c) = P(A^c) + P(B^c) - P(A^c B^c)$$

$$\leq P(A^c) + P(B^c) = 0$$

$$\text{故 } P(C^c) = 0 \Rightarrow P\left(\lim_{n \rightarrow \infty} \frac{X_n}{Y_n} = \frac{a}{b}\right) = 1$$

6. 由Kolmogorov 强大数定律:

$$P(\lim_{n \rightarrow \infty} \bar{X} = 2) = 1$$

$$P(\lim_{n \rightarrow \infty} \bar{Y} = 5) = 1$$

由5题知:

$$P(\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n Y_i} = \frac{2}{5}) = P(\lim_{n \rightarrow \infty} \frac{\bar{X}}{\bar{Y}} = \frac{2}{5}) = 1$$

7.  $P(X=20) = C_{40}^{20} (\frac{1}{2})^{40} \approx 0.12537$

$$X \overset{\text{近似}}{\sim} N(40 \times \frac{1}{2}, 40 \times \frac{1}{2} \times \frac{1}{2})$$

$$\text{即 } X \overset{\text{近似}}{\sim} N(20, 10)$$

$$P(19.5 \leq X \leq 20.5) = \Phi\left(\frac{20.5-20}{\sqrt{10}}\right) - \Phi\left(\frac{19.5-20}{\sqrt{10}}\right) \\ \approx 0.12563$$

8. 11) 设每份保险赔 $X$ 元.

$$E(X) = 10^{-3} \times 10^3 = 1 < 2.$$

合理.

12) 设这10000份保险赔付为 $Y$ 元

$$20000 - Y \geq 20000 \times 20\%$$

$$\Rightarrow Y \leq 16000$$

设共有 $M$ 次事故. 则 $Y = 10^3 M$ ,  $Y \leq 16000 \Leftrightarrow M \leq 16$

$$M \sim B(10000, 10^{-3}), \mu = 10, \sigma^2 = \frac{999}{100}$$

由CLT, 用正态估计 $M$ 的分布

$$P(M \leq 16) \approx \Phi\left(\frac{16.5-10}{\frac{\sqrt{999}}{10}}\right) \approx 0.9801$$

13)  $\Phi^{-1}(0.95) \approx 1.645 \Rightarrow \tilde{M} = \sigma \Phi^{-1}(0.95) + \mu \approx 15.2$

$$\Rightarrow \tilde{Y} = 10^3 M = 15200$$

$\Rightarrow$  以95%概率至少4800元毛利

9.<sup>(1)</sup> 设每次测量值与真值之差为  $\bar{X}_i$ . (iid)

有  $\bar{X}_i \sim U(-1, 1)$ ,  $\mu = 0$ ,  $\sigma^2 = \frac{1}{12} \times 4 = \frac{1}{3}$

由CLT, 认为  $P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq x\right) = \Phi(x)$

$$\begin{aligned} \text{所以 } P(|\bar{X}| < \varepsilon) &= P\left(\frac{-\varepsilon\sqrt{n}}{\frac{\sigma}{\sqrt{n}}} \leq \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \leq \frac{\varepsilon\sqrt{n}}{\frac{\sigma}{\sqrt{n}}}\right) \approx \Phi(\varepsilon\sqrt{3n}) - \Phi(-\varepsilon\sqrt{3n}) \\ &= 1 - 2\Phi(-\varepsilon\sqrt{3n}) \end{aligned}$$

令  $n = 25$ ,  $\varepsilon = 0.2$ .

$$\text{有 } P(|\bar{X}| < 0.2 | n=25) \approx 1 - 2\Phi(-\sqrt{3}) = 0.92$$

(2) 为使  $1 - 2\Phi(-\varepsilon\sqrt{3n}) \geq 1 - \alpha$

$$\text{有 } \Phi(-\varepsilon\sqrt{3n}) \leq \frac{\alpha}{2}$$

$$\Rightarrow -\varepsilon\sqrt{3n} \leq \Phi^{-1}\left(\frac{\alpha}{2}\right)$$

$$\sqrt{3n} \geq -\frac{1}{\varepsilon} \Phi^{-1}\left(\frac{\alpha}{2}\right)$$

$$n \geq \frac{1}{3\varepsilon^2} \left(\Phi^{-1}\left(\frac{\alpha}{2}\right)\right)^2$$

$$\text{令 } \varepsilon = 0.2, \alpha = 0.05,$$

$$\text{有 } n \geq 32.01$$

$\Rightarrow$  至少33次

(3) 由Chebyshev:  $P(|\bar{X} - \mu| \geq \varepsilon) \leq \frac{\text{Var}(\bar{X})}{\varepsilon^2} \leq \alpha$

$$\text{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{1}{3n\varepsilon^2} \leq \alpha$$

$$\Rightarrow n \geq \frac{1}{3\varepsilon^2\alpha} = 166.67$$

$\Rightarrow$  至少167次

CLT比Chebyshev结果更紧

10. 在这5000次实验中, 设示性变量  $I_n = \begin{cases} 1, & \text{合格} \\ 0, & \text{不合格} \end{cases}$

则  $\mu := E(I_n) = 0.8$   $\sigma^2 = 0.8(1-0.8) = 0.16$

由CLT进行估计.  $\text{Var}(\bar{I}) = \frac{\sigma^2}{n} = \frac{(0.4)^2}{n}$

$$P(|I_n - 0.8| \leq \varepsilon) = P\left(\frac{-\varepsilon\sqrt{n}}{\sigma} \leq \frac{I_n - \mu}{\sigma/\sqrt{n}} \leq \frac{\varepsilon\sqrt{n}}{\sigma}\right)$$

$$\approx 1 - 2\Phi\left(\frac{-\varepsilon\sqrt{n}}{\sigma}\right)$$

$$= 1 - 2\Phi(-125\sqrt{2} \cdot \varepsilon) \geq 1 - 0.01$$

$$\Rightarrow \varepsilon \geq -\frac{\Phi^{-1}(0.005)}{125\sqrt{2}} \approx 0.01457$$

$$\Rightarrow \bar{I} \in [\mu - \varepsilon, \mu + \varepsilon]$$

$$\Rightarrow n = 5000 \bar{I}$$

$$\therefore n = [3927, 4073] \cap \mathbb{Z}$$

11.

2020年6-7月关于唐纳德·特朗普前总统的民意调查<sup>[11][12][13]</sup>

| 民调机构                            | 日期            | 支持率 | 反对率 | 摇摆 | 净+/- | 样本容量 <sup>[13]</sup> | 样本范围 <sup>[13][14]</sup> |
|---------------------------------|---------------|-----|-----|----|------|----------------------|--------------------------|
| RMG Research <sup>[a][15]</sup> | 2020年7月9-11日  | 44% | 53% | 3% | -9%  | 1,200                | 登记选民                     |
| YouGov <sup>[16]</sup>          | 2020年7月8-10日  | 42% | 53% | 5% | -11% | 1,000                | 所有成年人                    |
| Morning Consult <sup>[17]</sup> | 2020年7月7-9日   | 38% | 57% | 5% | -19% | 2,200                | 所有成年人                    |
| 拉斯穆森报导 <sup>[b][20]</sup>       | 2020年7月7-9日   | 45% | 53% | 2% | -8%  | 1,500                | 活跃选民                     |
| 路透社/益普索 <sup>[21]</sup>         | 2020年7月6-7日   | 38% | 57% | 5% | -19% | 1,114                | 所有成年人                    |
| Research Co. <sup>[22]</sup>    | 2020年7月1-2日   | 40% | 56% | 4% | -16% | 1,200                | 所有成年人                    |
| IBD/TIPP <sup>[23]</sup>        | 2020年6月27-30日 | 39% | 56% | 5% | -17% | 1,213                | 所有成年人                    |

可见, 人数都是 1k~2k 的分布, 与课上案例类似

理由: 基于以下假设

抽样数量很小, 认为是独立的,

这时可使用独立同分布的随机

变量进行建模, 这时由CLT, 样本

均值近似服从正态分布,

故对于一定的精度和置信程度

抽样人数与总体大小无关.

12. 设  $\sum_n = \begin{cases} 1.7 & \text{上涨} \\ 0.5 & \text{下跌} \end{cases}$  独立同分布

$$\Rightarrow Y_n = Y_0 \prod_{i=1}^n \sum_i$$

$$\ln Y_n = \ln Y_0 + \sum_{i=1}^n \ln \sum_i, \text{ 设 } \mu = E(\ln \sum_1) = \frac{1}{2} \ln 0.85$$

则由 CLT:  $n \rightarrow \infty$  时:  $\begin{cases} \mu = E(\ln \sum_1) \\ \sigma^2 = \text{Var}(\ln \sum_1) \end{cases}$

$$\text{令 } M_n = \ln Y_n - \ln Y_0 = \sum_{i=1}^n \ln \sum_i$$

$$P\left(\frac{M_n - n\mu}{\sqrt{n}\sigma} \leq x\right) = \Phi(x)$$

$$\Rightarrow M_n \text{ 近似服从 } N(n\mu, n\sigma^2)$$

$$\Rightarrow \ln Y_n \text{ 近似服从 } N(n\mu, n\sigma^2)$$

(2)  $n \rightarrow \infty$  时:

$$\begin{aligned} E(Y_{n+1}) &= E(E(Y_{n+1} | Y_n)) \\ &= E\left(1.7Y_n \times \frac{1}{2} + 0.5Y_n \times \frac{1}{2}\right) \\ &= 1.1E(Y_n) \end{aligned}$$

$$E(Y_0) = Y_0 = 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} E(Y_n) = +\infty$$

(3) 由(1)知:  $E\left(\frac{\ln Y_n}{n}\right) = \mu = \frac{1}{2} \ln 0.85$

由 Kolmogorov 强大数定律:

$$P\left(\lim_{n \rightarrow \infty} \frac{\ln Y_n}{n} = \mu\right) = 1$$

$$\Rightarrow P\left(\lim_{n \rightarrow \infty} e^{\frac{\ln Y_n}{n}} = e^\mu\right) = 1$$

$$\text{即 } P\left(\lim_{n \rightarrow \infty} Y_n^{\frac{1}{n}} = \sqrt{0.85}\right) = 1$$

$$\text{而事件 } \lim_{n \rightarrow \infty} Y_n^{\frac{1}{n}} = \sqrt{0.85} \subseteq \text{事件 } \lim_{n \rightarrow \infty} Y_n = 0$$

$$\text{故 } 1 \leq P(\lim_{n \rightarrow \infty} Y_n = 0) \leq 1$$

$$\text{即 } P(\lim_{n \rightarrow \infty} Y_n = 0) = 1$$

(4) 我感到难以解释.....

13. <sup>(1)</sup> 对于  $\forall$  给定的  $x \in \mathbb{R}$

$$\text{考虑 } I_i(x) = \begin{cases} 1, & X_i \leq x \\ 0, & X_i > x. \end{cases}$$

$$\text{而 } P(I_i = 1) = F(x), E(I_i(x)) = F(x), \text{Var}(I_i(x)) = F(x)(1-F(x))$$

$$F_n(x) = \bar{I}(x)$$

$$\text{故 } E(F_n(x)) = F(x)$$

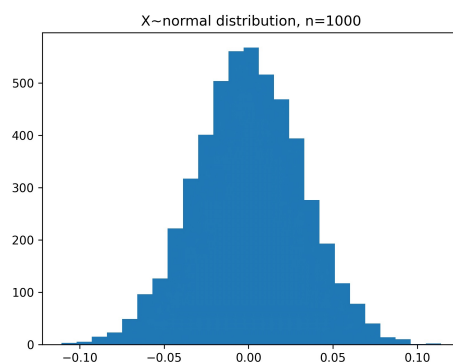
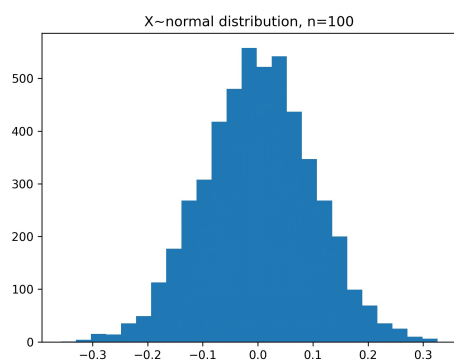
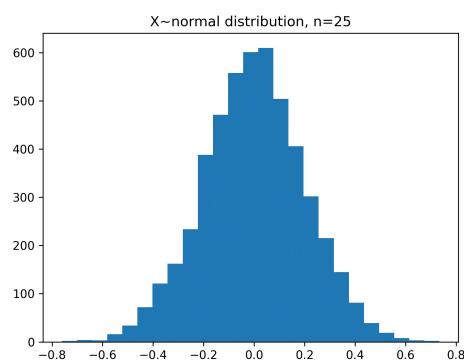
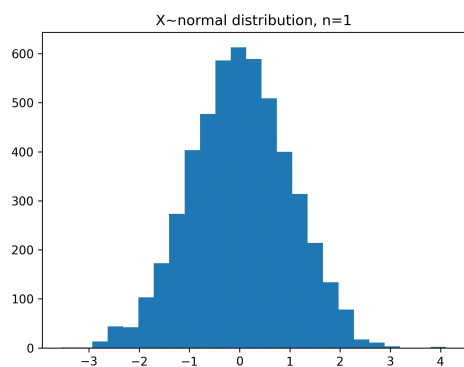
$$\text{Var}(\bar{I}(x)) = \frac{F(x)(1-F(x))}{n}$$

(3) 由 Kolmogorov 强大数定律, 对  $\forall$  给定  $x$ .

$$P\left(\lim_{n \rightarrow \infty} \bar{I}(x) = E(I_i(x))\right) = 1$$

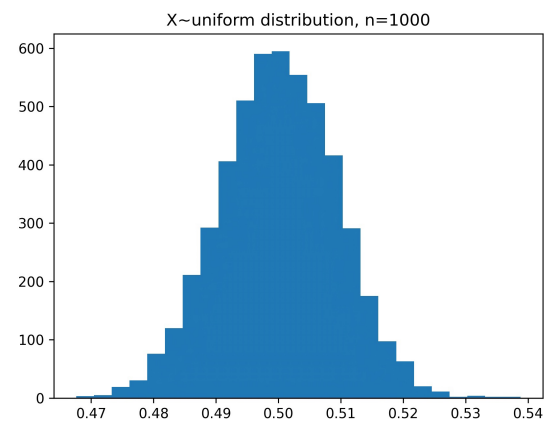
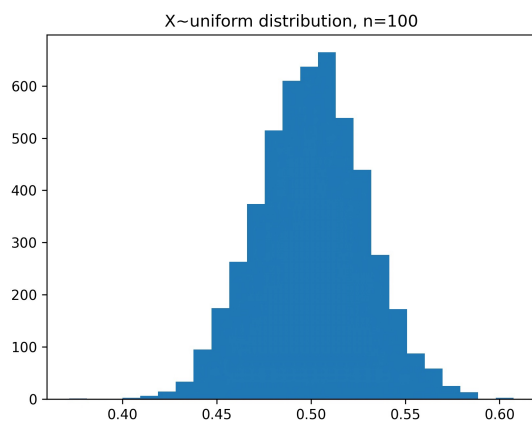
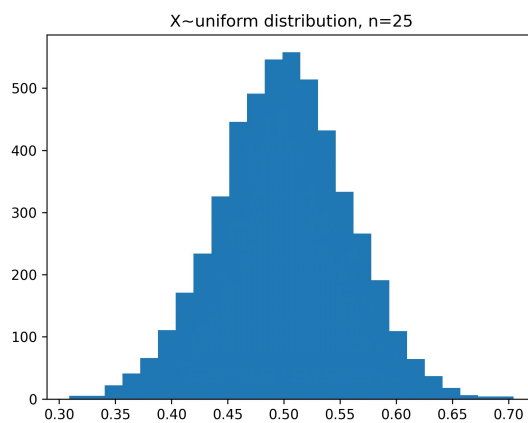
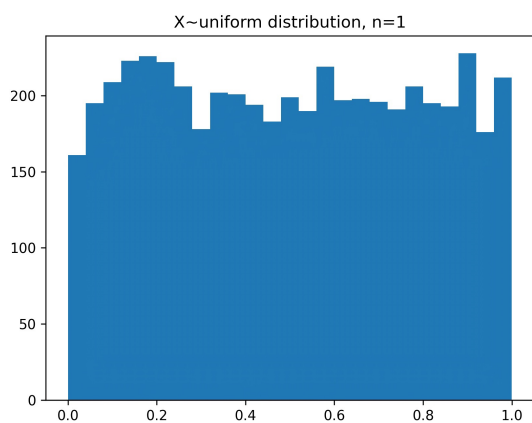
$$\text{即 } P\left(\lim_{n \rightarrow \infty} F_n(x) = F(x)\right) = 1$$

14. <sup>(1)</sup>

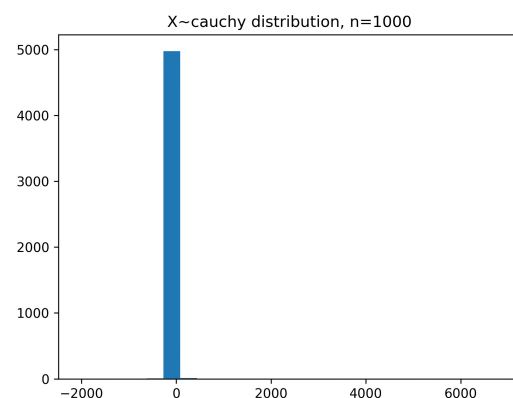
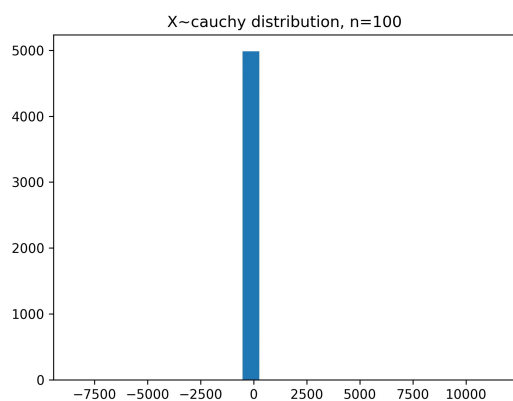
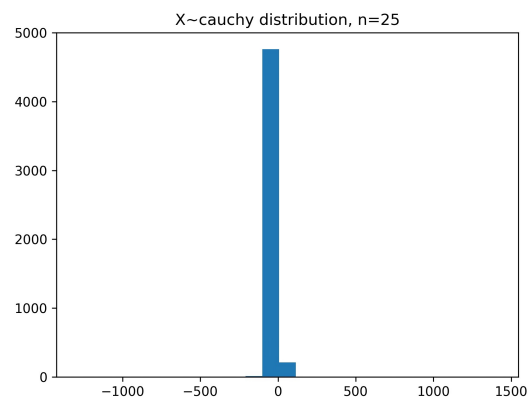
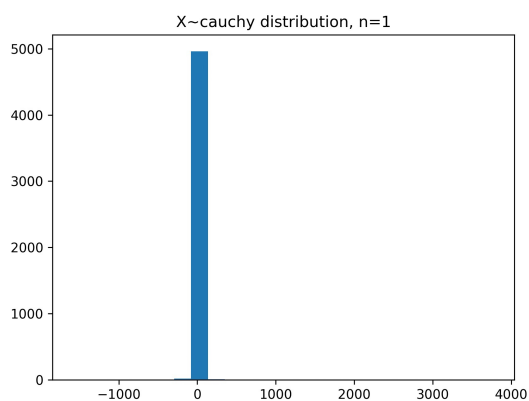




(2)



(3)



- 对(1)和(2):  
n越大,它越趋于正态分布,且方差渐小.
- 对(3):  
cauchy分布不存在期望、方差,所以不符合CLT.

```
1 import random
2 from matplotlib import pyplot as plt
3 import numpy as np
4
5 def normal(n):
6     return [random.normalvariate(0, 1) for ii in range(0, n)]
7
8 def uniform(n):
9     return [random.uniform(0, 1) for ii in range(0, n)]
10
11 def cauchy(n):
12     return np.random.standard_cauchy(n)
13
14 def graph(src, n, sample):
15     expectancy = [sum(src(n)) / n for ii in range(0, sample)]
16     plt.figure(dpi=300)
17     plt.hist(expectancy, bins=25)
18     plt.title(f" $X \sim \{src.__name__\}$  distribution,  $n=\{n\}$ ")
19     plt.savefig(f"{src.__name__}_{n}.jpg")
20
21 spl = 5000
22 for i in [1, 25, 100, 1000]:
23     graph(normal, i, spl)
24     graph(uniform, i, spl)
25     graph(cauchy, i, spl)
26
```

